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PLANAR BRACHISTOCHRONE OF A PARTICLE ATTRACTED IN VACUO BY AN INFINITE ROD

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Abstract. The authors analyze the planar brachistochrone in vacuo under the attraction of an infinite rod, adding a new closed form treatment to the known solutions collection. Accordingly, a nonlinear boundary value problem:

$$\begin{cases} y'(x) = -\sqrt{\dfrac{A^2}{\ln(ky_1) - \ln y} - 1}, \\ y(x_1) = y_1, \\ y(x_2) = y_2, \end{cases}$$

is met, where x_1, x_2, y_1, y_2, are fixed and A and k depend on x_i, y_i and on the initial speed. The solution's existence and uniqueness are proved noticing that the variational integrand meets the conditions of a Cesari's theorem. This problem, proposed by G.J. Tee in [**21**] and treated numerically, is solved here in closed form. The trajectory's parametric equations are obtained by means of a generalized, 2–variables, hypergeometric Lauricella confluent function, for the first time used in optimization.

1. The Problem of the Least Time Trajectory

The classic problem of the least time trajectory consists of finding the shape of the line down which a bead, sliding from the rest and accelerated by a force field, will slip from one point to another in the least time. The motion takes place on a prescribed surface connecting the endpoints.

In the case of a surface consisting of a vertical plane, with the force field (the weight only) belonging to it, the trajectory would properly termed "tachistoptotam" following Leibniz, but the more vague term "brachistochrone" of Johann Bernoulli prevailed.

The trajectory described by a particle:

(i) under the weight force only;

(ii) without any dry friction along the curve itself;

(iii) without any drag force by the surrounding medium;

(iv) with the geometrical link of planarity;

is an arc of upward cycloid whose cusp lies on the vertical line passing through the starting point.

The history of the problem begins in June 1696 when Johann Bernoulli challenged his contemporaries writing:

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> Datis in plano verticali duobus punctis A et B, assignare mobili M viam AMB, per quam gravitate descendens, et moveri incipiens a puncto A, brevissimo tempore perveniat ad alterum punctum B[1].

These lines appeared at the end of his article *Defectus Geometriae cartesianae circa Inventionem locorum*, pages 264–269, *Acta Eruditorum* 1696, for supporting his idea that the cartesian geometry alone wasn't enough for detecting all the geometric loci.

The cycloidal solutions to the brachistochrone problem went out simultaneously in *Acta Eruditorum* of May 1697 by: Leibniz, Newton, Jakob Bernoulli, Johann Bernoulli himself. That issue held Leibniz's solution on page 205 (*Solutio problematum a Jo. Bernoullio Geometris publice propositoru/*), Johann Bernoulli's one on pages 206 to 21 (*Curvatura radii in Diaphanis non uniformibus: solutio problematis a se propositi de invenienda linea brachistochrona*), Jakob Bernoulli's one on pages 211 to 214 (*Solutio problematum fraternorum una cum propositione reciproca aliorum*). On page 223, a latin translation appeared of the Newton's solution (*Solutio duorum problematum Mathematicorum a Jo. Bernoullio propositorum*, already published anonymous in the *Philosophical Transactions of the Royal Society* in January of the same year).

The solution of the Marquis de L'Hospital wasn't included there, and will be published only in 1988 when, nearly 300 years later, J. Pfeiffer presented it as Appendix to her edition of the letters of Johann Bernoulli.

A satisfactory outline of all the historical frame which these articles have to be set in, is given at [**3**].

Although all of the competitors reached the same conclusion, none of their solutions is entirely satisfactory; however that of Jakob Bernoulli admitted refinements and generalizations which became the Calculus of Variations, where the brachistochrone problems are nowadays studied. All this leads, in modern terminology, to the Euler–Lagrange equation, whose integration reveals the brachistochrone – under the (i), (ii), (iii), (iv) assumptions – to be just an "ordinary" cycloid, according to the solutions of 1697.

Furthermore the same problem has undergone several modern generalizations based upon removing one or more of the (i), (ii), (iii), (iv) assumptions. We survey below some recent studies.

If, e.g., (ii) is released, let $f > 0$ be the kinetic dry friction coefficient: the exact solution, [**24**], page 107, consists of a couple of parametric equations which, for $f \to 0^+$, will give back the cycloidal solution. Whilst, if f increases, the more realistic frictional brachistochrone will approach a straightline, [**26**], page 335.

Releasing the assumption (iii), when the motion is resisted by the surrounding medium, we get the brachistochrone in a resisting medium. It has been object of four papers by Euler, the first in 1734 and the remaining ones in 1780, but he did not succeed in defining in any way the shape of the line. The same conclusion follows from the XXII lesson – devoted to the subject-in Lagrange's *Calcul des fonctions* (1806). Anyhow, if $y = y(x)$ is the unknown equation of the least time

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[1] If in a vertical plane two points A and B are given, then it is required to specify the orbit AMB of the movable point M, along which it, starting from the quiet of A, and under the in fluence of its own weight, arrives at B in the shortest possible time.

trajectory under viscous drag, [**12**], x is tied to $y'(x)$ so that any exact integration seems to be hopeless.

Releasing the assumption (iv), we have the quickest curves in the space, i.e. on a sphere, on a cone, on an ellipsoid: their equations will lead [**7**] to hyperelliptic integrals.

Releasing the assumption (i), one can look for the brachistochrone under a central field of force: in [**20**] and in [**16**], such a problem is treated in a polar frame and solved numerically.

Releasing the assumption (i), and referring to the potential inside a gravitating, homogeneus, not–rotating sphere, the quickest trajectory, namely the shape of a tunnel through the Earth, was considered by Newton: the solution has been nowadays found, [**24**], page 1687, to be a different type of cycloid, namely a 3–cusped hypocycloid (*deltoid*). The same quickest descent tunneling path can be found by differential geometry method and applying the Pontryagin's maximum principle, see [**19**].

A formulation of all the main brachistochrone problems in the frame of modern optimum control theory can be seen at [**5**], where the controlling variable is the reaction force exerted by the curve on the bead.

Finally, keeping all the four assumptions, but positing the mass relativistic expression as a function of the speed, the relevant trajectory equation can be led, see [**2**], [**8**], to the quadratures.

Let us come to our article: its aim is the least time trajectory of a particle, keeping (ii), (iii), (iv), but under the attracting force exerted by a straight line. This force has a magnitude variable inversely as the distance from the attracting rod and is directed perpendicularly to it. The relevant Euler–Lagrange ode is highly nonlinear, and the brachistochrone will be found through a generalized, 2–variables, hypergeometric function of the Lauricella type, [**10**], in its confluent form.

2. The Brachistochrone Under Logarithmic Potential, and Some Literary Questions

G.J. Tee in [**21**] tackles the brachistochrone under the logarithmic potential, or, what is the same, under a force varying inversely as the distance, and treated numerically the relevant trajectory's ode. His article was negatively criticized by the mathscinet referee[2] because the problem would seem to have been treated by [**14**] already[3], according to an obscure quotation taken from Appell [**1**].

The problem in hand, which we are going to solve exactly, is the same of the Tee's paper. After a careful check of all the sources, we highlight that:

(1) the article [**21**] refers to an attractive logarithmic potential as due to a central force. But: if the force emanates from a centre attracting *à la Newton*, then the potential is not logarithmic. If, on the contrary, the potential is logarithmic, then the force shall depend upon a different, not central cause;

(2) we do not have any evidence someone analyzed the problem in reference before [**21**], even if Tee – who has a good command of the subject, see [**20**] – doesn't find the brachistochrone's equation in a closed form as we are going to do;

[2] See http://www.ams.org/mathscinet ref. 1877546

[3] Roger's paper is also available in electronic form at http://www.gallica.fr

(3) the referee seems to have misunderstood Appell [**1**], and not checked Roger, [**14**], whose paper is treating almost exclusively the brachistochrones on the surfaces of the second order, and not that due to an attracting line;

(4) as far as we are concerned, [**21**] treated first this problem, no other study being available, because the Roger thesis [**14**] has no relationship with it;

(5) both the AMS referee, and [**1**] were not aware of the (few, but appropriate) lines spent about the Roger paper by two popular treatises, [**22, 15**], both giving account that the Roger's treatment is not connected with the matter in hand;

(6) several modern papers like [**7, 12, 16, 20, 21, 24**], are used to treat the variational problems about the brachistochrones with different integrands $F(y, y')$, solving (if any) the relevant Euler–Lagrange equations *without proving the existence of a minimizing solution.* We will show this existence by means of Cesari's Theorem 14.3i, pages 428–429, [**4**].

The problem will then be formulated and solved at the next section.

3. The Brachistochrone Under a Force Varying Inversely as the Distance

A brachistochrone between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, is among the planar curves joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, that minimizes the travel time from P_1 to P_2, simultaneously satisfying the energy law.

Accordingly, let us consider the attraction of an infinitely long rod. It is well known that, assuming the rod as x–reference axis, and as Oy the upwards perpendicular to it at an arbitrary O, the attractive (not central) force exerted by the rod upon a particle moving on the plane xOy, will be perpendicular to the rod, oriented towards it, and with magnitude given [**13, 11**] by:

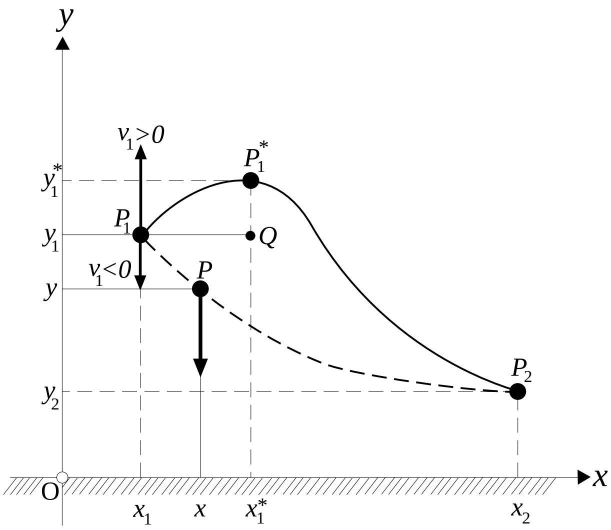
$$\frac{2G\mu}{y}.$$

Here G is the gravitational constant, y is the instantaneous distance of the m–particle from the line having μ as lineal (uniform) mass density. Under the sole attraction of the straight line, we seek for the trajectory joining P_1 to P_2 in the least time. The starting up is from P_1, and the surrounding medium doesn't exert any drag: therefore the Dynamics equation, being no force in the x–direction, will reduce to the y–scalar one:

$$\begin{cases} \dot{v} = -\dfrac{2G\mu}{m\,y}, \\ v(y_1) = v_1, \end{cases} \tag{3.1}$$

where $v = \dot{y}$, and the dot means derivative with respect to time.

Of course v is the function of y giving, for each position of the mobile, the y–component of the velocity vector $\mathbf{v}$. In such a way the prescribed initial value at P_1 can be $v_1 \gtreqless 0$ according to whether the motion starts upwards, by the rest, or downwards respectively (see the figure below). Being the y–axis upwards, $v_1 < 0$ means a motion starting towards the rod, and we will study it (arc P_1, P, P_2) solely. As a matter of fact, the case $v_1 > 0$ will allow a jump upwards (arc P_1, P_1^*) with a culmination given by $y_1 + \overline{QP_1^*}$: at P_1^* the speed becomes zero and the new motion



(arc P_1^*, P_2) would start by the rest. We can then restrict our study to $v_1 \leq 0$. Recalling that:

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{dy} v,$$

we will change problem (3.1) in that of detecting $v(y)$: then, integrating with respect to y, by the condition $v(y_1) = v_1$, we get:

$$v(y) \; = \; \sqrt{\frac{4G\mu}{m} (\ln y_1 - \ln y) + v_1^2}, \tag{3.2}$$

where $y_1 > y > 0$.

Inserting the positional speed (3.2) in the travel time T between the endpoints $P_1\,(x_1, y_1)$ and $P_2\,(x_2, y_2)$:

$$T = \int_{P_1}^{P_2} \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2(x)}}{v(y)}\, dx,$$

one obtains:

$$T \; = \; \sqrt{\frac{m}{4G\mu}} \int_{y_1}^{y_2} F\left(y, y'\right)\, dx,$$

where:

$$F\left(y, y'\right) \; = \; \sqrt{\frac{1 + y'^2}{\ln\left(k y_1\right) - \ln y}}\,, \tag{3.3}$$

is our specific integrand depending on y and y' but not on x, and to be varied in order to set its integral as a minimum as a minimum. Furthermore:

$$\ln k \; = \; \frac{m}{4G\mu} v_1^2. \tag{3.4}$$

By (3.3) it follows that the reality condition has to be met:

$$y < k y_1 \Leftrightarrow y < y_1 \exp\left(\frac{m}{4G\mu} v_1^2\right) = y_1^*.$$

This upper limit for y is of course effective only if $v_1 > 0$: in such a case, as a consequence of the initial $\boldsymbol{v}$ upwards slope, y can attain values $> y_1$, but not beyond y_1^*.

The integrand (3.3) meets the Cesari's theorem conditions [**4**], being: bounded from below, continuous and convex in y'. This implies that our optimization problem has only one solution we are going to compute. Nothwithstanding $F(y, y')$ is singular for $v_1 \to 0$, Cesari's theorem, as for the pure gravity brachistochrone, allows to infer the existence and uniqueness of the solution: we will be faced with a convergent improper integral, whose singularity will be eliminated by means of a suitable transformation (5.2).

Ought to x does not appear explicitly in F, a first integral of the Euler-Lagrange equation holds, namely the so called[4] *Beltrami identity*:

$$y' \frac{\partial F}{\partial y'} - F \; = \; C, \tag{3.5}$$

being the constant C to be determined by the boundary conditions. Plugging the expression (3.3) in (3.5) we find:

$$y' \frac{\partial F}{\partial y'} - F = -\frac{1}{\sqrt{[\ln(ky_1) - \ln y]\,(1 + y'^2)}} = C, \tag{3.6}$$

therefrom we infer the negativity of the Beltrami constant C in our problem. Moreover minding that $0 < y_2 \le y \le y_1$, one gets:

$$-\frac{1}{\sqrt{\ln k}} \; \le \; C < 0. \tag{3.7}$$

Solving for y' in (3.6), one obtains the brachistochrone's ode under attractive logarithmic potential:

$$\begin{cases} y'(x) = -\sqrt{\dfrac{1}{C^2\,[\ln(k\,y_1) - \ln y]} - 1}, \\ y(x_1) = y_1, \\ y(x_2) = y_2. \end{cases} \tag{3.8}$$

The minus sign has been chosen because, with $v_1 \le 0$, y shall decrease from y_1 to y_2 for connecting P_1 to P_2.

We postpone its integration to Section 5: let us premise some properties of a class of special functions.

4. The Lauricella Functions

We will recall in few lines the Lauricella functions[5] The best known ${}_2F_1\,(a, b; c; \xi)$ gaussian hypergeometric function has one *argument* ξ whose powers define the series development of the function itself, and three *parameters* a, b, c, appearing in the coefficients of the powers. Several generalizations of such a function are possible.

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[4]Eugenio Beltrami (1835–1900), professor of Bologna, Pavia, Rome, achieved an important role in Italian mathematics in the second half of the nineteenth century.

[5]Giuseppe Lauricella (1867–1913), professor of Rome University, published in 1893 a milestone–article about the generalized hypergeometric functions which took the name after him.

In 1893 G. Lauricella in [**10**] defined four types of hypergeometric functions of n arguments $(\xi_1,\dots,\xi_n)$ and $n+2$ parameters $(a;b_1,\dots,b_n;c)$. In particular what he called "D type" in its *complete* version is:

$$F_D^{(n)}(a;b_1,\dots,b_n;c;\xi_1,\dots,\xi_n) = \sum_{m_1,\cdots,m_n=0}^{\infty} \frac{(a,m_1+\cdots+m_n)(b_1,m_1)\cdots(b_n,m_n)}{(c,m_1+\cdots+m_n)} \frac{\xi_1^{m_1}\cdots\xi_n^{m_n}}{m_1!\cdots m_n!}$$

where (α,n), $\alpha\in\mathbb{C}$, $n\in\mathbb{N}$ denotes the Pochhammer symbol. The power series converges for $|\xi_1|,|\xi_2|,\dots,|\xi_n|<1$. Notice that $F_D^{(n)}$ reduces to ${}_2F_1$ if $n=1$; and to the Appell function of two variables F_1 if $n=2$.

The Lauricella *confluent* functions, first time introduced by Srivastava and Exton [**17**], can be obtained by a limiting process like that used in deducing the Kummer confluent hypergeometric function ${}_1F_1(-,a;c;z)$ from the Gauss one.

The "D type" Lauricella confluent function has the standard notation:

$$\Phi_D^{(n)}(a;b_1,\dots,b_{n-1};c;\xi_1,\dots,\xi_n) = \sum_{m_1,\cdots,m_n=0}^{\infty} \frac{(a,m_1+\cdots+m_{n-1})(b_1,m_1)\cdots(b_{n-1},m_{n-1})}{(c,m_1+\cdots+m_n)} \frac{\xi_1^{m_1}\cdots\xi_n^{m_n}}{m_1!\cdots m_n!},$$

with $n+1$ parameters and n variables, or "arguments" and the usual domanin of convergence $|\xi_1|,|\xi_2|,\dots,|\xi_n|<1$. In [**18**] and [**6**], the integral representation theorem is proved:

$$\Phi_D^{(n)}(a;b_1,\dots,b_{n-1};c;\xi_1,\dots,\xi_n) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \times \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-u\xi_1)^{-b_1}\cdots(1-u\xi_{n-1})^{-b_{n-1}} e^{u\xi_n}\,du, \quad (4.1)$$

which has been used by [**25**] and by Kalla [**9**], in an article about the stellar dynamics. For our purpose, we will restrict to the case $n=2$. Then the "D type" Lauricella confluent hypergeometric function for $n=2$ arguments, say ξ_1 and ξ_2, and 3 parameters: a, $b_1=b$, c, is commonly written as:

$$\Phi_D^{(2)}(a;b;c;\xi_1,\xi_2),$$

and its integral representation (4.1) becomes:

$$\Phi_D^{(2)}(a;b;c;\xi_1,\xi_2) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-u\xi_1)^{-b} e^{\xi_2 u}\,du. \quad (4.2)$$

The integral at (4.2) is meaningful for whichever ξ_1 and ξ_2 provided $\mathfrak{Re}(c-a-b)>0$ and $\mathfrak{Re}(a)>0$. With this choice, the analytic continuation unicity entitles us to define $\Phi_D^{(2)}$ over $\mathbb{C}^2$. By this we may solve our optimization problem without any restriction on the arguments. Of course, if one wishes to use the $\Phi_D^{(2)}$ power series expansion, the bounds on the variables shall be observed, as it will be explained later.

5. The Quickest Trajectory Effective Computation

Let us go back to (3.8). Separating the variables we obtain:

$$-\left(x-x_1\right) = \int_{y_1}^{y}\sqrt{\frac{\ln\left(ky_1\right)-\ln\eta}{C^{-2}-\ln\left(ky_1\right)+\ln\eta}}\,d\eta.$$

Notice that the integrand becomes infinite when evaluated at the lower integration bound. We define

$$z=z(y)=\ln\left(ky_1\right)-\ln y>0, \tag{5.1}$$

and then, performing the change of variables:

$$\eta=\eta(\psi)=ky_1\left(\frac{y}{ky_1}\right)^{\psi}, \tag{5.2}$$

we obtain:

$$x=x_1+y_1Cz^{3/2}\int_{w(z)}^{1}\frac{\sqrt{\psi}e^{-z\,\psi}}{\sqrt{1-zC^2\psi}}\,d\psi, \tag{5.3}$$

where the variable z is defined in (5.1), and:

$$w=w(z)=\frac{mv_1^2}{4G\mu}z. \tag{5.4}$$

The right hand side of (5.3), can be split as:

$$\begin{aligned}\int_{w(z)}^{1}\frac{\sqrt{\psi}\,e^{-z\,\psi}}{\sqrt{1-zC^2\psi}}\,d\psi &= \int_{0}^{1}\frac{\sqrt{\psi}\,e^{-z\,\psi}}{\sqrt{1-zC^2\psi}}\,d\psi-\int_{0}^{w(z)}\frac{\sqrt{\psi}\,e^{-z\,\psi}}{\sqrt{1-zC^2\psi}}\,d\psi\\ &= \int_{0}^{1}\frac{\sqrt{\psi}\,e^{-z\,\psi}}{\sqrt{1-zC^2\psi}}\,d\psi-w^{3/2}\int_{0}^{1}\frac{\sqrt{\psi}\,e^{-z\,w\psi}}{\sqrt{1-zC^2w\psi}}\,d\psi, \end{aligned}\tag{5.5}$$

namely a couple of abelian integrals, for no–one of which the integrand becomes infinite at the lower integration bound. Considering the first integral in the third side of (5.5), minding (4.2), we get:

$$a-1=\frac{1}{2};\ c-a-1=0,\ b=\frac{1}{2},\ \xi_1=z;\ \xi_2=-C^2z,$$

and for the second one:

$$a-1=\frac{1}{2};\ c-a-1=0,\ b=\frac{1}{2},\ \xi_1=zw;\ \xi_2=-C^2zw.$$

Being met all the conditions on a, b, c, the integral form of $\Phi_D^{(2)}$ can be used, and (5.5) becomes:

$$x-x_1=\frac{2y_1C}{3}z^{3/2}\left[\Phi_D^{(2)}\left(\frac{3}{2};\frac{1}{2};\frac{5}{2};z,-C^{-2}z\right)-\Phi_D^{(2)}\left(\frac{3}{2};\frac{1}{2};\frac{5}{2};zw,-C^{-2}zw\right)\right],$$

namely x as a function of the auxiliary variable z.

Minding the $(z,\,y)$ relationship (5.1), we eventually obtain the trajectory's parametric equations $x(z)$ and $y(z)$ through the parameter z subject to the limitations $0\leqslant z\leqslant\ln\left(ky_1\right)-\ln y$:

$$\begin{cases} x(z)\ =\ x_1+\dfrac{2y_1C}{3}z^{3/2}\left[\Phi_D^{(2)}\left(\dfrac{3}{2};\dfrac{1}{2};\tfrac{5}{2};z,-\dfrac{z}{C^2}\right)-\Phi_D^{(2)}\left(\dfrac{3}{2};\dfrac{1}{2};\tfrac{5}{2};zw,-\dfrac{zw}{C^2}\right)\right],\\ y(z)=ky_1e^{-z}. \end{cases}$$

In such a way the parametric equations $x = x(z)$, $y = y(z)$ of our planar brachistochrone under logarithmic potential have been computed completely for $v_1 > 0$.

The brachistochrone of a particle moving by the rest, $v_1 = 0$, as asked by Johann Bernoulli: "et moveri incipiens a puncto A", can be obtained - see (3.4) – putting $k = 1$ in the parametric equations $x(z)$, $y(z)$.

As for the cycloidal brachistochrone, one could inquire about the cause bending the quickest trajectory as to stretch it along the Ox sense even in the absence of x–forces. The answer is that the stretching effect is due to the boundary link

$$y(x_2) \;=\; y_2,\, y_2 < y_1,\, x_2 > x_1$$

of reaching the endpoint P_2.

Recalling that the Lauricella series $\Phi_D^{(2)}$ converges when $|\xi_1| < 1$, $|\xi_2| < 1$ and it is represented by means of the integral (4.2) being $\Re\mathfrak{e}\,(a)$ and $\Re\mathfrak{e}(c-a-b)$ positive, the variables involved in our problem and which have to be bounded are $\xi_1 = z$, $\xi_2 = -C^2 z$ and $\xi_1 = kz^2$, $\xi_2 = -kC^2z^2$. About them we observe that:

(i) the knowledge of C is necessary for the boundedness check of ξ_2 and we supplied the "a priori" estimation (3.7);

(ii) a suitable scaling can be recommended in order to keep the ξ_i bounded for $y_2 \le y \le y_1$;

(iii) a computation *restart* can be done in order to assume as "new initial conditions" the values:

$$z_{\text{last}},\; x(z_{\text{last}}),\; y(z_{\text{last}}),\; v\big(y(z_{\text{last}})\big)$$

computed at the end of the first solution which provides the (x, y) couples after the prescribd boundary conditions:

$$x_1,\, y_1,\, v_1;\; x_2,\, y_2$$

and so on.

6. The Determination of the Beltrami Constant

Looking for a brachistochrone, one has to solve partially a Cauchy problem, namely to integrate the dynamics equation, in obtaining the function $v(y)$ to be plugged in the integral to be varied. Next, the second order boundary value problem (Euler–Lagrange equation) requires two integration constants to be computed, what is often disregarded, even if far from being a trivial one.

For instance, see [**27**], the classic cycloidal brachistochrone requires, in order to compute the integration constants, to solve the φ–transcendental equation:

$$\frac{1-\cos\varphi}{\varphi-\sin\varphi} \;=\; \text{constant.}$$

The same subject has been faced also by Bolza, see [**28**], equation (12).

Our second constant of integration has been evaluated tacitly "in itinere", by means of (5.5) where it appears as a definite integral.

In order to achieve the first integration constant, namely the Beltrami one, let us posit the least time trajectory to meet the final position $P_2(x_2, y_2)$. Then we will obtain the C–resolvent:

$$\frac{3\,(x_2-x_1)}{2y_1}=\left(\ln\frac{ky_1}{y_2}\right)^{3/2}\left[\Phi_D^{(2)}\left(\frac{3}{2};\frac{1}{2};\frac{5}{2};\ln\frac{ky_1}{y_2},-\frac{1}{C^2}\ln\frac{ky_1}{y_2}\right)\right.$$
$$\left.-\,\Phi_D^{(2)}\left(\frac{3}{2};\frac{1}{2};\frac{5}{2};\ln k\left(\ln\frac{ky_1}{y_2}\right)^2,-\frac{1}{C^2}\ln k\left(\ln\frac{ky_1}{y_2}\right)^2\right)\right],$$

holding the integration constant C as a sole unknown. Of course the estimation (3.7) may be useful to this aim.

Such a transcendental problem is of course much more difficult than the Vivanti's one, and requires a specific analysis and numerical treatment, outside of the authors' interest.

7. Conclusions

The planar brachistochrone in vacuo under the attraction of an infinite rod has been considered. The use of a first integral of the Euler–Lagrange ode, namely the Beltrami identity, leads to solve a nonlinear first order boundary value problem.

Its solution is proved to exist unique because the variational integrand meets a sufficient condition by Cesari.

The first order nonlinear ode, leading to an abelian integral, has been solved in closed form by means of a generalized, 2–variables, hypergeometric Lauricella function, specifically the confluent $\Phi_D^{(2)}$ one.

The trajectory parametric equations $x(z)$, $y(z)$ have then been obtained, defining our brachistochrone completely. Such a solution describes it for both nonzero and zero initial speed v_1 of the moving bead in vacuo.

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