A LOOP MODULE OF THE EXTENDED AFFINE LIE ALGEBRA OF TYPE A

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Abstract. We shall construct a loop module for the extended affine Lie algebra over a quantum torus.

1. Introduction

Extended affine Lie algebras form a new class of infinite dimensional Lie algebras, which were first introduced by Høegh–Krohn and Torresani [9] as a generalization of the finite dimensional simple Lie algebras and the affine Kac–Moody Lie algebras, and systematically studied in the book [1].

In this note, we give some irreducible representations for the extended affine Lie algebra of type $A$ coordinated by a quantum torus. This loop module–like was motivated by Chari’s work [5]. Representations for extended affine Lie algebras have been constructed by a number of people. The approach in this note is very elementary and straightforward in some sense.

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2. An Extended Affine Lie Algebra

In this section we present our Lie algebra obtained by using a quantum torus.

Let $\mathbb{C}_q = \mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ be the quantum torus which is an associative unital algebra over the complex field $\mathbb{C}$ with generators $x^{\pm 1}$ and $y^{\pm 1}$, subject to the following relations:

$$xx^{-1} = x^{-1}x = 1$$
$$yy^{-1} = y^{-1}y = 1$$
$$yx = qxy.$$

Then $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}] = \oplus_{m,n \in \mathbb{Z}} \mathbb{C}x^m y^n$ which we will write as $\mathbb{C}_q$. We need some preliminary stuff before we define our algebra. Let $d_x$ and $d_y$ be the degree operators.

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— that is,
\[ d_x x^m y^n = m x^m y^n \]
\[ d_y x^m y^n = n x^m y^n. \]

The degree operators \( d_x \) and \( d_y \) are derivations of \( \mathbb{C}_q \) which can be lifted to be derivations of \( \mathfrak{M}_n(\mathbb{C}_q) \).

We now define a \( \mathbb{C} \)–linear function \( \varepsilon : \mathbb{C}_q \to \mathbb{C} \) by
\[
\varepsilon(x^m y^n) = \begin{cases} 1 & \text{if } m = n = 0, \\ 0 & \text{otherwise.} \end{cases}
\]

Recall that for an associative algebra \( R \), the matrix algebra \( \mathfrak{M}_n(R) \) is an associative algebra. Then the general linear Lie algebra over \( R \) is \( \mathfrak{gl}_n(R) = \mathfrak{M}_n(R) \) where the Lie bracket is given by \([A,B] = AB - BA\) for \( A, B \in R \). Consider the Lie algebra \( \mathfrak{gl}_n(\mathbb{C}_q) \).

Let \( c_x \) and \( c_y \) be symbols. Set
\[ \mathfrak{g} \mathfrak{l}_n(\mathbb{C}_q) = \mathfrak{gl}_n(\mathbb{C}_q) \oplus \mathbb{C}c_x \oplus \mathbb{C}c_y \]
where \([A,B] = AB - BA + \varepsilon(\text{tr}((d_x A)B)) c_x + \varepsilon(\text{tr}((d_y A)b)) c_y \) and
\[
[c_x, c_x] = 0, 
[c_y, c_y] = 0, 
[c_x, c_y] = 0, 
[c_x, \mathfrak{gl}_n(\mathbb{C}_q)] = [\mathfrak{gl}_n(\mathbb{C}_q), c_x] = 0, 
[c_y, \mathfrak{gl}_n(\mathbb{C}_q)] = [\mathfrak{gl}_n(\mathbb{C}_q), c_y] = 0.
\]

Claim. \( \mathfrak{g} \mathfrak{l}_n(\mathbb{C}_q) \) is a Lie algebra.

This can be verified by noting that \( \varepsilon(\varepsilon(a)) = 0 = \varepsilon(\varepsilon(a)) = 0 \) for \( a \in (\mathbb{C}_q), \]
\( \varepsilon(\text{tr}(AB)) = \varepsilon(\text{tr}(BA)), B, A \in \mathfrak{M}_n(\mathbb{C}_q), \) and the fact that \( d_x, d_y \) are derivations.

We will now form a semi–direct product of the Lie algebra \( \mathfrak{g} \mathfrak{l}_n(\mathbb{C}_q) \) with the degree operators \( d_x, d_y \).
\[ \mathfrak{g} \mathfrak{l}_n(\mathbb{C}_q) = \mathfrak{g} \mathfrak{l}_n(\mathbb{C}_q) \oplus \mathbb{C}d_x \oplus \mathbb{C}d_y \]
subject to the following brackets:
\[
[d_x, c_x] = 0, 
[d_y, c_x] = 0, 
[d_x, d_y] = 0, 
[d_x, A] = d_x A, 
[d_y, A] = d_y A.
\]

The Lie algebra \( \mathfrak{g} \mathfrak{l}_n(\mathbb{C}_q) \) is called an extended affine Lie algebra.

3. \( M_n(\mathbb{C}_q) \) — Modules

In this section we will introduce two modules of \( \mathfrak{M}_n(\mathbb{C}_q) \) which would naturally give modules for the extended affine Lie algebra \( \mathfrak{g} \mathfrak{l}_n(\mathbb{C}_q) \).

Let \( W \) be a \( \mathbb{C}_q \)–module and \( V \) a \( \mathfrak{M}_n(\mathbb{C}) \)–module. We know that \( V \otimes W \) is an \( \mathfrak{M}_n(\mathbb{C}) \)–module.

We will now construct our modules:
Construction. Let $V = \mathbb{C}^n$ be the natural module for $M_n(\mathbb{C})$ and let $W_1 = \mathbb{C}_q$ with left multiplication as the module action. Then $V \otimes W_1$ is a $M_n(\mathbb{C}_q)$-module.

Construction. Let $V = \mathbb{C}^n$ be the natural module for $M_n(\mathbb{C})$ and let $W_2 = \mathbb{C}[x, x^{-1}]$, with $\mathbb{C}_q$-module action defined by $x$ as left multiplication such that $x^m y^n f(x) = x^m f(q^n x)$. Then $V \otimes W_2$ is a $M_n(\mathbb{C}_q)$-module. Moreover, if $q$ is not a root of unity, $V \otimes W_2$ is irreducible.

We will prove our claim in Construction 2, which will be a direct consequence of the following two lemmas:

**Lemma 1.** If $V = \mathbb{C}^n$ is the natural representation of $M_n(\mathbb{C})$ and $W$ is irreducible as a $\mathbb{C}_q$-module, then $V \otimes W$ is an irreducible $M_n(\mathbb{C}_q)$-module.

**Proof.** Let $N$ be a submodule of $V \otimes W$. If $U = \sum_{i=1}^n e_i \otimes w_i \neq 0 \in N$ where $\{e_i\}$ is the standard basis of $\mathbb{C}^n$ then $e_k e_i = \delta_{ik} e_k$ where $\delta_{ik}$ is the Kronecker delta and $e_i$ is the standard matrix unit. So $e_k e_i = \sum_{i=1}^n e_k e_i \otimes w_i = e_k \otimes w_j$. Hence $e_k \otimes w_j \in N$. If $U \neq 0$ then $e_j \otimes w_j \in N$, $w_j \neq 0$ for some $j$. Now $e_i(a)(e_j \otimes w_j) = e_i \otimes aw_j \in N$ for all $i$ and $a \in \mathbb{C}_q$. Since $W$ is $\mathbb{C}_q$-irreducible, $e_i \otimes W \subseteq N$ for all $i$. This implies that $V \otimes W \subseteq N$, which implies that $N = V \otimes W$. Therefore $V \otimes W$ is irreducible. \qed

**Lemma 2.** $W_2 = \mathbb{C}[x, x^{-1}]$ is $\mathbb{C}_q$-irreducible if $q$ is not a root of unity.

**Proof.** Let $N$ be a submodule of $W_2$. Let $f(x) = \sum_{i=0}^n a_i x^i \neq 0 \in W_2$. We may assume that $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in N$. Now $y^n f(x) = f(q^n x) \in N$ for all $m \in \mathbb{Z}$. Consider the following $n + 1$ equations:

\[
\begin{align*}
a_0 + a_1 x + \ldots + a_n x^n &= f(x) \\
a_0 + a_1 q x + \ldots + a_n q^n x^n &= f(qx) \\
a_0 + a_1 q^2 x + \ldots + a_n q^{2n} x^n &= f(q^2 x) \\
&\vdots \\
a_0 + a_1 q^n x + \ldots + a_n q^{n^2} x^n &= f(q^n x).
\end{align*}
\]

Consider the coefficient matrix which is a Vandermonde matrix of these $n + 1$ equations:

\[
P = \begin{pmatrix} 1 & 1 & \ldots & 1 \\
1 & q & \ldots & q^n \\
1 & q^2 & \ldots & q^{2n} \\
\vdots & \vdots & \ddots & \vdots \\
1 & q^n & \ldots & q^{n^2} \end{pmatrix}.
\]

So

\[
P \begin{pmatrix} a_0 \\ a_1 x \\ \vdots \\ a_n x^n \end{pmatrix} = \begin{pmatrix} f(x) \\ f(qx) \\ \vdots \\ f(q^n x) \end{pmatrix}.
\]
Since \( q \) is not a root of unity, \( P \) is invertible. Hence
\[
\begin{pmatrix}
a_0 \\ a_1x \\ \vdots \\ a_nx^n
\end{pmatrix} = P^{-1} \begin{pmatrix}
f(x) \\ f(qx) \\ \vdots \\ f(q^n x)
\end{pmatrix}.
\]
Therefore \( a_i x^i \in \mathbb{N} \) for all \( i = 0, 1, \ldots, n \). We assumed at \( f(x) \neq 0 \) so \( a_j x^j \neq 0 \) for some \( j \). Hence \( a_j^{-1} x^{-1} a_j x^j = 1 \in \mathbb{N} \). From this it follows that \( W_2 = \mathbb{N} \) and hence \( W_2 \) is \( \mathbb{C}_q \)-irreducible. \( \square \)

So from these two lemmas we immediately have:

**Theorem 3.** If \( V = \mathbb{C}^n \), the natural module for \( M_n(\mathbb{C}) \) and \( W = \mathbb{C}[x, x^{-1}] \), a \( \mathbb{C}_q \)-module, and a module action given by \( x^m y^n f(x) = x^{mn} f(q^n x) \), then \( V \otimes W \) is a \( M_n(\mathbb{C}_q) \)-module if \( q \) is not a root of unity.

4. \( \tilde{g}_n(\mathbb{C}_q) \)–Modules

We now state and prove the main results of this paper.

**Theorem 4.** \( V \otimes W_1 \) is an \( \tilde{g}_n(\mathbb{C}_q) \)-module. Moreover, \( V \otimes W_1 \) is irreducible.

**Proof.** The actions are defined in the natural way. For example, \( c_x \) and \( c_y \) act trivially. The proof follows from Lemma 1. \( \square \)

**Theorem 5.** \( V \otimes W_2 \times \mathbb{C}[y, y^{-1}] \) is an \( \tilde{g}_n(\mathbb{C}_q) \)-module. Moreover, if \( q \) is not a root of unity, \( V \otimes W_2 \) is irreducible.

**Proof.** The actions are defined naturally. The actions \( c_x \) and \( c_y \) are trivial. The proof follows from Lemma 1 and Theorem 1. \( \square \)

**References**


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