LOCALLY CYCLIC PROJECTIVE MODULES

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Abstract. Let $R$ be a commutative ring with identity and $M$ an $R$-module. If $M$ is either locally cyclic projective or faithful multiplication then $M$ is locally either zero or isomorphic to $R$. We investigate locally cyclic projective modules and the properties they have in common with faithful multiplication modules. Our main tool is the trace ideal. We see that the module structure of a locally cyclic projective module and its trace ideal are closely related. We prove cancellation laws involving projective modules and their trace ideals. Among various applications, we show that the product of a prime ideal and a locally cyclic projective module is a prime submodule.

Introduction

All rings are commutative with identity and all modules are unital.

Let $R$ be a ring. An $R$-module $M$ is a multiplication module if every submodule $N$ of $M$ has the form $IM$ for some ideal $I$ of $R$. Equivalently, $N = \{N : M\}M$. Anderson [7] defined $\theta(M) = \sum_{m \in M} [Rm : M]$ and showed that if $M$ is multiplication then $M = \theta(M)$. Anderson and Al–Shaniafi also proved [5, Theorem 2.3] that if $M$ is faithful multiplication then $\theta(M)$ is a pure ideal of $R$ (equivalently $\theta(M)$ is an idempotent multiplication ideal of $R$, [1]). Let $M$ be a multiplication module. Then $M = \theta(M)$. If $m \in M$ then $Rm = \{Rm : M\}M = \{Rm : M\} \theta(M)M = \theta(M)\{Rm : M\}M = \theta(M)m$. The converse is also true: Suppose $P$ is a maximal ideal of $R$ and $Rm = \theta(M)m$ for each $m \in M$. If $\theta(M)$ is contained in $P$ then $Rm = Pm$, and hence $(Rm)P = (Pm)p$. By Nakayama’s Lemma $(Rm)_P = 0_P$, and hence $M_P = 0_P$. Otherwise $\theta(M) \not\subseteq P$, and hence there exist $m \in M$ and $p \in P$ such that $(1 - p)m \subseteq Rm$. It follows by [6, Theorem 2.1] that $M$ is multiplication. See also [3, Corollary 1.4(2)] and [20, Theorem 2]. The trace ideal of an $R$-module $M$ is $\text{Tr}(M) = \sum_{f \in \text{Hom}(M,R)} f(M)$. If $M$ is projective then $M = \text{Tr}(M)M$, $\text{ann} M = \text{ann} \text{Tr}(M)$, and $\text{Tr}(M)$ is a pure ideal of $R$, [11, Proposition 3.30],[21], and [22].

Locally cyclic projective modules (for example projective ideals) and faithful multiplication modules have many common properties. If $M$ is a locally cyclic projective module or a faithful multiplication module then $M$ is locally either zero or isomorphic to $R$. If $M$ is a locally cyclic projective (resp. faithful multiplication) $R$-module, then $M$ is finitely generated if and only if $\text{Tr}(M)$ (resp. $\theta(M)$) is finitely generated, [2, Corollary 2.6] and [5, Corollary 2.2]. More generally, if $M$ is locally cyclic projective (resp. faithful multiplication), then for each $r \in \text{Tr}(M)$,
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In the first part of this paper we study the relationship between projective and multiplication modules and give some properties of locally cyclic projective modules similar to those of faithful multiplication modules, see Theorems 1.2, 1.3 and Corollary 1.4. We investigate the ring of endomorphisms of projective modules in Proposition 1.6.

It is evident from results proved in the second part of this paper that the module structures of a locally cyclic projective module and its trace ideal are closely related even though they are not necessarily isomorphic, for example see [19, pp. 2606-7].

We prove that if $M$ is a projective module which is finitely cogenerated (resp. has uniform dimension), then $\text{Tr}(M)$ is finitely cogenerated (resp. has uniform dimension), and the converse is true if $M$ is moreover locally cyclic. We also show that if every submodule of a projective module $M$ is large then every ideal contained in $\text{Tr}(M)$ is large, and the converse is true if $M$ is locally cyclic, Theorem 2.1 and Proposition 2.2.

Theorem 2.3 gives some cancellation laws involving submodules of a projective module $M$ and ideals of $R$ contained in $\text{Tr}(M)$. In case $M$ is locally cyclic, there is a lattice isomorphism between the submodules of $M$ and the ideals contained in $\text{Tr}(M)$. As a consequence of Theorems 2.1 and 2.3, we give a formula for the socle and Jacobson radical of a locally cyclic projective module similar to one for faithful multiplication modules, [10, Corollary 2.14]. We conclude with Corollary 2.6 which shows that if $P$ is a prime ideal of $R$ and $M$ is a locally cyclic projective $R$–module such that $M \neq PM$, then $PM$ is a prime submodule of $M$.

For the basic concepts used, see [11], [12], [14], [15] and [16].

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1. Projective Modules and Multiplication Modules

In [1] we defined a submodule $N$ of $M$ to be idempotent in $M$ if $N = [N : M]N$. For convenience we collect several observations from Theorems 1.1, 1.6 and 2.1 of that paper.

Lemma 1.1. Let $R$ be a ring and $M$ a multiplication $R$–module such that $\text{ann} M$ is a pure ideal, and let $N$ be a submodule of $M$. Then statements (1), (2) and (3) below are equivalent. If $M$ is furthermore finitely generated and faithful, then (4) and (5) are true.

1. $N$ is pure in $M$.
2. $N$ is multiplication and idempotent in $M$.
3. $N_P = 0$ or $N_P = M_P$ for each maximal ideal $P$ of $R$.
4. If $N$ is pure in $M$, then $(\bigcap_{\lambda \in \Lambda} I_{\lambda})N = \bigcap_{\lambda \in \Lambda} I_{\lambda}N$.
5. If $N$ is pure in $M$, then $\text{Tr}(N) = [N : M]$.

Let $M$ be an $R$–module and $P$ any maximal ideal of $R$. $M$ is $P$–torsion, [20], if for each $m \in M$, there exists $p \in P$ such that $(1-p)m = 0$. P.F. Smith [20, Lemma 6] proved that a multiplication module $M$ is $P$–torsion if and only if $PM = M$.

The next result gives several properties of the trace of projective modules. Compare it with [10, Lemma 3.2 and Proposition 3.4] and [20, Lemma 6].
Theorem 1.2. Let $R$ be a ring and $M$ a projective $R$-module. Then

1. $Rm = \text{Tr}(M)m$ for each $m \in M$.
2. For every prime ideal $P$ of $R$, $M$ is $P$-torsion if and only if $M = PM$.
3. For every prime ideal $P$ of $R$, either $\text{Tr}(M) \subseteq P$ or $\text{Tr}(M) + P = R$.
4. For every proper ideal $I$ of $\text{Tr}(M)$, $IM \neq M$.
5. If $M$ is non-zero then there exists a maximal ideal $P$ of $R$ such that $\text{Tr}(PM)$ is a maximal submodule of $\text{Tr}(M)$.
6. If $M$ is locally cyclic and for each minimal prime ideal $P$ of $R$, $PM \neq M$, then $M$ is finitely generated.

Proof. (1) Let $m \in M$. Let $P$ be a maximal ideal of $R$. As $\text{Tr}(M)$ is a pure ideal of $R$, it follows by Lemma 1.1(3) that either $\text{Tr}(M)_P = 0_P$, in which case both sides of the equality $(Rm)_P = (\text{Tr}(M)m)_P$ collapse to $0_P$, or $\text{Tr}(M)_P = R_P$, in which case each side is $(Rm)_P$. So the equality is true locally and hence globally.

(2) Suppose that $P$ is prime and $M = PM$. Then $\text{Tr}(M) = P\text{Tr}(M)$, and hence $\text{Tr}(M) \subseteq P$. Let $m \in M$. By (1), $Rm = \text{Tr}(M)m \subseteq Pm$, so that $Rm = Pm$, and hence there exists $p \in P$ such that $(1-p)m = 0$, and $M$ is $P$-torsion. Conversely, assume that $M$ is $P$-torsion. Then $M_P = 0_P$, [6], and hence $\text{Tr}(M)_P \cong \text{Tr}(M_P) = 0_P$, and therefore $\text{Tr}(M) \subseteq P$. This finally gives that $M = \text{Tr}(M)M \subseteq PM \subseteq M$, so that $M = PM$.

(3) Let $P$ be any prime ideal of $R$. If $M = PM$, then $\text{Tr}(M) \subseteq P$. Suppose that $M \neq PM$. Then $\text{Tr}(M) \nsubseteq P$, and hence there exists $p \in P$ such that $1-p \in \text{Tr}(M)$. Since $\text{Tr}(M)$ is a pure ideal of $R$, $R(1-p) = (1-p)\text{Tr}(M)$, and hence $1 = p + (1-p)\text{Tr}(M) \subseteq P + \text{Tr}(M)$, so that $R = P + \text{Tr}(M)$.

(4) Suppose not. Then there exists an ideal $I$ properly contained in $\text{Tr}(M)$ such that $M = IM$. Then

$$\text{Tr}(M) = I\text{Tr}(M) \subseteq I \subseteq \text{Tr}(M),$$

and hence $\text{Tr}(M) = I$, a contradiction.

(5) Let $0 \neq m \in M$. Then $\text{ann}(m) \neq R$, and hence there exists a maximal ideal $P$ of $R$ such that $\text{ann}(m) \subseteq P$. Suppose that $M = PM$. Then by (2), $M$ is $P$-torsion and hence there exists $p \in P$ such that $(1-p)m = 0$. It follows that $1-p \in \text{ann}(m) \subseteq P$, a contradiction. Therefore $M \neq PM$, and hence $\text{Tr}(M) \neq \text{Tr}(PM) = P\text{Tr}(M)$. Suppose that $I$ is an ideal of $R$ such that $P\text{Tr}(M) \subseteq I \subseteq \text{Tr}(M)$. Then

$$P \subseteq [P\text{Tr}(M) : \text{Tr}(M)] \subseteq [I : \text{Tr}(M)] \subseteq R,$$

and hence either $P = [I : \text{Tr}(M)]$ or $[I : \text{Tr}(M)] = R$. This gives that either $P\text{Tr}(M) = [I : \text{Tr}(M)]\text{Tr}(M) = I$, or $I = \text{Tr}(M)$, and the result is now clear.

(6) Suppose $M$ is not finitely generated. It follows by [22, Lemma 1.2] that $\text{Tr}(M)$ is not finitely generated, see also [2, Corollary 3.6]. Hence $\text{Tr}(M) \neq R$, and hence there exists a maximal ideal $Q$ of $R$ such that $\text{Tr}(M) \subseteq Q$. Let $P$ be any
minimal prime ideal of \( R \) such that \( P \subseteq Q \). It follows by (3) that \( R = \text{Tr}(M) + P \subseteq Q \), a contradiction. \( \square \)

Multiplication modules are locally cyclic \([8]\), but the converse is not true. For example the \( \mathbb{Z} \)-module \( \bigoplus_p \mathbb{Z}_p \), where the sum is taken over an infinite collection of primes \( p \), is locally cyclic but not multiplication. On the other hand if \( M \) is a multiplication module, then \( M = \theta(M)M \), \([7, \text{Proposition 1}]\), and hence \( \text{Tr}(M) = \theta(M) \text{Tr}(M) \), so that \( \text{Tr}(M) \subseteq \theta(M) \). Obviously, \( \text{ann} M \subseteq \theta(M) \), and therefore \( \text{Tr}(M) + \text{ann} M \subseteq \theta(M) \). The reverse inclusion is not true. Let \( R = \mathbb{Z}[\sqrt{5}] \), \( I = R2 + R(\sqrt{5} - 1) \), and \( K = R(\sqrt{5} - 1) \). Let \( M = R/K \) and \( N = I/K \). \( N \) is a cyclic submodule of \( M \) (hence multiplication), and therefore \( \theta(N) = R \). Also \( \text{Tr}(N) = 0 \). As \( N \) is a non–zero module, \( \text{ann}(N) \neq R \), and hence \( \theta(N) \notin \text{Tr}(N) + \text{ann} N \).

As we mentioned above, if \( M \) is multiplication, then \( M = \theta(M)M \). The converse is not true. For example, take \( M \) to be a maximal ideal of a non–discrete valuation ring. Then \( M = \theta(M) \), and \( M = M^2 = \theta(M)M \), but \( M \) is not multiplication, \([7, \text{p. 464}]\).

The next theorem shows that the converses of the above facts are true if the multiplication module is projective.

**Theorem 1.3.** Let \( R \) be a ring and \( M \) a projective \( R \)-module. Then the following are equivalent:

1. \( M \) is locally cyclic.
2. \( \theta(M) = \text{Tr}(M) + \text{ann} M \).
3. \( M = \theta(M)M \).
4. \( M \) is multiplication.

**Proof.**

(1) \( \implies \) (2): Let \( m \in M \). By Theorem 1.1(1), \( Rm = \text{Tr}(M)m \), and hence \( R = \text{Tr}(M) + [0 : Rm] \). Therefore

\[
[Rm : M] = [Rm : M][\text{Tr}(M) + [0 : Rm][Rm : M]] \subseteq \text{Tr}(M) + \text{ann} M,
\]

so that \( \theta(M) \subseteq \text{Tr}(M) + \text{ann} M \). For the reverse inclusion, let \( z = x + y \) with \( x \in \text{Tr}(M) \) and \( y \in \text{ann} M \). Then \( zM = xM \) is a finitely generated locally cyclic \( R \)-module, and by \([7, \text{Theorem 1}]\), \( \theta(zM) = R \). Next,

\[
[Rzm : zM] \subseteq [Rm : zM] \subseteq [[Rm : M] : Rz],
\]

and hence

\[
z[Rzm : zM] \subseteq [[Rm : M] : Rz]Rz \subseteq [Rm : M],
\]

from which we obtain \( z\theta(zM) \subseteq \theta(M) \), and hence \( z \in \theta(M) \).

(2) \( \implies \) (3): Let \( \theta(M) = \text{Tr}(M) + \text{ann} M \). Then

\[
M = \text{Tr}(M)M = (\text{Tr}(M) + \text{ann} M)M = \theta(M)M.
\]

(3) \( \implies \) (4): Let \( M = \theta(M)M \). Then \( \text{Tr}(M) = \text{Tr}(\theta(M)M) = \theta(M) \text{Tr}(M) \). Let \( m \in M \). Then

\[
Rm = \text{Tr}(M)m = \theta(M)\text{Tr}(M)m = \theta(M)m,
\]

and \( M \) is multiplication.

(4) \( \implies \) (1): \([8, \text{Proposition 4}]\). \( \square \)
An $R$–module $M$ is torsion free if for every non–zero divisor $s \in R$ and every $0 \neq m \in M$, $sm \neq 0$. Let

$$Z(M) = \{ r \in R : rm = 0 \text{ for some } 0 \neq m \in M \}. $$

The next result gives two properties of locally cyclic projective modules. Compare with [10, Lemmas 4.1 and 4.3].

**Corollary 1.4.** Let $R$ be a ring and $M$ a locally cyclic projective $R$–module. Then

1. $M$ is torsion free.
2. $Z(M) = Z(\text{Tr}(M))$.

**Proof.** (1) Suppose not. There exists a non–zero divisor $s$ and $0 \neq m \in M$ such that $sm = 0$. By Theorem 1.3, $M$ is multiplication and hence

$$Rm = [Rm : M]M = [Rm : M]\text{Tr}(M)M.$$

Since $[Rm : M]\text{Tr}(M) \neq 0$, $s[Rm : M]\text{Tr}(M) \neq 0$. But $0 = Rsm = s[Rm : M] \text{Tr}(M)M$ which implies that

$$s[Rm : M] \text{Tr}(M) \subseteq \text{Tr}(M) \cap \text{ann} M = \text{Tr}(M) \cap \text{ann}(\text{Tr}(M)) = 0,$$

so that $s[Rm : M] \text{Tr}(M) = 0$, a contradiction.

(2) Let $a \in Z(\text{Tr}(M))$. There exists $0 \neq b \in \text{Tr}(M)$ with $ab = 0$. Since $\text{Tr}(M)$ is a pure ideal of $R$ and $b \in \text{Tr}(M)$, $Rb = b \text{Tr}(M)$, and hence $b\text{Tr}(M) \neq 0$. It follows that $a(bM) = 0$ and $bM \neq 0$, so $a \in Z(M)$. Conversely, suppose that $M$ is locally cyclic. By Theorem 1.3, $M$ is multiplication. Let $r \in Z(M)$. Then there exists $0 \neq m \in M$ such that $rm = 0$. Now $Rm = [Rm : M]M = [Rm : M]\text{Tr}(M)M$, and hence $0 = Rrm = r[Rm : M] \text{Tr}(M)M$. This gives that $r[Rm : M] \text{Tr}(M) \subseteq \text{Tr}(M) \cap \text{ann} M = 0$, so that $r[Rm : M] \text{Tr}(M) = 0$. But $[Rm : M] \text{Tr}(M) \neq 0$. Thus $r \in Z(\text{Tr}(M))$, and the result follows.

The next lemma may be compared with [10, Theorem 1.6].

**Lemma 1.5.** Let $R$ be a ring and $M$ an $R$–module. Let $I_\lambda (\lambda \in \Lambda)$ be a non–empty collection of ideals of $R$. If $M$ is faithful multiplication or $M$ is projective then $\cap_{\lambda \in \Lambda} I_\lambda M = \cap_{\lambda \in \Lambda} I_\lambda$.  

**Proof.** Assume that $M$ is a faithful multiplication $R$–module. Then $M = \theta(M)M$ [6, Proposition 1], and $\theta(M)$ is a pure ideal of $R$ [5, Theorem 2.3]. It follows by Lemma 1.1 that $\cap_{\lambda \in \Lambda} I_\lambda \theta(M) = (\cap_{\lambda \in \Lambda} I_\lambda \theta(M))M = (\cap_{\lambda \in \Lambda} I_\lambda \theta(M)) = (\cap_{\lambda \in \Lambda} I_\lambda \theta(M))M$, and hence $\cap_{\lambda \in \Lambda} I_\lambda \theta(M)M = \cap_{\lambda \in \Lambda} I_\lambda M$. We need to show that $\cap_{\lambda \in \Lambda} I_\lambda \theta(M)M = \cap_{\lambda \in \Lambda} I_\lambda M$. We do this locally. Thus we assume that $R$ is a local ring. If $\theta(M) = 0$, then $M = 0$ and both sides of the equality collapse to 0. If $\theta(M) = R$, then the equality is obviously satisfied. Thus the result is true locally and hence globally. In case $M$ is projective, $M = \text{Tr}(M)M$ and $\text{Tr}(M)$ is a pure ideal of $R$, and the proof is similar.

Naoum [18, Corollary 3.3] proved that if $M$ is a finitely generated multiplication module then the ring of endomorphisms of $M$ is commutative (in fact $\text{End}_R(M) \cong R/\text{ann} M$). Since every submodule $N$ of a multiplication module (not necessarily finitely generated) is fully invariant in the sense that $f(N) \subseteq N$ for each $f \in \text{Hom}(M,R)$, the ring of endomorphisms of an arbitrary multiplication
module is commutative [9]. Singh and Al-Shaniafi [19] investigated the ring of endomorphisms of faithful multiplication modules. They showed that the ring of endomorphisms of a faithful multiplication module over a von Neumann regular ring is again von Neumann regular [19, Theorem 5.3]. They also gave an example of a faithful multiplication module over a von Neumann regular ring that can not be embedded in $R$ and whose ring of endomorphisms is $R$. In an earlier investigation of endomorphism rings of projective modules, $R$. Ware [23] proved that a finitely generated projective module $M$ is regular (i.e. every submodule is pure) if and only if $\text{End}_R(M)$ is von Neumann regular [23, Corollary 3.10]. Compare the next result with [19, Theorems 5.2 and 5.3].

**Proposition 1.6.** Let $R$ be a ring and $M$ a projective $R$–module. Let $S = \text{End}_R(M)$.

1. If $R$ is von Neumann regular then so too is $S$.
2. If $M$ is locally cyclic then for any prime ideal $P$ of $R$, if $M \neq PM$ then $S_P \cong R_P$ as $R$–modules.
3. If $M$ is locally cyclic and $R$ has no nilpotent elements, then $S$ has no nilpotent elements.

**Proof.** (1) Kaplansky proved in 1958, [13, Lemma 4], that every finitely generated submodule of a projective module over a von Neumann regular ring is a direct summand and hence is projective. However we apply Lemma 1.5 to give a different proof which may be of interest. Let $N$ be a finitely generated submodule of $M$. Since $R$ is von Neumann regular, it follows by [11, Theorem 11.24] that $N$ is a pure submodule of $M$. Since $M$ is projective and hence flat, it follows by [11, Corollary 11.21] that for each ideal $I$ of $R$, $IN = N \cap IM$. In particular, $[N : M]N = N \cap [N : M]M = N$, since $[N : M]M \subseteq N$. It follows by [14, Theorem 76] that $R = [N : M] + \text{ann} N$, and hence

$$[N : M] \cap \text{ann} N = [N : M] \text{ann} N \subseteq [(\text{ann} N)N : M] = \text{ann} M.$$  

By Lemma 1.5,

$$N \cap (\text{ann} N)M = [N : M]N \cap (\text{ann} N)M \subseteq [N : M]\cap (\text{ann} N)M = ([N : M] \cap \text{ann} N)M = 0,$$

so that $N \cap (\text{ann} N)M = 0$. But

$$M = [N : M]M + (\text{ann} N)M \subseteq N + (\text{ann} N)M \subseteq M,$$

so that $M = N + (\text{ann} N)M$ and hence $M = N \oplus (\text{ann} N)M$. Since $M$ is projective, so too is $N$. The proof now proceeds the same as that of [19, Theorem 5.3].

(2) Since $M \neq PM$, it follows by Proposition 1.2 that $M_P \neq 0_P$, and hence $M_P \cong R_P$. Hence $S_P = \text{End}_R(M)_P \cong \text{End}_{R_P}(M_P) \cong \text{End}_{R_P}(R_P) \cong R_P$.

(3) Let $f \in S$ with $f^n = 0$ for some positive integer $n$. Since $f(M) \subseteq M$ and $M$ is multiplication by Theorem 1.3,

$$f(M) = [f(M) : M]M = [f(M) : M] \text{Tr}(M).$$
It follows that \( f^n(M) = ([f(M) : M]Tr(M))^nM = 0 \), and hence 
\([f(M) : M]Tr(M)\) \subseteq \text{ann}M \cap \text{Tr}(M) = 0 \), so that 
\([f(M) : M]Tr(M)\) \subseteq 0. Since \( R \) has no nilpotent elements, we infer that 
\([f(M) : M]Tr(M) = 0 \), and hence 
\( f(M) = 0 \). This implies that \( f = 0 \). \( \square \)

The condition that \( M \) is projective in part (3) above cannot be discarded. For 
example, the \( \mathbb{Z} \)-module \( M = \mathbb{Z}_8 \) is locally cyclic (in fact multiplication) 
but not projective. The ring \( \text{End}_\mathbb{Z}(M) \equiv \mathbb{Z}/\text{ann}(M) \equiv M \) has nilpotent elements 
but \( \mathbb{Z} \) does not.

2. The Trace of Projective Modules

In this section we explore the relationship between the submodules of a locally 
cyclic projective module and the ideals of \( R \) that are contained in the trace of that 
module.

An \( R \)-module \( M \) is called finitely cogenerated if for every non-empty collection 
of submodules \( N_\lambda (\lambda \in \Lambda) \) of \( M \) with \( \bigcap_{\lambda \in \Lambda} \text{ann}N_\lambda = 0 \), there exists a finite subset \( \Lambda' \) 
of \( \Lambda \) such that \( \bigcap_{\lambda \in \Lambda'} \text{ann}N_\lambda = 0 \). A submodule \( N \) of \( M \) is called large (or essential) 
in \( M \) if for all submodules \( K \) of \( M \), \( N \cap K = 0 \) implies \( K = 0 \). Dually, \( N \) is small 
(or superfluous) in \( M \) if for all submodules \( K \) of \( M \), \( N + K = M \) implies \( K = M \). 
For properties of finitely cogenerated, large and small modules and ideals, see for 
example [15]. Compare the next result with [10, Corollary 1.8 and Theorem 2.13].

**Theorem 2.1.** Let \( R \) be a ring and \( N \) a submodule of an \( R \)-module \( M \).

1. Let \( M \) be projective. If \( M \) is finitely cogenerated then \( \text{Tr}(M) \) is finitely cogenerated, 
and the converse is true if \( M \) is locally cyclic.

2. Let \( M \) be projective. If every submodule of \( M \) is large then every ideal contained 
in \( \text{Tr}(M) \) is large, and the converse is true if \( M \) is locally cyclic.

3. Let \( M \) be finitely generated faithful multiplication and \( N \) pure in \( M \). Then \( N \) 
is finitely cogenerated if and only if \( \text{Tr}(N) \) is finitely cogenerated.

4. Let \( M \) be finitely generated faithful multiplication and \( N \) pure in \( M \). Then \( N \) 
is large (small) in \( M \) if and only if \( \text{Tr}(N) \) is a large (small) ideal of \( R \).

**Proof.** (1) Let \( I_\lambda (\lambda \in \Lambda) \) be a non-empty collection of ideals of \( R \) contained in 
\( \text{Tr}(M) \) such that \( \bigcap_{\lambda \in \Lambda} I_\lambda = 0 \). By Lemma 1.5, \( \bigcap_{\lambda \in \Lambda} I_\lambda M = (\bigcap_{\lambda \in \Lambda} I_\lambda) M = 0 \), and hence 
there exists a finite subset \( \Lambda' \) of \( \Lambda \) such that \( \bigcap_{\lambda \in \Lambda'} I_\lambda M = 0 \). It follows that 
\( (\bigcap_{\lambda \in \Lambda'} I_\lambda) M = 0 \), and hence \( \bigcap_{\lambda \in \Lambda'} I_\lambda \subseteq \text{ann} M = \text{ann} \text{Tr}(M) \). But 
\( \bigcap_{\lambda \in \Lambda'} I_\lambda \subseteq \text{Tr}(M) \). Thus \( \bigcap_{\lambda \in \Lambda'} I_\lambda \subseteq \text{ann}(\text{Tr}(M)) \cap \text{Tr}(M) = 0 \), so that 
\( \bigcap_{\lambda \in \Lambda'} I_\lambda = 0 \), and \( \text{Tr}(M) \) is finitely cogenerated. Assume now that \( M \) is a locally cyclic projective 
module. By Theorem 1.3, \( M \) is multiplication. Let \( N_\lambda (\lambda \in \Lambda) \) be a non-empty collection of submodules of \( M \) with 
\( \bigcap_{\lambda \in \Lambda} \text{ann}N_\lambda = 0 \). Then 
\[ \bigcap_{\lambda \in \Lambda} [N_\lambda : M] = \bigcap_{\lambda \in \Lambda} N_\lambda : M = \text{ann} M = \text{ann} \text{Tr}(M), \]
and hence \( (\bigcap_{\lambda \in \Lambda} [N_\lambda : M]) \text{Tr}(M) = 0 \). As \( \text{Tr}(M) \) is a pure ideal of \( R \), we infer from 
Lemma 1.1 that \( \bigcap_{\lambda \in \Lambda} [N_\lambda : M] \text{Tr}(M) = 0 \). Since \( \text{Tr}(M) \) is finitely cogenerated,
there exists a finite subset $\Lambda'$ of $\Lambda$ such that $\bigcap_{\lambda \in \Lambda'} [N_\lambda : M] \text{Tr}(M) = 0$, and hence

$$\bigcap_{\lambda \in \Lambda'} N_\lambda = \bigcap_{\lambda \in \Lambda'} [N_\lambda : M] M = \bigcap_{\lambda \in \Lambda'} [N_\lambda : M] \text{Tr}(M) M = 0,$$

and $M$ is finitely cogenerated.

(2) Let $I$ be an ideal of $R$ contained in $\text{Tr}(M)$ and assume that $I \cap J = 0$ for some $J \subseteq \text{Tr}(M)$. It follows by Lemma 1.5 that $IM \cap JM = (I \cap J)M = 0$. Then either $IM = 0$ or $JM = 0$, and hence either $I \subseteq \text{Tr}(M) \cap \text{ann} M = 0$, or $J \subseteq \text{Tr}(M) \cap \text{ann} M = 0$, i.e. $I = 0$ or $J = 0$.

Conversely assume that $M$ is a locally cyclic projective module. Let $N$ be a submodule of $M$ and suppose that $N \cap K = 0$ for some submodule $K$ of $M$. Then

$$[N : M] \cap [K : M] = ([N \cap K] : M) = \text{ann} M = \text{ann} \text{Tr}(M),$$

and hence by Lemma 1.1, $[N : M] \text{Tr}(M) \cap [K : M] \text{Tr}(M) = 0$. Then either $[N : M] \text{Tr}(M) = 0$ or $[K : M] \text{Tr}(M) = 0$, and hence either $N = [N : M]M = [N : M] \text{Tr}(M) M = 0$ or $K = [K : M] \text{Tr}(M) M = 0$.

(3) By Lemma 1.1, $\text{Tr}(N) = [N : M]$. Let $I_\lambda(\lambda \in \Lambda)$ be a collection of ideals of $R$ contained in $\text{Tr}(N)$ such that $\bigcap_{\lambda \in \Lambda} I_\lambda = 0$. By Lemma 1.1, $\bigcap_{\lambda \in \Lambda} I_\lambda N = (\bigcap_{\lambda \in \Lambda} I_\lambda) N = 0$, and hence there exists a finite subset $\Lambda'$ of $\Lambda$ such that $(\bigcap_{\lambda \in \Lambda'} I_\lambda) N = \bigcap_{\lambda \in \Lambda'} I_\lambda N = 0$. It follows that $\bigcap_{\lambda \in \Lambda'} I_\lambda \subseteq \text{ann} N \cap \text{Tr}(N) = 0$, so that $\bigcap_{\lambda \in \Lambda'} I_\lambda = 0$ and $\text{Tr}(N)$ is finitely cogenerated. Conversely, let $N_\lambda(\lambda \in \Lambda)$ be a collection of submodules of $N$ with $\bigcap_{\lambda \in \Lambda} N_\lambda = 0$. Then

$$\bigcap_{\lambda \in \Lambda} [N_\lambda : M] = \bigcap_{\lambda \in \Lambda} [N_\lambda : M] = \text{ann} M = 0,$$

and hence there exists a finite subset $\Lambda'$ of $\Lambda$ such that $\bigcap_{\lambda \in \Lambda'} [N_\lambda : M] = 0$. It follows from Lemma 1.1 that

$$\bigcap_{\lambda \in \Lambda'} N_\lambda = \bigcap_{\lambda \in \Lambda'} [N_\lambda : M] M = \bigcap_{\lambda \in \Lambda'} [N_\lambda : M] M = 0,$$

and $N$ is finitely cogenerated.

(4) Let $N$ be a large submodule of $M$. Let $\text{Tr}(N) \cap I = 0$ for some ideal $I$ of $R$. Then by Lemma 1.5,

$$0 = (\text{Tr}(N) \cap I) M = \text{Tr}(N) M \cap IM = N \cap IM,$$

and hence $IM = 0$. Since $M$ is faithful, $I = 0$, and $\text{Tr}(N)$ is large. Conversely, let $\text{Tr}(N)$ be a large ideal of $R$, and let $N \cap K = 0$ for some submodule $K$ of $M$. Then

$$0 = \text{ann} M = [N \cap K : M] = [N : M] \cap [K : M] = \text{Tr}(N) \cap [K : M],$$

and hence $[K : M] = 0$. This implies that $K = [K : M]M = 0$, and $N$ is large in $M$. Finally, suppose that $N$ is small in $M$ and let $\text{Tr}(N) + J = R$ for some ideal $J$ of $R$. Then $N + JM = \text{Tr}(N) M + JM = M$, and hence $JM = M$. As $M$ is finitely generated faithful multiplication, $J = R$ and $\text{Tr}(N)$ is a small ideal of $R$. Conversely, let $\text{Tr}(N)$ be a small ideal of $R$ and let $N + K = M$ for some submodule $K$ of $M$. By [20, Corollary 3(i) and Proposition 4] and see also the remark after [3, Corollary 1.2],

$$\text{Tr}(N) + [K : M] = [N : M] + [K : M] = [N + K : M] = R,$$
An \( R \)-module \( M \) is uniform if the intersection of any two non-zero submodules of \( M \) is non-zero. Using Theorem 2.1(1) it is clear that if a projective \( R \)-module \( M \) is uniform then so too is \( \text{Tr}(M) \), and the converse is true if \( M \) is locally cyclic. \( M \) has finite uniform dimension if it does not contain an infinite direct sum of non-zero submodules. Compare the next result with \([10, \text{Theorem 2.15}]\).

**Proposition 2.2.** Let \( R \) be a ring and \( M \) a projective \( R \)-module. If \( M \) has finite uniform dimension then so too has \( \text{Tr}(M) \). The converse is true if \( M \) is locally cyclic.

**Proof.** Suppose that \( M \) has finite uniform dimension. If \( \text{Tr}(M) \) contains a direct sum of submodules \( I_\lambda(\lambda \in \Lambda) \), then by Lemma 1.5 it follows that \( \sum_{\lambda \in \Lambda} I_\lambda M \) is direct, and hence all but a finite number of the submodules \( I_\Lambda M \) are zero. If \( I_\Lambda M = 0 \) then \( I_\Lambda \subseteq \text{ann} \, M \cap \text{Tr}(M) = 0 \). Hence \( \text{Tr}(M) \) has finite uniform dimension.

Conversely, suppose \( M \) is locally cyclic and \( \text{Tr}(M) \) has finite uniform dimension. If \( M \) contains a direct sum of submodules \( N_\lambda(\lambda \in \Lambda) \), then using Lemmas 1.5 and 1.1 we see that \( \sum_{\lambda \in \Lambda} [N_\lambda : M] \text{Tr}(M) \) is direct, and hence all but a finite number of the submodules \( [N_\lambda : M] \text{Tr}(M) \) are zero. If \( [N_\lambda : M] \text{Tr}(M) = 0 \) then \( N_\lambda = [N_\lambda : M]M = [N_\lambda : M] \text{Tr}(M)M = 0 \). Hence \( M \) has finite uniform dimension. \( \square \)

An \( R \)-module \( M \) is a cancellation module if for all ideals \( I, J \) of \( R \), \( IM = JM \) implies that \( I = J \). Finitely generated faithful multiplication modules are cancellation modules, \([20, \text{Theorem 9 Corollary 1}]\). In the Boolean ring of eventually constant sequences of elements of \( \mathbb{Z}_2 \), the maximal ideal of finitely non-zero sequences is an example of a faithful multiplication module (not finitely generated) which is not cancellation, \([6, \text{p.2580}]\). A faithful multiplication \( R \)-module \( M \) is cancellation if and only if \( \theta(M) = R \). On the other hand, projective modules are not necessarily cancellation: a projective \( R \)-module \( M \) is cancellation if and only if \( \text{Tr}(M) = R \). If \( M \) is a multiplication \( R \)-module which contains a finitely generated faithful submodule \( N \), then \( N = \theta(M)N \) \([5, \text{Lemma 1.1}]\), and by \([14, \text{Theorem 76}]\) \( R = \theta(M) + \text{ann} \, N = \theta(M) \). Hence \( M \) is finitely generated \([7, \text{Theorem 1}]\). Moreover \( M \) is faithful since \( \text{ann} \, M \subseteq \text{ann} \, N = 0 \). Hence \( M \) is a cancellation module. Similarly, if \( M \) is projective and contains a finitely generated faithful submodule \( N \) then by Proposition 2.2, \( N = \text{Tr}(M)N \), and by \([14, \text{Theorem 76}]\), \( R = \text{Tr}(M) \), so that \( M \) is a cancellation module.

The next theorem lists several properties of cancellation for projective modules. Compare it with \([4, \text{Theorems 1.4 and 1.5}]\).

**Theorem 2.3.** Let \( R \) be a ring, \( M \) a projective \( R \)-module and \( I, J \) ideals of \( R \).

1. If \( I, J \) are contained in \( \text{Tr}(M) \), then \( IM = JM \) if and only if \( I = J \).
2. \( IM = JM \) if and only if \( I \cap \text{Tr}(M) = J \cap \text{Tr}(M) \).
3. \( IM = JM \) if and only if \( [I : \text{Tr}(M)] = [J : \text{Tr}(M)] \).
4. \( I \cap \text{Tr}(M) = I \text{Tr}(M) \).
(5) $M/IM$ is a faithful $R/[I : \text{Tr}(M)]$–module, and $I \cap \text{Tr}(M) = [I : \text{Tr}(M)] \cap \text{Tr}(M)$.

(6) If $M$ is locally cyclic, then $I \leftrightarrow IM$ is a lattice isomorphism between the ideals $I$ contained in $\text{Tr}(M)$ and the submodules of $M$.

**Proof.** (1) Suppose $I, J$ are ideals contained in $\text{Tr}(M)$. Since $\text{Tr}(M)$ is a pure ideal, $I = I\text{Tr}(M)$ and $J = J\text{Tr}(M)$. Suppose that $IM = JM$. Then $I\text{Tr}(M) = J\text{Tr}(M)$, and hence $I = J$.

(2) $M = \text{Tr}(M)M$ implies $IM = IM \cap \text{Tr}(M)M = (I \cap \text{Tr}(M))M$.

$IM = JM$ implies $(I \cap \text{Tr}(M))M = (J \cap \text{Tr}(M))M$, and by (1), $I \cap \text{Tr}(M) = J \cap \text{Tr}(M)$. The converse is clear.

(3) $I \cap \text{Tr}(M) = J \cap \text{Tr}(M)$ by (2). Hence $[I : \text{Tr}(M)] = [(I \cap \text{Tr}(M)) : \text{Tr}(M)] = [(J \cap \text{Tr}(M)) : \text{Tr}(M)] = [J : \text{Tr}(M)]$.

Conversely, since $\text{Tr}(M)$ is a multiplication ideal of $R$, we obtain that $I \cap \text{Tr}(M) = [I : \text{Tr}(M)]\text{Tr}(M) = [J : \text{Tr}(M)]\text{Tr}(M) = J \cap \text{Tr}(M)$, and the result follows.

(4) Let $I$ be an ideal of $R$. Since $\text{Tr}(M)$ is a pure ideal of $R$, $I \cap \text{Tr}(M) = (I \cap \text{Tr}(M))\text{Tr}(M)$, and by Lemma 1.1(4), $I \cap \text{Tr}(M) = I\text{Tr}(M)$.

(5) Since $\text{ann}(M/IM) = [IM : M]$, we have that $\text{ann}(M/IM)M = [IM : M]M \subseteq IM$, and hence $\text{ann}(M/IM)\text{Tr}(M) \subseteq I\text{Tr}(M) \subseteq I$. It follows that $\text{ann}(M/IM) \subseteq [I : \text{Tr}(M)]$. Conversely, $[I : \text{Tr}(M)]M = [I : \text{Tr}(M)]\text{Tr}(M)M \subseteq IM$.

Hence $[I : \text{Tr}(M)] \subseteq [IM : M] = \text{ann}(M/IM)$, so that $\text{ann}(M/IM) = [I : \text{Tr}(M)]$, and therefore, $M/IM$ is a faithful $R/[I : \text{Tr}(M)]$–module. For the second part, $[I : \text{Tr}(M)]M = [I : \text{Tr}(M)]\text{Tr}(M)M \subseteq IM \subseteq [I : \text{Tr}(M)]M$, and so $IM = [I : \text{Tr}(M)]M$. By (2), $I \cap \text{Tr}(M) = [I : \text{Tr}(M)] \cap \text{Tr}(M)$.

(6) $M$ is multiplication by Theorem 1.3, and every submodule of $M$ therefore has the form $IM$ for some ideal $I$ of $R$. Since $(I \cap J)M = IM \cap JM$, the result follows by (1).

Let $M$ be an $R$–module. The *socle* of $M$, $\text{soc} M$, is the sum of all the simple submodules of $M$. Equivalently, $\text{soc} M$ is the intersect of the large submodules of $M$, [10, pp. 766-767]. The *Jacobson radical*, $\text{rad} M$, of $M$ is the intersection of all maximal submodules of $M$ if any, otherwise $M$. Equivalently, $\text{rad} M$ is the sum of all small submodules of $M$.

Compare the first part of the next result with [10, Corollary 2.14(i)] and [4, Corollary 3.3].

**Corollary 2.4.** Let $R$ be a ring and $M$ a locally cyclic projective $R$–module. Then
(1) \( \text{soc} M = \text{Tr}(M)\text{soc} M = \text{soc}(\text{Tr}(M))M = (\text{soc} R)M \).
(2) \( \text{rad} M = \text{Tr}(M)\text{rad} M = \text{rad}(\text{Tr}(M))M = (\text{rad} R)M \).

**Proof.** (1) \( \text{soc} M \) is the intersection of all large submodules \( N \) of \( M \). As \( M \) is projective, it follows by Proposition 1.2 that \( Rm = \text{Tr}(M)m \) for each \( m \in M \), and hence \( N = \text{Tr}(M)N \) for all submodules \( N \) of \( M \). Since \( \text{Tr}(M) \) is a pure ideal, it is locally either 0 or \( R \), so that the equation \( \bigcap \text{Tr}(M)N = (\bigcap N)\text{Tr}(M) \) is true locally and hence globally. Therefore \( \text{soc} M = \text{Tr}(M)\text{soc} M \). Next, Theorem 2.3(6) gives that any submodule of \( M \) is minimal if and only if it is of the form \( IM \), where \( I \) is a minimal ideal of \( R \) contained in \( \text{Tr}(M) \). So \( \text{soc} M = (\text{soc} \text{Tr}(M))M \). Finally, \( \text{soc} R \) is the intersection of all large ideals \( I \) of \( R \). Hence by Lemma 1.1
\[ (\text{soc} R)\text{Tr}(M) = (\bigcap I)\text{Tr}(M) = \bigcap I\text{Tr}(M) . \]
Since \( I \) is large in \( R \), \( I\text{Tr}(M) \) is large in \( \text{Tr}(M) \). For if \( J \) is an ideal of \( R \) contained in \( \text{Tr}(M) \), then by the purity of \( \text{Tr}(M) \),
\[ I\text{Tr}(M) \cap J = I \cap \text{Tr}(M) \cap J = I \cap J , \]
and so \( I\text{Tr}(M) \cap J = 0 \) if and only if \( J = 0 \), since \( I \) is large. This shows that
\[ (\text{soc} R)\text{Tr}(M) = \text{soc} \text{Tr}(M) . \] and hence \( \text{soc} M = (\text{soc} \text{Tr}(M))M = (\text{soc} R)\text{Tr}(M))M = (\text{soc} R)M \), as required.

(2) \( \text{rad} M \) is the sum of all small submodules \( N \) of \( M \). Since \( N = \text{Tr}(M)N \), \( \text{rad} M = \text{Tr}(M)\text{rad} M \). Theorem 2.3(6) also gives that a submodule of \( M \) is maximal if and only if it is of the form \( IM \), where \( I \) is a maximal ideal of \( R \) contained in \( \text{Tr}(M) \). So \( \text{rad} M = (\text{rad} \text{Tr}(M))M \). Next, \( \text{rad} R \) is the sum of all small ideals of \( R \). Hence \( (\text{rad} R)\text{Tr}(M) = \sum_{I \text{ small}} I\text{Tr}(M) \). \( I\text{Tr}(M) \) is small in \( \text{Tr}(M) \) if \( I \) is small in \( R \). In fact if \( J \) is any ideal contained in \( \text{Tr}(M) \) such that \( R = I\text{Tr}(M) + J \), then \( R = I + J \) and hence \( R = J \), so that \( \text{Tr}(M) = J\text{Tr}(M) = J \). This gives that
\[ (\text{rad} R)\text{Tr}(M) = \text{rad} \text{Tr}(M) , \]
and hence
\[ \text{rad} M = (\text{rad} \text{Tr}(M))M = (\text{rad} R)\text{Tr}(M))M = (\text{rad} R)\text{Tr}(M) . \]

Let \( R \) be a ring and \( M \) an \( R \)-module. A proper submodule \( N \) of \( M \) is called a prime submodule of \( M \) if for all \( r \in R \), \( m \in M \), if \( rm \in N \) then either \( r \in [N : M] \) or \( m \in N \), see [17]. It is clear that if \( P \) is a prime submodule of \( M \) then \( \{ P : M \} \) is a prime ideal of \( R \). If \( M \) is a faithful multiplicative \( R \)-module and \( P \) is a prime ideal of \( R \) such that \( PM \neq M \), then \( PM \) is a prime submodule of \( M \), [10, p. 765]. In this case, \( MP \) is a local \( Rp \)-module, and \( (PM)P \cong P \) is the only maximal submodule of \( MP \). We show that this property of faithful multiplication modules is shared by locally cyclic projective modules. First, we give a lemma.

**Lemma 2.5.** Let \( P \) be a prime ideal of a ring \( R \) and \( M \) a locally cyclic projective \( R \)-module. For all \( a \in R \), \( x \in M \), if \( ax \in PM \) then \( a \in [PM : M] \) or \( x \in PM \).

**Proof.** Suppose that \( a \notin [PM : M] \). Then \( a \notin P \). Let \( K = \{ r \in R | rx \in PM \} \). Suppose that \( K \neq R \). There exists a maximal ideal \( Q \) of \( R \) such that \( K \subseteq Q \). We discuss two cases:

**Case 1:** \( \text{Tr}(M) \subseteq Q \). Then \( \text{Tr}(M)Q = 0_Q \), and hence \( MQ = 0_Q \). \( M \) is \( Q \)-torsion, and hence there exists \( q \in Q \) with \( (1 - q)x = 0 \in PM \). This gives that \( 1 - q \in K \subseteq Q \), a contradiction.
Case 2: $\text{Tr}(M) \not\subseteq Q$. There exists $q' \in Q$ with $1 - q' \in \text{Tr}(M)$. By Theorem 1.3, $\theta(M) = \text{Tr}(M) + \text{ann } M \not\subseteq Q$. Hence there exist $m \in M$ and $q'' \in Q$ such that $1 - q'' \in [Rm : M]$, and hence $(1 - q'')(M) \subseteq Rm$. Let $1 - q = (1 - q')(1 - q'')$. Then $q \in Q$, $1 - q \in \text{Tr}(M)$, and $(1 - q)M \subseteq Rm$. It follows that $(1 - q)\text{ann}(m) \subseteq \text{ann } M = \text{ann } \text{Tr}(M)$, and hence $(1 - q)\text{ann}(m) \subseteq \text{ann } \text{Tr}(M) \cap \text{Tr}(M) = 0$, so that $(1 - q)\text{ann}(m) = 0$. Now since $(1 - q)M \subseteq Rm$, we infer that $(1 - q)x = sm$ and $(1 - q)ax = pm$ for some $s \in R$, $p \in P$. Hence $as - p \in \text{ann}(m)$, and $(as - p)(1 - q) \in (1 - q)\text{ann}(m) = 0$. This implies that $(1 - q)as = (1 - q)p \in P$. Since $P$ is a prime ideal, $(1 - q)a \in P$ or $s \in P$. But if $(1 - q)a \in P$ then $1 - q \in P \subseteq K \subseteq Q$, a contradiction. Thus $s \in P$ and $(1 - q)x = sm \in PM$, which finally gives that $1 - q \in K \subseteq Q$, a contradiction. \[\square\]

The proof of the next result is now immediate.

**Corollary 2.6.** Let $R$ be a ring and $M$ a locally cyclic projective $R$–module. If $P$ is a prime ideal of $R$ such that $PM \neq M$, then $PM$ is a prime submodule of $M$ and $MP$ is a local $R_P$–module with unique maximal submodule $(PM)_P$.

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