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RANDOM APPROXIMATION AND RANDOM FIXED POINT THEORY FOR RANDOM NON–SELF MULTIMAPS

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Abstract. This paper presents new random fixed point theorems and random Leray–Schauder alternatives for a variety of maps (e.g., $\mathcal{B}^\kappa$, $\mathcal{U}_c^\kappa$, KKM and PK maps). A random Kransnoselskii cone compression theorem for $\mathcal{U}_c^\kappa$ maps is also given. Various random approximation theorems for the above classes are proved and as applications several random fixed point theorems are also derived.

1. Introduction

Probabilistic operator theory is the branch of probabilistic analysis concerned with the study of random operators and their properties. It is required for the study of various models in the applied sciences. The theory of random fixed point is the core of this area and lies at the intersection of nonlinear analysis and probability theory. Although its systematic study was initiated by the Prague school of probabilists in the middle of the 20th century, most of the work has been done during the last 25 years (see [**1**], [**5**], [**12**], [**14**], [**19**], [**21**], [**22**], [**23**] and references therein).Most random fixed point theorems for multimaps in Banach spaces consider either convex-valued maps or acyclic–valued maps (cf., [**12**], [**19**], [**20**], [**22**]). Of course it is of interest to obtain random fixed point theory for maps which are neither convex–valued nor acyclic–valued. In this paper we prove several random fixed point theorems for a general class of maps, namely the $\mathcal{U}_c^\kappa$ maps (other types of maps are also considered). It is worth mentioning that the class of $\mathcal{U}_c^\kappa$ maps includes the Kakutani maps, the acyclic maps, the O'Neill maps, the approximable maps, and also the maps admissible in the sense of Gorniewicz. We begin with random fixed point theory for $\mathcal{U}_c^\kappa$ maps in hyperconvex spaces. We also present new random fixed point results for a variety of maps (e.g., $\mathcal{B}^\kappa$, $\mathcal{U}_c^\kappa$, KKM, PK and inward maps). Random Leray–schauder alternatives and Furi–Pera type theorems are also mentioned. Then we establish a random Krasnoselskii compression theorem for $\mathcal{U}_c^\kappa$ maps. Finally, we prove several random approximation theorems for $\mathcal{U}_c^\kappa$, KKM and PK maps and derive, as applications, various random fixed point theorems for such maps.

Next in this section we present some preliminary results which will be needed. Let $(\Omega, \mathcal{A})$ be a measurable space and C a nonempty subset of a metric space $X = (X, d)$. Let 2^C denote the family of nonempty subsets of C and $CD(C)$ the family of all nonempty closed subsets of C. A mapping $G : \Omega \to 2^C$ is said to be measurable if

$$G^{-1}(U) \ = \ \{w \in \Omega : \ G(w) \cap U \neq \varnothing\} \in \mathcal{A}$$

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for each open subset U of C. A mapping $\xi : \Omega \to C$ is called a measurable selector of the measurable mapping $G : \Omega \to 2^C$ if ξ is measurable and $\xi(w) \in G(w)$ for each $w \in \Omega$. A mapping $F : \Omega \times C \to 2^X$ is called a random operator if, for any fixed $x \in C$, the map $F(\,.\,, x) : \Omega \to 2^X$ is measurable. A measurable mapping $\xi : \Omega \to C$ is said to be a random fixed point of a random operator $F : \Omega \times C \to 2^X$ if $\xi(w) \in F\big(w, \xi(w)\big)$ for each $w \in \Omega$. Let $P_B(X)$ be the bounded subsets of X. The Kuratowskii measure of noncompactness is the map $\alpha : P_B(X) \to [0, \infty)$ defined by

$$\alpha(A) = \inf \{\varepsilon > 0 : \ A \subseteq \cup_{i=1}^n X_i \quad \text{and} \quad \operatorname{diam}(X_i) \le \varepsilon\};$$

here $A \in P_B(X)$. Let C be a nonempty subset of X, and for each $x \in X$ define $d(x, C) = \inf_{y \in C} d(x, y)$. Let $F : C \to 2^X$. F is called

(i) (countably) k–set contractive ($k \ge 0$) if $F(C)$ is bounded and $\alpha\big(F(Y)\big) \le k\,\alpha(Y)$ for all (countably) bounded sets Y of S;

(ii) (countably) condensing if $F(C)$ is bounded and $\alpha\big(F(Y)\big) < \alpha(Y)$ for all (countably) bounded sets Y of C with $\alpha(Y) \ne 0$;

(iii) hemicompact if each sequence $(x_n)_{n=1}^\infty$ in C has a convergent subsequence whenever $d\big(x_n, F(x_n)\big) \to 0$ as $n \to \infty$. F is said

(iv) to satisfy condition (A) if for any sequence $(x_n)_{n=1}^\infty$ in C, $D \in CD(C)$ such that $d(x_n, D) \to 0$ and $d\big(x_n, F(x_n)\big) \to 0$, there exists an $x_0 \in D$ with $x_0 \in F(x_0)$.

We note that every continuous hemicompact map satisfies condition (A). For details of hemicompact maps and maps satisfying condition (A), we refer the reader to [**21**], [**23**].

A random operator $F : \Omega \times C \to CD(X)$ is said to be continuous ((countably) k–set contractive etc.) if for each $w \in \Omega$, the map $F(w,\,.\,) : C \to CD(X)$ is continuous ((countably) k–set contractive etc.).

Next we state a well known result of Shahzad [**21**].

Theorem 1.1. *Let $(\Omega, \mathcal{A})$ be a measurable space and Z a nonempty separable complete subset of a metric space $X = (X, d)$. Suppose the map $F : \Omega \times Z \to CD(X)$ is a continuous random operator satisfying condition (A). If F has a deterministic fixed point then F has a random fixed point.*

Remark 1.2. A single valued map $\phi : \Omega \to X$ is said to be a deterministic fixed point of F if $\phi(w) \in F\big(w, \phi(w)\big)$ for each $w \in \Omega$. In other words, F has a deterministic fixed point if the set $\{x \in X : x \in F(\omega, x)\} \ne \varnothing$ for each $\omega \in \Omega$.

Remark 1.3. In Theorem 1.1, if $Y \subset Z$ is closed and $\{x \in Y : x \in F(\omega, x)\} \ne \emptyset$ for each $\omega \in \Omega$, then F has a random fixed piont $\xi : \Omega \to Z$ such that $\xi(\omega) \in Y$.

The following convergence result [**5**] is well known.

Theorem 1.4. *Let (X, d) be a Fréchet space, D a closed subset of X and $F : D \to 2^X$ a condensing map. Then F is hemicompact. If, in addition, F is continuous, then it satisfies condition (A).*

In view of Theorem 1.1 it is easy to use well known fixed point theory to establish random fixed point theory. Before we do this we need to describe these deteministic fixed point theorems. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively. We will look at maps $F : X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say $F : X \to K(Y)$ is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Y)$ is *acyclic* if F is upper semicontinuous with acyclic values. $F : X \to K(Y)$ is said to be an *O'Neill* map if F is continuous and if the values of F consist of one or m acyclic components (here m is fixed).

Given two open neighborhoods U and V of the origins in E_1 and E_2 repectively, a (U,V)–approximate continuous selection [**8**] of $F : X \to K(Y)$ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (F[(x+U) \cap X] + V) \cap Y \qquad \text{for every} \qquad x \in X.$$

We say $F : X \to K(Y)$ is *approximable* if it is a closed map and if its restriction $F|_K$ to any compact subset K of X admits a (U,V)–approximate continuous selection for every open neighborhood U and V of the origins in E_1 and E_2 repectively.

For our next definition let X and Y be metric spaces. A continuous single valued map $p : Y \to X$ is called a Vietoris map if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic

(ii) p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Definition 1.5. A multifunction $\phi : X \to K(Y)$ is *admissible* (strongly) in the sense of Gorniewicz, if $\phi : X \to K(Y)$ is upper semicontinuous, and if there exists a metric space Z and two continuous maps $p : Z \to X$ and $q : Z \to Y$ such that

(i) p is a Vietoris map

and

(ii) $\phi(x) = q\big(p^{-1}(x)\big)$ for any $x \in X$.

Remark 1.6. It should be noted [**11**, pp. 179] that ϕ upper semicontinuous is redundant in Definition 1.5

Definition 1.7. We say $G \in \mathcal{B}(X,Y)$ (here X is a nonempty, convex subset of a Hausdorff topological vector space E and Y a topological space) if $G : X \to 2^Y$ is such that for any polytope P in X and any continuous function $g : G(P) \to P$, the composition $g(G|_P) \to 2^P$ has a fixed point.

Definition 1.8. We say $F \in \mathcal{B}^\kappa(X,Y)$ (i.e., F is $\mathcal{B}^\kappa$–admissible) if $F : X \to 2^Y$ is such that for any compact, convex subset K of X, there exists a closed map $G \in \mathcal{B}(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Suppose X and Y are Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions;

(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and

(iii) for any polytope P, $F \in \mathcal{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

Definition 1.9. $F \in \mathcal{U}_c^\kappa(X,Y)$ if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Examples of $\mathcal{U}_c^\kappa$ maps are the Kakutani maps, the acyclic maps, the O'Neill maps, and the maps admissible in the sense of Gorniewicz.

Let Q be a subset of a Hausdorff topological space X. We let $\overline{Q}$ (respectively, ∂Q, $int(Q)$) to denote the closure (respectively, boundary, interior) of Q.

A map $F : X \to 2^Y$ is called upper hemicontinuous if for each $f \in Y^*$ (the dual of Y) and for any $\alpha \in \boldsymbol{R}$, the set $x \in X : \sup Re\, f(fx) < \alpha\}$ is open in X.

Definition 1.10. $F \in \mathcal{H}(X,Y)$ (i.e., $\mathcal{H}$–admissible) if F is upper hemicontinuous with nonempty, closed, convex values if E is locally convex and with nonempty, compact, convex values if E is not locally convex.

Definition 1.11. Let X be a convex subset of a Hausdorff topological vector space and Y a topological space. If $S,T : X \to 2^Y$ are two set–valued maps such that $T\big(\mathrm{co}(A)\big) \subseteq S(A)$ for each finite subset A of X, then we say that S is a generalized KKM map w.r.t. T. The map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM w.r.t. T map $S : X \to 2^Y$, the family

$$\{\overline{S(x)} : x \in X\}$$

has the finite intersection property. We let

$$KKM(X,Y) \;=\; \{T : X \to 2^Y : T \,\text{has the KKM property}\}\,.$$

Remark 1.12. Let X be a convex subset of a linear space, and let Y and Z be two topological spaces. If $F \in KKM(X,Y)$ and $f \in \mathcal{C}(Y,Z)$ (i.e. f is a single valued continuous map), then $f \circ F \in KKM(X,Z)$ (see, [**13**]).

Definition 1.13. Let Z and W be subsets of Hausdorff topological vector spaces E_1 and E_2 and F a set–valued map. We say that $F \in PK(Z,W)$ if W is convex, and there exists a map $S : Z \to W$ with

$$Z \;=\; \cup\{\mathrm{int} S^{-1}(w) : w \in W\}\,,\ \mathrm{co}\big(S(x)\big) \subset F(x) \quad \text{for} \quad x \in Z\,,$$

and $S(x) \neq \varnothing$ for each $x \in Z$; here $S^{-1}(w) = \{z : w \in S(z)\}$.

Remark 1.14. Suppose Z is paracompact, W is convex, and $F \in PK(Z,W)$. Then there exists a continuous (single valued) mapping $f : Z \to W$ such that $f(x) \in F(x)$ for each $x \in Z$ (see [**13**]).

By a space we mean a Hausdorff topological space. In what follows Q denotes a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X,K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

The following fixed point result was established in [**4**].

Theorem 1.15. *Let $X \in ES(compact)$ and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact map. Then F has a fixed point.*

Corollary 1.16. *Let* $X \in AR$ *and* $F \in \mathcal{U}_c^\kappa(X,X)$ *a compact map. Then* F *has a fixed point.*

Remark 1.17. Recall a space Z is called an absolute retract (written $Z \in AR$) if Z is metrizable and for any metrizable W and for any embedding $h : Z \to W$ the set $h(Z)$ is a retract of W. Note if $X \in AR$ then $X \in ES$(compact). To see this notice that we know from the Arens–Eells theorem that X is r–dominated by a normed space E so there exists maps $r : E \to X$ and $s : X \to E$ with $r\,s = 1$. Now since any normed space is ES(compact), it follows immediately that $X \in ES$(compact).

Let (E,d) be a pseudometric space. For any $C \subseteq E$, let $B(C,\varepsilon) = \{x \in E : d(x,C) \le \varepsilon\}$, here $\varepsilon > 0$ The measure of noncompactness of the set $M \subseteq E$ is defined by $\alpha(M) = \inf Q(M)$, where

$$Q(M) \;=\; \{\varepsilon > 0 : M \subseteq B(A,\varepsilon) \qquad \text{for some finite subset } A \text{ of } E\}.$$

Let C be a subset of a locally convex Hausdorff topological vector space E, and let P be a defining system of seminorms on E. Suppose $F : C \to 2^E$. Then F is called countably P–concentrative mapping if $F(C)$ is bounded, and for $p \in P$ and each countably bounded subset S of C, we have $\alpha_p\big(F(S)\big) \le \alpha_p(S)$, and for $p \in P$ for each countably bounded non–p–paracompact subset S of C (i.e., S is not precompact in the pseudonorm space (E,p)) we have $\alpha_p(F(S)) < \alpha_p(S)$; here $\alpha_p(.)$ denotes the measure of noncompactness in the pseudonorm space (E,p). We remark that fixed point results for countably P–concentrative maps are still valid for countably condensing maps.

The following results are taken from [**2**].

Theorem 1.18. *Let* C *be a closed, convex, bounded subset of a* Fréchet *space* E *(P is a defining system of seminorms) with* $x_0 \in C$*. Suppose* $F \in \mathcal{B}^\kappa(C,C)$ *is a countably* P*–concentrative mappings. Then* F *has a fixed point in* C*.*

Let C be a subset of a Hausdorff topological vector space E and $x \in X$. Then the inward set $I_C(x)$ is defined by

$$I_C(x) \;=\; \{x + r(y - x) : y \in C, r \ge 0\}.$$

If C is convex and $x \in C$, then

$$I_C(x) \;=\; x + \{r(y - x) : y \in C, r \ge 1\}.$$

Theorem 1.19. *Let* C *be a closed, convex, bounded subset of a* Fréchet *space* E *(*P *is a defining system of seminorms) with* $x_0 \in C$*. Suppose either*

$$F \in \mathcal{H}(C,E) \qquad with \qquad F(x) \cap I_C(x) \ne \varnothing \quad forall \quad x \in C$$

or

$$F \in \mathcal{U}_c^\kappa(C,E) \quad with \quad F(x) \subseteq I_C(x) \quad forall \quad x \in C$$

ocurrs. Also assume F *is a countably* P*–concentrative mapping. Then* F *has a fixed point in* C*.*

The following results were established in [**17**].

Theorem 1.20. *Let* C *be a nonempty, closed, convex subset of a* Fréchet *space* E *(P is a defining system of seminorms). Suppose* $F \in \mathcal{U}_c^\kappa(C,C)$ *is a countably* P*–concentrative mappings. Then* F *has a fixed point in* C*.*

Theorem 1.21. *Let C be a nonempty, closed, convex subset of a* Fréchet *space E (P is a defining system of seminorms). Suppose $F \in KKM(C,C)$ is a countably P–concentrative, closed mapping.*

Theorem 1.22. *Let C be a nonempty, closed, convex subset of a* Fréchet *space E (P is a defining system of seminorms). Suppose $F \in PK(C,C)$ is a countably P–concentrative mapping. Then F has a fixed point in C.*

Theorem 1.23. *Let C be a closed, convex subset of a Hilbert space H with $0 \in C$. Suppose $F \in \mathcal{U}_c^\kappa(C,H)$ is a closed countably condensing map with $F(C)$ bounded. In addition, assume the following conditions holds:*

$$\left\{ \begin{array}{l} \textit{if } \{(x_j,\lambda_j)\}_{j=1}^{\infty} \textit{ is a sequence in } \partial C \times [0,1] \textit{ converging to } (x,\lambda) \textit{ with} \\ x \in \lambda F(x) \textit{ and } 0 \le \lambda < 1, \textit{ then } \{\lambda_j F(x)\} \subseteq C \textit{ for } j \textit{ sufficiently large.} \end{array} \right.$$

Then F has a fixed point in C.

Theorem 1.24. *Let C be a closed, convex subset of a Hilbert space H with $0 \in C$. Suppose $F \in PK(C,H)$ is a countably condensing map with $F(C)$ bounded. In addition, assume the following conditions holds:*

$$\left\{ \begin{array}{l} \textit{if } \{(x_j,\lambda_j)\}_{j=1}^{\infty} \textit{ is a sequence in } \partial C \times [0,1] \textit{ converging to } (x,\lambda) \textit{ with} \\ x \in \lambda F(x) \textit{ and } 0 \le \lambda < 1, \textit{ then } \{\lambda_j F(x)\} \subseteq C \textit{ for } j \textit{ sufficiently large.} \end{array} \right.$$

Then F has a fixed point in C.

Remark 1.25. In Theorem 1.23 and Theorem 1.24, by countably condensing map F we mean $\alpha\big(F(B)\big) < \alpha(B)$ for all countably bounded sets B of C with $\alpha(B) \neq 0$ and $\alpha\big(F(D)\big) \le \alpha(D)$ for all countably bounded sets D of C, where $\alpha(\,.\,)$ is the Kuratowski measure of noncompactness.

Let C be a cone in a normed space $E = (E, ||\,.\,||)$. For $\rho > 0$, let

$$B_\rho = \{x \in E : ||x|| < \rho\}, \qquad \overline{B_\rho} = \{x \in E : ||x|| \le \rho\}$$

with

$$S_\rho = \{x \in E : ||x|| = \rho\}, \qquad EB_\rho = \{x \in E : ||x|| \ge \rho\}.$$

We state some results established in [**15**].

Theorem 1.26. *Let C be a cone in a Banach space E and let r, R be constants with $0 < r < R$. Suppose $F \in \mathcal{U}_c^\kappa(\overline{B_\rho} \cap C, C)$ is compact with*

$$F(S_r \cap C) \subseteq EB_r \cap C \quad \textit{and} \quad F(S_R \cap C) \subseteq \overline{B_R} \cap C\,.$$

Then F has a fixed point x in C such that $r \le ||x|| \le R$.

Theorem 1.27. *Let E be an infinite dimensional Banach space and let r, R be constants with $0 < r < R$. Suppose $F \in \mathcal{U}_c^\kappa(\overline{B_\rho}, E)$ is countably k–set-contractive, $0 \le k < \frac{1}{k_0}$ (here k_0 is a Lipschtiz constant of the retraction $r_0 : \overline{B_r} \to S_r$), with*

$$F(S_r) \subseteq EB_r \quad \textit{and} \quad F(S_R) \subseteq \overline{B_R}\,.$$

Then F has a fixed point $x \in E$ such that $r \le ||x|| \le R$.

Theorem 1.28. *Let E be an infinite dimensional Banach space and let U_1 and U_2 be open convex subsets of E with $0 \in U_1$ with $\overline{U_1} \subset U_2$. Suppose $F \in \mathcal{U}_c^\kappa(\overline{U_2}, E)$ is compact with*

$$F(\partial U_1) \subseteq E \setminus U_1 \quad \text{and} \quad F(\partial U_2) \subseteq \overline{U_2}\,.$$

Then F has a fixed point $x \in \overline{U_2} \setminus U_1$.

Finally we recall the following deterministic results from [**17**].

Theorem 1.29. *Let E be a* Fréchet *space (P a defining systems of seminorms), C a closed, convex subset of E, $U \subseteq C$ an open, convex, subset of E, and $0 \in U$. Suppose $F \in \mathcal{U}_c^\kappa(\overline{U}, C)$ is a countably P–concentrative mapping. If in addition*

$$x \notin \lambda F(x) \quad \text{for} \quad x \in \partial U \quad \text{and} \quad \lambda \in (0,1)\,.$$

Then F has a fixed point in $\overline{U}$.

Theorem 1.30. *Let E be a* Fréchet *space (P a defining systems of seminorms), C a closed, convex subset of E, $U \subseteq C$ an open, convex, subset of E, and $0 \in U$. Suppose $F \in KKM(\overline{U}, C)$ is a countably P–concentrative, upper semicontinuous mapping with closed values. If in addition*

$$x \notin \lambda F(x) \quad \text{for} \quad x \in \partial U \quad \text{and} \quad \lambda \in (0,1)\,.$$

Then F has a fixed point in $\overline{U}$.

Theorem 1.31. *Let E be a* Fréchet *space (P a defining systems of seminorms), C a closed, convex subset of E, $U \subseteq C$ an open, convex, subset of E, and $0 \in U$. Suppose $F \in PK(\overline{U}, C)$ is a countably P–concentrative mapping. If in addition*

$$x \notin \lambda F(x) \quad \text{for} \quad x \in \partial U \quad \text{and} \quad \lambda \in (0,1)\,.$$

Then F has a fixed point in $\overline{U}$.

2. Random Fixed Point Theory in Hyperconvex Spaces

A metric space (X,d) is hyperconvex if $\cap_\alpha B(x_\alpha, r_\alpha) \neq \varnothing$ for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in X for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$. We begin by presenting a fixed point result which enables us to improve considerably most results in the literature. The class of maps we consider is very general and contains for example acyclic, approachable and permissible maps.

Theorem 2.1. *Let X be a hyperconvex compact space and $F \in \mathcal{U}_c^k(X,X)$. Then F has a fixed point.*

Proof. Since X is hyperconvex and compact then $X \in AR$ (see [**6**, p. 422]). Now Theorem 1.15 guarantees that F has a fixed point. □

We next replace the compactness of the space with the compactness (or condensingness) of the map. Indeed the argument to establish this is now standard (see [**9**]) but for completeness we include it here. We first however need the following concepts. A mapping of metric spaces $e : X \to E$ is called a hyperconvex hull of X if E is hyperconvex, e is an isometric embedding, and no hyperconvex proper subspace of E contains $e(X)$. A function $f \in C(X)$ (continuous functions from X to $\boldsymbol{R}$) is an extremal function over X if for all $x,\ y \in X$ we have

$f(x)+f(y) \geq d(x,y)$ and f is pointwise minimal $\big($i.e. if g is another function with the same property such that $g(x) \leq f(x)$ for all $x \in X$ then $g=f\big)$. We let

$$\varepsilon X \;=\; \{f \in C(X) : f \quad \text{is extremal}\};$$

we refer the reader to [**8**] for a discussion of the above ideas.

Theorem 2.2. *Let X be a hyperconvex, bounded metric space and let $F \in \mathcal{U}_c^k(X,X)$ be condensing. Then F has a fixed point.*

Proof. Fix $x_0 \in X$, and let

$$\Sigma \;=\; \{A : x_0 \in A,\; A \subseteq X,\; A \text{ hyperconvex and } F(A) \subseteq A\}.$$

Note $X \in \Sigma$ so $\Sigma \neq \varnothing$. We may now apply Zorn's Lemma since it is immediate from [**7**, Theorem 7] (or [**9**, Theorem 3]) that every chain in Σ has a lower bound. As a result there exists a minimal element Y of Σ. Now [**9**, Lemma 4] guarantees that there exists a subset B of X isometric to $\varepsilon\big(F(Y) \cup \{x_0\}\big)$ with B hyperconvex and $F(Y) \cup \{x_0\} \subseteq B \subseteq Y$. This immediately implies $F(B) \subseteq F(Y) \subseteq B$, and so $x_0 \in B$ with B hyperconvex, $F(B) \subseteq B$ and $B \subseteq Y$. As a result $B = Y$. Next notice

$$\alpha(Y) = \alpha(B) = \alpha\big(\varepsilon\,(F(Y) \cup \{x_0\})\big). \tag{1}$$

Also [**9**, Corollary pp. 135] yields

$$\alpha\left(\varepsilon\left(F(Y) \cup \{x_0\}\right)\right) \;=\; \alpha\big(F(Y) \cup \{x_0\}\big)$$

and this together with (1) gives

$$\alpha(Y) \;=\; \alpha\big(F(Y)\big).$$

Now since F is condensing we have that $\overline{Y}$ is compact. In fact since hyperconvex spaces are closed [**6**] we have Y compact. Thus Y is a compact hyperconvex space with $F(Y) \subseteq Y$. In addition since $\mathcal{U}_c$ is closed under compositions we have $F|_Y \in \mathcal{U}_c^k(Y,Y)$. Now Theorem 2.1 establishes the result. □

Theorem 2.3. *Let $(\Omega, \mathcal{A})$ be a measurable space, X a hyperconvex, separable space and $F : \Omega \times X \to CD(X)$ a random continuous, condensing operator with $F(w,\,.\,) \in \mathcal{U}_c^\kappa(X,X)$ for each $w \in \Omega$. Then F has a random fixed point.*

Proof. Now Theorem 1.4 guarantees that $F : \Omega \times X \to CD(X)$ satisfies condition (A) and Theorem 2.2 guarantees that F has a deterministic fixed point. The result now follows from Theorem 1.1 □

Theorem 2.4. *Let $(\Omega, \mathcal{A})$ be a measurable space, X a hyperconvex, separable space and $F : \Omega \times X \to CD(X)$ a random continuous, condensing operator with $F(w,\,.\,) \in \mathcal{U}_c^\kappa(X,X)$ for each $w \in \Omega$. Then F has a random fixed point.*

Proof. Now Theorem 1.4 guarantees that $F : \Omega \times X \to CD(X)$ satisfies condition (A) and Theorem 2.2 guarantees that F has a deterministic fixed point. The result now follows from Theorem 1.1. □

3. Random Fixed Point Theory for Admissible Multimaps

In this section we present random fixed theory for admissible maps.

Theorem 3.1. *Let C be a separable, closed, convex, bounded subset of a* Fréchet *space E (P is a defining system of seminorms) with $x_0 \in C$ and $F : \Omega \times C \to CD(C)$ a continuous countably P–concentrative random operator satisfying condition (A). Suppose $F(\omega, .) \in \mathcal{B}^{\kappa}(C, C)$ for each $\omega \in \Omega$. Then F has a random fixed point.*

Proof. By Theorem 1.18, F has a deterministic fixed point. Since F satisfies condition (A), the result follows by Theorem 1.1. □

Theorem 3.2. *Let C be a nonempty, separable, closed, convex subset of a* Fréchet *space E (P is a defining system of seminorms) and $F : \Omega \times C \to CD(C)$ a continuous countably P–concentrative random operator satisfying condition (A). Suppose $F(\omega, .) \in \mathcal{U}_c^{\kappa}(C, C)$ for each $\omega \in \Omega$. Then F has a random fixed point.*

Proof. By Theorem 1.20, F has a deterministic fixed point. Since F satisfies condition (A), the result follows by Theorem 1.1. □

Theorem 3.3. *Let C be a separable, closed, convex, bounded subset of a* Fréchet *space E (P is a defining system of seminorms) with $x_0 \in C$ and $F : \Omega \times C \to CD(E)$ a continuous countably P–concentrative random operator satisfying condition (A). Suppose, for each $\omega \in \Omega$ either*

$$F(\omega, .) \in \mathcal{H}(C, E) \quad \text{with} \quad F(\omega, x) \cap I_C(x) \neq \varnothing \text{ for all} \quad x \in C$$

or

$$F(\omega, .) \in \mathcal{U}_c^{\kappa}(C, E) \quad \text{with} \quad F(\omega, x) \subseteq I_C(x) \quad \text{for all} \quad x \in C$$

ocurrs. Then F has a random fixed point.

Proof. By Theorem 1.19, F has a deterministic fixed point. Since F satisfies condition (A), an application of Theorem 1.1 yields that F has a random fixed point. □

Remark 3.4. Using Theorem 3.3, one may obtain random homotopy and random Leray–Schauder results for inward multimaps parallel to the deterministic results in [**16**].

Theorem 3.5. *Let C be a nonempty, separable, closed, convex subset of a* Fréchet *space E (P is a defining system of seminorms)and $F : \Omega \times C \to CD(C)$ a continuous countably P–concentrative random operator satisfying condition (A). Suppose $F(\omega, .) \in KKM(C, C)$ for each $\omega \in \Omega$. Then F has a random fixed point.*

Proof. The result follows from Theorem 1.1 and Theorem 1.21. □

Theorem 3.6. *Let C be a nonempty, separable closed, convex subset of a* Fréchet *space E (P is a defining system of seminorms) and $F : \Omega \times C \to CD(C)$ a continuous countably P–concentrative random operator satisfying condition (A). Suppose $F(\omega, .) \in PK(C, C)$ for each $\omega \in \Omega$. Then F has a random fixed point.*

Proof. The result follows from Theorem 1.1 and Theorem 1.22. □

Remark 3.7. If, in Theorem 3.1–Theorem 3.6, $F : \Omega \times X \to CD(E)$ is countably condensing, then condition (A) is satisfied automatically.

Theorem 3.8. *Let C be a separable, closed, convex subset of a Hilbert space H with $0 \in C$ and $F : \Omega \times C \to CD(E)$ a continuous countably condensing random operator. Suppose, for each $\omega \in \Omega$, $F(\omega, .) \in \mathcal{U}_c^\kappa(C, H)$ and $F(\omega, C)$ is bounded. In addition, assume the following conditions holds: for each $\omega \in \Omega$,*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^\infty \text{ is a sequence in } \partial C \times [0,1] \text{ converging to } (x, \lambda) \text{ with} \\ x \in \lambda F(\omega, x) \text{ and } 0 \le \lambda < 1, \text{ then } \{\lambda_j F(\omega, x)\} \subseteq C \text{ for } j \text{ sufficiently large.} \end{cases}$$

Then F has a random fixed point.

Proof. By Theorem 1.23, F has a deterministic fixed point. Since F is countably condensing, by Theorem 1.4, it is hemicompact and so satisfies condition (A). The result now follows from Theorem 1.1. □

Theorem 3.9. *Let C be a separable, closed, convex subset of a Hilbert space H with $0 \in C$ and $F : \Omega \times C \to CD(E)$ a continuous countably condensing random operator. Suppose, for each $\omega \in \Omega$, $F(\omega, .) \in PK(C, H)$ and $F(\omega, C)$ is bounded. In addition, assume the following conditions holds: for each $\omega \in \Omega$,*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^\infty \text{ is a sequence in } \partial C \times [0,1] \text{ converging to } (x, \lambda) \text{ with} \\ x \in \lambda F(\omega, x) \text{ and } 0 \le \lambda < 1, \text{ then } \{\lambda_j F(\omega, x)\} \subseteq C \text{ for } j \text{ sufficiently large.} \end{cases}$$

Then F has a random fixed point.

Proof. The result follows from Theorem 1.1 and Theorem 1.24. □

4. Random Leray–Schauder Alternatives

In this section we present random Leray-Schauder alternative for different types of maps.

Theorem 4.1. *Let E be a separable Fréchet space (P a defining systems of seminorms), C a closed, convex subset of E, $U \subseteq C$ an open, convex, subset of E with $0 \in U$ and $F : \Omega \times \overline{U} \to CD(C)$ a continuous countably P–concentrative random operator satisfying condition (A). Suppose for each $\omega \in \Omega$, $F(\omega, .) \in \mathcal{U}_c^\kappa(\overline{U}, C)$ and*

$$x \notin \lambda F(\omega, x) \quad \text{for} \quad x \in \partial U \quad \text{and} \quad \lambda \in (0, 1)\,.$$

Then F has a random fixed point ξ with $\xi(\omega) \in \overline{U}$ for each $\omega \in \Omega$.

Proof. Fix $\omega \in \Omega$ and notice Theorem 1.29 guarantees that $F(\omega, .)$ has a fixed point in $\overline{U}$. Since F satisfies condition (A), by Theorem 1.1 (cf. Remark 1.3) F has a random fixed point ξ with $\xi(\omega) \in \overline{U}$ for each $\omega \in \Omega$. □

Theorem 4.2. *Let E be a separable Fréchet space (P a defining systems of seminorms), C a closed, convex subset of E, $U \subseteq C$ an open, convex, subset of E with $0 \in U$, and $F : \Omega \times \overline{U} \to CD(C)$ a continuous countably P–concentrative random operator satisfying condition (A). Suppose for each $\omega \in \Omega$, $F(\omega, .) \in KKM(\overline{U}, C)$ and*

$$x \notin \lambda F(\omega, x) \text{ for } x \in \partial U \quad \text{and} \quad \lambda \in (0, 1)\,.$$

Then F has a random fixed point ξ with $\xi(\omega) \in \overline{U}$ for each $\omega \in \Omega$.

Proof. Let $\omega \in \Omega$ be fixed. Now Theorem 1.30 guarantees that $F(\omega, .)$ has a fixed point in $\overline{U}$. Since F satisfies condition (A), by Theorem 1.1 F has a random fixed point ξ with $\xi(\omega) \in \overline{U}$ for each $\omega \in \Omega$. □

Theorem 4.3. *Let E be a Fréchet space (P a defining systems of seminorms), C a closed, convex subset of E, $U \subseteq C$ an open, convex, subset of E with $0 \in U$ and $F : \Omega \times \overline{U} \to CD(C)$ a continuous countably P-concentrative random operator satisfying condition (A). Suppose $\omega \in \Omega$, $F(\omega, .) \in PK(\overline{U}, C)$ and*

$$x \notin \lambda F(\omega, x) \text{ for } x \in \partial U \quad \text{and} \quad \lambda \in (0, 1).$$

Then F has a random fixed point ξ with $\xi(\omega) \in \overline{U}$ for each $\omega \in \Omega$.

Proof. This follows from Theorem 1.1 and Theorem 1.31. □

5. Krasnoselskii Cone Compression Theorems for Random Multimaps

We obtain a random Krasnoselskii cone compression theorem.

Theorem 5.1. *Let C be a cone in a Banach space E, let r, R be constants with $0 < r < R$ and let $F : \Omega \times (\overline{B_\rho} \cap C) \to CD(C)$ be a continuous compact random operator. Suppose $\overline{B_\rho} \cap C$ is separable and for each $\omega \in \Omega$, $F(\omega, .) \in \mathcal{U}_c^\kappa(\overline{B_\rho} \cap C, C)$ with*

$$F(\omega, S_r \cap C) \subseteq EB_r \cap C \text{ and } F(\omega, S_R \cap C) \subseteq \overline{B_R} \cap C\,.$$

Then F has a fixed point ξ such that $r \leq ||\xi(\omega)|| \leq R$ for each $\omega \in \Omega$.

Proof. Let $B_{r,R} = \{x \in E : r \leq ||x|| \leq R\}$ and consider $G(\omega) = \{x \in B_{r,R} : x \in F(\omega, x)\}$. Then, by Theorem 1.26, $G(\omega) \neq \varnothing$ for each $\omega \in \Omega$. Since F is continuous and compact, it is hemicompact and so satisfies condition (A). Now Theorem 1.1 (cf. Remark 1.3) guarantees that F has a random fixed point ξ such that $r \leq ||\xi(\omega)|| \leq R$ for each $\omega \in \Omega$. □

Remark 5.2. Theorem 5.1 remains valid (see [15]) if "$F(\omega, .) \in \mathcal{U}_c^\kappa(\overline{B_\rho} \cap C, C)$ for each $\omega \in \Omega$" is replaced by "$F(\omega, .) \in \mathcal{A}(\overline{B_\rho} \cap C, C)$ for each $\omega \in \Omega$" where $\mathcal{A}$ is a subclass of $\mathcal{B}^\kappa$ maps satisfying a composition condition: if $G \in \mathcal{A}(X_1, X_3)$ and $g \in C(X_2, X_1)$, then $G \circ g \in \mathcal{B}^\kappa(X_2, X_3)$ for any topological spaces X_1, X_2 and X_3.

Remark 5.3. In Theorem 5.1, the condition $F(\omega, S_R) \subseteq \overline{B_R}$ may be replaced (see [**15**]) by the following condition

$$x \notin \lambda F(\omega, x) \quad \text{for} \quad x \in S_R\, quad \text{and} \quad \lambda \in (0, 1)\,.$$

Theorem 5.4. *Let E be an infinite dimensional Banach space and let r, R be constants with $0 < r < R$ and let $F : \Omega \times \overline{B_\rho} \to CD(E)$ be a continuous countably k-set-contractive random operator with $0 \leq k < \frac{1}{k_0}$ (here k_0 is as described in Theorem 1.27). Suppose $\overline{B_\rho}$ is separable and for each $\omega \in \Omega$, $F(\omega, .) \in \mathcal{U}_c^\kappa(\overline{B_\rho}, E)$ with*

$$F(\omega, S_r) \subseteq EB_r \quad \text{and} \quad F(\omega, S_R) \subseteq \overline{B_R}\,.$$

Then F has a random fixed point ξ such that $r \leq ||\xi(\omega)|| \leq R$ for each $\omega \in \Omega$.

Proof. Define $G(\omega) = \{x \in B_{r,R} : x \in F(\omega, x)\}$. Then, by Theorem 1.27, $G(\omega) \neq \varnothing$ for each $\omega \in \Omega$. Note [**10**, Chapter 21] that the Lipschtiz constant k_0 of a Lipschtizian retraction $r_0 : \overline{B_r} \to S_r$ always satisfies $k_0 > 1$. Since F is continuous and countably k–set–contractive with $0 \leq k < \frac{1}{k_0}$, it is countably condensing and so, by Theorem 1.4, satisfies condition (A). The result now follows from Theorem 1.1 (cf. Remark 1.3). □

Theorem 5.5. *Let E be an infinite dimensional Banach space, let U_1 and U_2 be open convex subsets of E with $0 \in U_1$ with $\overline{U_1} \subset U_2$, and let $F : \Omega \times \overline{U_2} \to CD(E)$ be a continuous compact random operator. Suppose $\overline{U_2}$ is separable and for each $\omega \in \Omega$, $F(\omega, .) \in \mathcal{U}_c^{\kappa}(\overline{U_2}, E)$ with*

$$F(\omega, \partial U_1) \subseteq E \setminus U_1 \quad \textit{and} \quad F(\omega, \partial U_2) \subseteq \overline{U_2}\,.$$

Then F has a random fixed point ξ such that $\xi(\omega) \in \overline{U_2} \setminus U_1$ for each $\omega \in \Omega$.

Proof. Define $G(\omega) = \{x \in \overline{U_2} \setminus U_1 : x \in F(\omega, x)\}$. Then, by Theorem 1.28, $G(\omega) \neq \varnothing$ for each $\omega \in \Omega$. Since F is continuous and compact, it satisfies condition (A). The result immediately follows from Theorem 1.1 (cf. Remark 1.3). □

Remark 5.6. In Theorem 5.5, we may replace $F(\omega, \partial U_2) \subseteq \overline{U_2}$ by (see [**15**])

$$x \notin \lambda F(\omega, x) \text{ for } x \in \partial U_2 \quad \text{and} \quad \lambda \in (0, 1)\,.$$

6. Random Approximation and Random Fixed Point Theorems

In this section, we prove some random approximation and random fixed point theorems for a variety of maps.

Let C be a convex subset of a Banach space E with $0 \in int(C)$. The Minkowski functional $p : E \to [0, \infty)$ of C is defined by

$$p(x) = \inf\{r > 0 : x \in rC\}\,.$$

The following properties of the Minkowski functional are well known:

(i) p is continuous on E;

(ii) $p(x + y) \leq p(x) + p(y),\ x, y \in E$;

(iii) $p(\lambda x) = \lambda p(x)$, $\lambda \geq 0$, $x \in E$;

(iv) $0 \leq p(x) < 1$ if $x \in int(C)$;

(v) $p(x) > 1$, if $x \notin \overline{C}$;

(vi) $p(x) = 1$, if $x \in \partial C$.

Let $x \in E$. We let

$$d_p(x, C) = \inf\{p(x - y) : y \in C\}\,.$$

The following is a random version of Theorem 3.2 of [**18**].

Theorem 6.1. *Let C be a separable, closed, convex subset of a Banach space E with $0 \in int(C)$ and $F : \Omega \times C \to CD(E)$ a continuous countably condensing random operator. Suppose $F(\omega, .) \in \mathcal{U}_c^\kappa(C, E)$ for each $\omega \in \Omega$. Then there exist a measurable mapping $\xi : \Omega \to C$ and a mapping $\eta : \Omega \to E$ such that for each $\omega \in \Omega$ we have*

$$\eta(\omega) \in F(\omega, \xi(\omega))$$

and

$$p\big(\eta(\omega) - \xi(\omega)\big) \;=\; d_p\big(\eta(\omega), C\big) \;=\; d_p\left(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\right) \;;$$

here p is the Minkowski functional of C in E.

Proof. Let $r : E \to C$ be defined by

$$r(x) = \begin{cases} x & \text{if } x \in C \\ \frac{x}{p(x)} & \text{if } x \notin C \,. \end{cases}$$

Then r is continuous and

$$r(A) \subseteq \overline{co}(A \cup \{0\})$$

for each bounded subset A of C. This gives $\alpha(r(A)) \le \alpha(A)$. Therefore, r is a 1-set-contractive map. Since $F(\omega, \,.\,)$ is countably condensing for each $\omega \in \Omega$, $G(\omega, .) = r \circ F(\omega, \,.\,)$ is countably condensing. Since $\mathcal{U}_c^\kappa$ is closed under composition, $G(\omega, .) \in \mathcal{U}_c^\kappa(C, C)$. By Theorem 3.2, there exists a measurable mapping $\xi : \Omega \to C$ such that $\xi(\omega) \in G\big(\omega, \xi(\omega)\big)$ for each $\omega \in \Omega$. Now, fix $\omega \in \Omega$. Then there exists some $\eta(\omega) \in F\big(\omega, \xi(\omega)\big)$ such that $\xi(\omega) = r\big(\eta(\omega)\big)$.

There are two cases to consider, either $\big(\eta(\omega)\big) \in C$ or $\eta(\omega) \notin C$:

If $\eta(\omega) \in C$, then

$$\xi(\omega) = r\big(\eta(\omega)\big) = \eta(\omega)$$

and so

$$p\big(\eta(\omega) - \xi(\omega)\big) = 0 = d_p\big(\eta(\omega), C\big) \,.$$

On the other hand, if $\eta(\omega) \notin C$, then

$$\xi(\omega) = r\big(\eta(\omega)\big) = \frac{\eta(\omega)}{p\big(\eta(\omega)\big)} \,.$$

Thus, for any $x \in C$,

$$\begin{aligned} p\big(\eta(\omega) - \xi(\omega)\big) \;&=\; p\left(\eta(\omega) - \frac{\eta(\omega)}{p\big(\eta(\omega)\big)}\right) \\ &=\; \frac{\big(p\big(\eta(\omega)\big) - 1\big)}{p\big(\eta(\omega)\big)} p\big(\eta(\omega)\big) \\ &=\; p\big(\eta(\omega)\big) - 1 \le p\big(\eta(\omega)\big) - p(x) \le p\big(\eta(\omega) - x\big). \end{aligned}$$

Consequently,

$$p\big(\eta(\omega) - \xi(\omega)\big) \;=\; d_p\big(\eta(\omega), C\big)$$

for each $\omega \in \Omega$.

Now we claim that

$$p\big(\eta(\omega) - \xi(\omega)\big) \;=\; d_p\left(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\right)$$

for each $\omega \in \Omega$. Indeed, fix $\omega \in \Omega$. Let $z \in I_C\big(\xi(\omega)\big) \setminus C$. Then there exist $y \in C$ and $\lambda > 1$ such that $z = \xi(\omega) + \lambda\big(y - \xi(\omega)\big)$. Suppose

$$p\big(\eta(\omega) - z\big) < p\big(\eta(\omega) - \xi(\omega)\big)\,.$$

Since $(\frac{1}{\lambda})z + \big(1 - \frac{1}{\lambda}\big)\,\xi(\omega) \in C$ we have

$$\begin{aligned} p\big(\eta(\omega) - y\big) &= p\left[\frac{1}{\lambda}\big(\eta(\omega) - z\big) + \left(1 - \frac{1}{\lambda}\right)\big(\eta(\omega) - \xi(\omega)\big)\right] \\ &\le \frac{1}{\lambda}p\big(\eta(\omega) - z\big) + \left(1 - \frac{1}{\lambda}\right)p\big(\eta(\omega) - \xi(\omega)\big) \\ &< p\big(\eta(\omega) - \xi(\omega)\big), \end{aligned}$$

a contradiction. Therefore,

$$p\big(\eta(\omega) - \xi(\omega)\big) \;\le\; p\big(\eta(\omega) - z\big)$$

for all $z \in I_C\big(\xi(\omega)\big)$.
Hence

$$p\big(\eta(\omega) - \xi(\omega)\big) \;=\; d_p\big(\eta(\omega), C\big) \;=\; d_p\left(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\right)\,,$$

for each $\omega \in \Omega$. □

For $R > 0$, set $B_R = \{x \in E : ||x|| \le R\}\,, \partial B_R = \{x \in E : ||x|| = R\}$. Since $p(x) = \frac{||x||}{R}$ is the Minkowski functional of B_R in E, the following result follows immediately from Theorem 6.1.

Corollary 6.2. *Let B_R be separable in a Banach space E and $F : \Omega \times B_R \to CD(E)$ a continuous countably condensing random operator. Suppose $F(\omega, .) \in \mathcal{U}_c^{\kappa}(B_R, E)$ for each $\omega \in \Omega$. Then there exist a measurable mapping $\xi : \Omega \to B_R$ and a mapping $\eta : \Omega \to E$ such that for each $\omega \in \Omega$ we have*

$$\eta(\omega) \in F\big(\omega, \xi(\omega)\big)$$

and

$$||\eta(\omega) - \xi(\omega)|| \;=\; d(\eta(\omega), B_R) \;=\; d\left(\eta(\omega), \overline{I_{B_R}\big(\xi(\omega)\big)}\right)\,.$$

Remark 6.3. Corollary 6.2 extends Theorem 1 of Lin [**14**] to countably condensing multimaps.

Theorem 6.4. *Let C be a separable, closed, convex subset of a Banach space E with $0 \in int(C)$ and $F : \Omega \times C \to CD(E)$ a continuous countably condensing random operator. Suppose $F(\omega, .) \in KKM(C, E)$ for each $\omega \in \Omega$. Then there exist a measurable mapping $\xi : \Omega \to C$ and a mapping $\eta : \Omega \to E$ such that for each $\omega \in \Omega$ we have*

$$\eta(\omega) \in F(\omega, \xi(\omega))$$

and

$$p\big(\eta(\omega) - \xi(\omega)\big) \;=\; d_p\big(\eta(\omega), C\big) \;=\; d_p\left(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\right)\,;$$

here p is the Minkowski functional of C in E.

Proof. Let $r : E \to C$ be as in Theorem 6.1. Then, as in Theorem 6.1, r is continuous and for each $\omega \in \Omega$, $G(\omega, .) = r \circ F(\omega, .)$ is countably condensing. By Remark 1.4, $G(\omega, .) \in KKM(C,C)$ for each $\omega \in \Omega$. Now Theorem 3.5 implies that G has a random fixed point $\xi : \Omega \to C$. Hence, as in Theorem 6.1, there exist a mapping $\eta : \Omega \to E$ such that

$$\eta(\omega) \in F\big(\omega, \xi(\omega)\big)$$

and

$$p\big(\eta(\omega) - \xi(\omega)\big) = d_p\big(\eta(\omega), C\big) = d_p\left(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\right)$$

for each $\omega \in \Omega$. □

Let $\mathcal{A}$ be a subclass of $\mathcal{B}^\kappa$ maps satisfying a composition condition: if $G \in \mathcal{A}(X_1, X_2)$ and $g \in \mathcal{C}(X_2, X_3)$, then $g \circ G \in \mathcal{B}^\kappa(X_1, X_3)$ for any topological spaces X_1, X_2 and X_3.

Theorem 6.5. *Let C be a separable, closed, bounded, convex subset of a Banach space E with $0 \in int(C)$ and $F : \Omega \times C \to CD(E)$ a continuous countably condensing random operator. Suppose $F(\omega, .) \in \mathcal{A}(C, E)$ for each $\omega \in \Omega$. Then there exist a measurable mapping $\xi : \Omega \to C$ and a mapping $\eta : \Omega \to E$ such that for each $\omega \in \Omega$ we have*

$$\eta(\omega) \in F(\omega, \xi(\omega))$$

and

$$p(\eta(\omega) - \xi(\omega)) = d_p(\eta(\omega), C) = d_p(\eta(\omega), \overline{I_C(\xi(\omega))});$$

here p is the Minkowski functional of C in E.

Proof. As above, r is continuous and for each $\omega \in \Omega$, $G(\omega, .) = r \circ F(\omega, .)$ is countably condensing. Now, $G(\omega, .) \in \mathcal{B}^\kappa(C, C)$ for each $\omega \in \Omega$. An application of Theorem 3.1 gives that G has a random fixed point $\xi : \Omega \to C$. Thus, as in Theorem 6.1, there exist a mapping $\eta : \Omega \to E$ such that

$$\eta(\omega) \in F\big(\omega, \xi(\omega)\big)$$

and

$$p\big(\eta(\omega) - \xi(\omega)\big) = d_p\big(\eta(\omega), C\big) = d_p\left(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\right)$$

for each $\omega \in \Omega$. □

Theorem 6.6. *Let C be a separable, closed, convex subset of a Banach space E with $0 \in int(C)$ and $F : \Omega \times C \to CD(E)$ a continuous countably condensing random operator. Suppose $F(\omega, .) \in PK(C, E)$ for each $\omega \in \Omega$. Then there exist a measurable mapping $\xi : \Omega \to C$ and a mapping $\eta : \Omega \to E$ such that for each $\omega \in \Omega$ we have*

$$\eta(\omega) \in F\big(\omega, \xi(\omega)\big)$$

and

$$p\big(\eta(\omega) - \xi(\omega)\big) = d_p\big(\eta(\omega), C\big) = d_p\left(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\right);$$

here p is the Minkowski functional of C in E.

Proof. As before, we have r is continuous and 1–set–contractive. Let $\omega \in \Omega$ be fixed. Since metrizable spaces are paracompact, C is paracompact. By Remark 1.14, there exists a continuous (single valued) mapping $f(\omega, .) : C \to E$ such that $f(\omega, x) \in F(\omega, x)$ for each $x \in C$. Let $g(\omega, .) = r \circ f(\omega, .)$. Clearly $g(\omega, .) : C \to C$ is continuous countably condensing mapping. Now Theorem 1.20 guarantees that there is a $x \in C$ such that $x = g(\omega, x) \in G(\omega, x)$, where $G(\omega, .) = r \circ F(\omega, .)$. Clearly G has a deterministic fixed point. Theorem 1.1 further implies that G has a random fixed point $\xi : \Omega \to C$. Thus, as in Theorem 6.1, there exist a mapping $\eta : \Omega \to E$ such that

$$\eta(\omega) \in F\big(\omega, \xi(\omega)\big)$$

and

$$p\big(\eta(\omega) - \xi(\omega)\big) \;=\; d_p\big(\eta(\omega), C\big) \;=\; d_p\left(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\right)$$

for each $\omega \in \Omega$. □

We now apply random approximation results to derive some random fixed point theorems.

Theorem 6.7. *Let C be a separable, closed, convex subset of a Banach space E with $0 \in int(C)$ and $F : \Omega \times C \to CD(E)$ a continuous countably condensing random operator. Suppose $F(\omega, .) \in \mathcal{U}_c^\kappa(C, E)$ for each $\omega \in \Omega$ and satisfies any one of the following conditions for any $\omega \in \Omega$ and $x \in \partial C \setminus F(\omega, x)$:*

(i) *For each $y \in F(\omega, x)$, $p(y - z) < p(y - x)$ for some $z \in \overline{I_C(x)}$;*

(ii) *For each $y \in F(\omega, x)$, there exist λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_C(x)}$;*

(iii) *$F(\omega, x) \subseteq \overline{I_C(x)}$;*

(iv) *For each $\lambda \in (0, 1)$, $x \notin \lambda F(\omega, x)$;*

(v) *For each $y \in F(\omega, x)$, there exist $\alpha \in (1, \infty)$ such that $p^\alpha(y) - 1 \le p^\alpha(y - x)$;*

(vi) *For each $y \in F(\omega, x)$, there exist $\beta \in (0, 1)$ such that $p^\beta(y) - 1 \ge p^\beta(y - x)$.*

Then F has a random fixed point.

Proof. By Theorem 6.1, then there exist a measurable mapping $\xi : \Omega \to C$ and a mapping $\eta : \Omega \to E$ such that for each $\omega \in \Omega$ we have

$$\eta(\omega) \in F\big(\omega, \xi(\omega)\big) \quad \text{with} \quad \xi(\omega) = r\big(\eta(\omega)\big)$$

and

$$p\big(\eta(\omega) - \xi(\omega)\big) \;=\; d_p\big(\eta(\omega), C\big) \;=\; d_p\left(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\right),$$

where p is the Minkowski functional of C in E. Moreover, it is clear that if $d_p\left(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\right) > 0$ for some $\omega \in \Omega$, then $\xi(\omega) \in \partial C$ and $p\big(\eta(\omega)\big) > 1$ $\big($note, for fixed ω, if $\xi(\omega) \in int\,(C)$ then it is well known that $\overline{I_C\big(\xi(\omega)\big)} = E$ and so $d_p\big(\xi(\omega, \overline{I_C\big(\xi(\omega)\big)}\big) = 0$, a contradiction$\big)$.

Suppose F satisfies condition (i). Assume there is some $\omega \in \Omega$ such that $\xi(\omega) \notin F\big(\omega, \xi(\omega)\big)$. Then, by condition (i), we have $p\big(\eta(\omega) - z\big) < p\big(\eta(\omega) - \xi(\omega)\big)$ for some $z \in \overline{I_C\big(\xi(\omega)\big)}$. But this contradicts the choice of ξ. Hence $\xi(\omega) \in F\big(\omega, \xi(\omega)\big)$ for all $\omega \in \Omega$ so F has a random fixed point.

Suppose F satisfies condition (ii). Assume there is some $\omega \in \Omega$ such that $\xi(\omega) \notin F(\omega, \xi(\omega))$. Then, by condition (ii), there exists λ with $|\lambda| < 1$ such that $\lambda\xi(\omega) + (1-\lambda)\eta(\omega) \in \overline{I_C(\xi(\omega))}$. This implies that

$$\begin{aligned} p(\eta(\omega) - \xi(\omega)) &\leq p\left(\eta(\omega) - \big(\lambda\xi(\omega) + (1-\lambda)\eta(\omega)\big)\right) \\ &= p\left(\lambda\big(\eta(\omega) - \xi(\omega)\big)\right) \\ &= |\lambda| p\big(\eta(\omega) - \xi(\omega)\big) \\ &< p\big(\eta(\omega) - \xi(\omega)\big), \end{aligned}$$

which is a contradiction. Hence F has a random fixed point.

If F satisfies condition (iii), then it satisfies condition (ii) by letting $\lambda = 0$.

Suppose F satisfies condition (iv). Assume there is some $\omega \in \Omega$ such that $\xi(\omega) \notin F(\omega, \xi(\omega))$. Then $\xi(\omega) \in \partial C$ and so, by condition (iv), $\xi(\omega) \notin \lambda F(\omega, \xi(\omega))$ for each $\lambda \in (0,1)$. It further implies that $\xi(\omega) \neq \lambda\eta(\omega)$ for each $\lambda \in (0,1)$. But we have $\xi(\omega) = \frac{\eta(\omega)}{p(\eta(\omega))}$ and $p\big(\eta(\omega)\big) > 1$, a contradiction. Hence F has a random fixed point.

Suppose F satisfies condition (v). Assume there is some $\omega \in \Omega$ such that $\xi(\omega) \notin F\big(\omega, \xi(\omega)\big)$. Then condition (v) implies that there exists $\alpha \in (1, \infty)$ with $p^\alpha\big(\eta(\omega)\big) - 1 \leq p^\alpha\big(\eta(\omega) - \xi(\omega)\big)$. Let $\lambda_0 = \frac{1}{p(\eta(\omega))}$. Then $\lambda_0 \in (0,1)$ and

$$\begin{aligned} \frac{\big(p\big(\eta(\omega)\big) - 1\big)^\alpha}{p^\alpha\big(\eta(\omega)\big)} &= (1-\lambda_0)^\alpha < 1 - \lambda_0^\alpha \\ &= \frac{p\big(\eta(\omega)\big)^\alpha - 1}{p^\alpha\big(\eta(\omega)\big)} \leq \frac{p^\alpha\big(\eta(\omega) - \xi(\omega)\big)}{p^\alpha\big(\eta(\omega)\big)}, \end{aligned}$$

which gives $p\big(\eta(\omega) - \xi(\omega)\big) > p\big(\eta(\omega)\big) - 1$. This contradicts the fact that $p\big(\eta(\omega) - \xi(\omega)\big) = p\big(\eta(\omega)\big) - 1$ (see the proof of Theorem 6.1 when $\eta(\omega) \notin C$).

Finally suppose F satisfies condition (vi). Then, as above (proof of (v)), it is easy to see that F has a random fixed point. □

Remark 6.8. Theorem 6.6 is a random version of Theorem 3.5 of [**18**]. It can also be derived from Theorem 1.1. We have included its proof here as application of random approximation result.

Corollary 6.9. *Let B_R be separable in a Banach space E and $F : \Omega \times B_R \to CD(E)$ a continuous countably condensing random operator. Suppose $F(\omega, .) \in \mathcal{U}_c^\kappa(B_R, E)$ for each $\omega \in \Omega$ and satsfies any one of the following conditions for any $\omega \in \Omega$ and $x \in \partial B_R \setminus F(\omega, x)$:*

(i) *For each $y \in F(\omega, x)$ $||y - z|| < ||y - x||$ for some $z \in \overline{I_{B_R}(x)}$;*

(ii) *For each $y \in F(\omega, x)$, there exist λ with $|\lambda| < 1$ such that $\lambda x + (1-\lambda)y \in \overline{I_{B_R}(x)}$;*

(iii) *$F(\omega, x) \subseteq \overline{I_{B_R}(x)}$;*

(iv) *For each $\lambda \in (0,1)$, $x \notin \lambda F(\omega, x)$;*

(v) *For each $y \in F(\omega, x)$, there exist $\alpha \in (1, \infty)$ such that $||y||^\alpha - R \leq ||y - x||^\alpha$;*

(vi) *For each $y \in F(\omega, x)$, there exist $\beta \in (0,1)$ such that $||y||^\beta - R \geq ||y - x||^\beta$.*

Then F has a random fixed point.

Remark 6.10. Corollary 6.7 generalizes Theorem 4 of Lin [**14**] to countably condensing multimaps.

Using arguments similar to those of Theorem 6.6, we can obtain the following results.

Theorem 6.11. *Let C be a separable, closed, convex subset of a Banach space E with $0 \in int(C)$ and $F : \Omega \times C \to CD(E)$ a continuous countably condensing random operator. Suppose $F(\omega, .) \in KKM(C, E)$ for each $\omega \in \Omega$ and satisfies any one of the following conditions for any $\omega \in \Omega$ and $x \in \partial C \setminus F(\omega, x)$:*

(i) *For each $y \in F(\omega, x)$, $p(y - z) < p(y - x)$ for some $z \in \overline{I_C(x)}$;*

(ii) *For each $y \in F(\omega, x)$, there exist λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_C(x)}$;*

(iii) *$F(\omega, x) \subseteq \overline{I_C(x)}$;*

(iv) *For each $\lambda \in (0, 1)$, $x \notin \lambda F(\omega, x)$;*

(v) *For each $y \in F(\omega, x)$, there exist $\alpha \in (1, \infty)$ such that $p^\alpha(y) - 1 \le p^\alpha(y - x)$;*

(vi) *For each $y \in F(\omega, x)$, there exist $\beta \in (0, 1)$ such that $p^\beta(y) - 1 \ge p^\beta(y - x)$.*

Then F has a random fixed point.

Theorem 6.12. *Let C be a separable, closed, bounded, convex subset of a Banach space E with $0 \in int(C)$ and $F : \Omega \times C \to CD(E)$ a continuous countably condensing random operator. Suppose $F(\omega, .) \in \mathcal{A}(C, E)$ for each $\omega \in \Omega$ and satisfies any one of the following conditions for any $\omega \in \Omega$ and $x \in \partial C \setminus F(\omega, x)$:*

(i) *For each $y \in F(\omega, x)$, $p(y - z) < p(y - x)$ for some $z \in \overline{I_C(x)}$;*

(ii) *For each $y \in F(\omega, x)$, there exist λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_C(x)}$;*

(iii) *$F(\omega, x) \subseteq \overline{I_C(x)}$;*

(iv) *For each $\lambda \in (0, 1)$, $x \notin \lambda F(\omega, x)$;*

(v) *For each $y \in F(\omega, x)$, there exist $\alpha \in (1, \infty)$ such that $p^\alpha(y) - 1 \le p^\alpha(y - x)$;*

(vi) *For each $y \in F(\omega, x)$, there exist $\beta \in (0, 1)$ such that $p^\beta(y) - 1 \ge p^\beta(y - x)$.*

Then F has a random fixed point.

Theorem 6.13. *Let C be a separable, closed, convex subset of a Banach space E with $0 \in int(C)$ and $F : \Omega \times C \to CD(E)$ a continuous countably condensing random operator. Suppose $F(\omega, .) \in PK(C, E)$ for each $\omega \in \Omega$ and satisfies any one of the following conditions for any $\omega \in \Omega$ and $x \in \partial C \setminus F(\omega, x)$:*

(i) *For each $y \in F(\omega, x)$, $p(y - z) < p(y - x)$ for some $z \in \overline{I_C(x)}$;*

(ii) *For each $y \in F(\omega, x)$, there exist λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_C(x)}$;*

(iii) *$F(\omega, x) \subseteq \overline{I_C(x)}$;*

(vi) *For each $\lambda \in (0, 1)$, $x \notin \lambda F(\omega, x)$;*

(v) *For each* $y \in F(\omega, x)$ *, there exist* $\alpha \in (1, \infty)$ *such that* $p^\alpha(y) - 1 \leq p^\alpha(y - x)$*;*

(vi) *For each* $y \in F(\omega, x)$ *, there exist* $\beta \in (0, 1)$ *such that* $p^\beta(y) - 1 \geq p^\beta(y - x)$*.*

Then F *has a random fixed point.*

Finally, following the ideas of the above results, it is possible to obtain other random approximation and random fixed point theorems in the Hilbert space setting. We only state the results and leave the details to the reader. We must mention that, in these cases, the mapping r is replaced by the nearest point projection (i.e. proximity map) r and the condition that $0 \in int(C)$ is redundant.

Theorem 6.14. *Let* C *be a nonempty, separable, closed, convex subset of a Hilbert space* H *and* $F : \Omega \times C \to CD(H)$ *a continuous countably condensing random operator. Suppose* $F(\omega, .) \in \mathcal{U}_c^\kappa(C, H)$ $\big(F(\omega, .) \in KKM(C, H)$ *or* $F(\omega, .) \in PK(C, H)\big)$ *for each* $\omega \in \Omega$*. Then there exist a measurable mapping* $\xi : \Omega \to C$ *and a mapping* $\eta : \Omega \to H$ *such that for each* $\omega \in \Omega$ *we have*

$$\eta(\omega) \in F\big(\omega, \xi(\omega)\big)$$

and

$$||\eta(\omega) - \xi(\omega)|| \,=\, d\big(\eta(\omega), C\big) \,=\, d\Big(\eta(\omega), \overline{I_C\big(\xi(\omega)\big)}\Big)\ ;$$

here $||.||$ *is the norm induced by the inner product.*

Theorem 6.15. *Let* C *be a nonempty, separable, closed, convex subset of a Hilbert space* H *and* $F : \Omega \times C \to CD(H)$ *a continuous countably condensing random operator. Suppose* $F(\omega, .) \in \mathcal{U}_c^\kappa(C, H)$ $\big(F(\omega, .) \in KKM(C, H)$ *or* $F(\omega, .) \in PK(C, H)\big)$ *for each* $\omega \in \Omega$ *and satisfies any one of the following conditions for any* $\omega \in \Omega$ *and* $x \in \partial C \setminus F(\omega, x)$ *:*

(i) *(i). For each* $y \in F(\omega, x)$ *,* $||y - z|| < ||y - x||$ *for some* $z \in \overline{I_C(x)}$*;*

(ii) *For each* $y \in F(\omega, x)$ *, there exist* λ *with* $|\lambda| < 1$ *such that* $\lambda x + (1 - \lambda)y \in \overline{I_C(x)}$*;*

(iii) $F(\omega, x) \subseteq \overline{I_C(x)}$*.*

Then F *has a random fixed point.*

Remark 6.16. Theorem 6.14 and Theorem 6.15 also hold when $F(\omega, .) \in \mathcal{A}(C, H)$ for each $\omega \in \Omega$. However, for this, we need to assume that C is bounded.

Remark 6.17. Theorem 6.11 extends Theorem 2 of Lin [**14**] whereas Theorem 6.12 generalizes Theorem 5 of Lin [**14**] to countably condensing multimaps.

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