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# SOLUTION OF THE HAMMERSTEIN EQUATIONS UNDER NON–MONOTONE PERTURBATIONS

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Abstract. The aim of this note is to study convergence and convergence rates of the regularized solutions for the operator equation of Hammerstein type $x + F_2F_1(x) = f$ in reflexive Banach spaces under the non-monotone perturbations $F_2^h$ and $F_1^h$ of the operators $F_2$ and $F_1$, respectively.

## 1. Introduction

Let $X$ be a real reflexive Banach space and $X^*$ be its dual. For the sake of simplicity, the norms of $X$ and $X^*$ will be denoted by the symbol $\|.\|$. We write $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let $F_1 : X \to X^*$ and $F_2 : X^* \to X$ be nonlinear, monotone, bounded (i.e. image of any bounded subset is bounded) and continuous operators.

Consider the operator equation of Hammerstein type

$$x + F_2F_1(x) = f, \qquad f \in X. \tag{1.1}$$

In the linear case of $F_2$, this equation had been intensively investigated (see [**1**], [**2**], [**6**], [**20**]) because of its importance in the theory of partial differential equations (see [**8**], [**14**]), the theory of optimal control, mechanics and, in particular, in technical problems (see [**19**]). Moreover, several problems in the theory of bifucation $A(x) = \lambda x$, where $A$ is some operator in $X$ and $\lambda$ is a parameter (see [**9**], [**12**], [**21**]) are transformed into equation (1.1) in the form $x + \gamma F_2F_1(x) = 0$ where $\gamma$ is some parameter. In this reason, equation (1.1) has been generalized for the case that both the operators $F_1$ and $F_2$ are nonlinear (see [**3**]–[**5**], [**7**], [**14**]–[**16**], [**18**]).

Let $S_0 \neq \varnothing$, where $S_0$ denotes the set of solutions of (1.1). In [**4**], for (1.1) we studied the regularized operator equation

$$x + F_{2\alpha}F_{1\alpha}(x) = f, \tag{1.2}$$

where $F_{1\alpha} = F_1 + \alpha U_1$, $U_1$ is the standard dual mapping of $X$ (see [**20**]), i.e. $U_1$ is a mapping from $X$ onto $X^*$ satisfying the condition

$$\langle U_1(x), x\rangle = \|U_1(x)\| \, \|x\| = \|x\|^2, \qquad \forall\, x \in X,$$

$F_{2\alpha} = F_2 + \alpha U_2$, $U_2$ is the standard dual mapping of $X^*$, and $\alpha > 0$ is a small parameter. For every $\alpha > 0$ equation (1.2) has a unique solution $x_\alpha$, and the sequence $\{x_\alpha\}$ converges to a solution $x_0$ of (1.1) as $\alpha \to 0$. Moreover, this solution $x_\alpha$, for every fixed $\alpha > 0$, depends continuously on $f$.

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Let $P_n$ be a linear projection from $X$ onto its finite–dimensional subspace $X_n$ such that $X_n \subset X_{n+1}$, $P_nx \to x$ for any $x \in X$, as $n \to \infty$, and $P_n^*$ be the dual of $P_n$ with $\|P_n^*\|$ $(= \|P_n\|) \leq \tilde{c} =$ constant for all $n$. Then, the finite–dimensional problems

$$x + F_{2\alpha n}F_{1\alpha n}(x) \; = \; f_n, \qquad x \in X_n,$$

where $F_{2\alpha n} = P_nF_{2\alpha}P_n^*$, $F_{1\alpha n} = P_n^*F_{1\alpha}P_n$, $f_n = P_nf$, have a unique solution $x_{\alpha n}$, and the sequence $\{x_{\alpha n}\}$ converges to $x_\alpha$, as $n \to \infty$. When $F_2$ is linear, the convergence rates of the sequences $\{x_\alpha\}$ and $\{x_{\alpha n}\}$ are given in our paper [**6**] under that $-1$ is not an eigenvalue of the operator $F_2F_1'(x_0)^*$. This condition has been proposed in [**10**] for investigating the collocation–type method, and it is equivalent to $R\big(I + F_2F_1'(x_0)^*\big) = X$, where $R(F)$ denotes the range of any operator $F$, and $I$ denotes the identity operator in $X$. Recently, it is replaced by a weaker condition requiring only that $R\big(I + F_2'(x_0^*)^*F_1'(x_0)^*\big)$ contains some element of $X$ even if both the operators $F_i$ are nonlinear, where $x_0^* = F_1(x_0)$ (see [**7**]). If instead of $F_1$ and $F_2$ we know their continuous approximations $F_1^h$ and $F_2^h$, respectively, such that

$$\|F_1^h(x) - F_1(x)\| \;\leq\; hg_1(\|x\|), \qquad \forall\, x \in X,$$

$$\|F_2^h(x^*) - F_2(x^*)\| \;\leq\; hg_2(\|x^*\|), \qquad \forall x^* \in X^*,$$

$$g_i(t) \leq M_it + N_i,\; M_i, \quad N_i > 0, \quad t \geq 0, \quad h \geq 0,$$

where $g_i(t)$, $i = 1, 2$, are the real, nondecreasing and continuous functions with $g_i(0) = 0$, $g_i(t) \to +\infty$ as $t \to +\infty$, then the regularized solutions are constructed by the regularized equation (see [**5**])

$$x + F_{2\alpha}^hF_{1\alpha}^h(x) \; = \; f, \tag{1.3}$$

where $F_{i\alpha}^h = F_i^h + \alpha U_i$, for the case when $F_i^h$, $i = 1, 2$, are monotone. If one of $F_i^h$ is not monotone, equation (1.3), perhaps, does not have solution. So, we have in some way to determine an element in $X$ depending on $h$, $\alpha$ as regularized solution for (1.1). In Section 2, following the idea in [**11**] this element is defined, and its convergence and convergence rates as $h, \alpha \to 0$, are considered in combination with finite–dimensional approximations of the space $X$.

Below, by "$a \sim b$" we mean "$a = O(b)$ and $b = O(a)$", and the symbols $\rightharpoonup$ and $\to$ denote weak convergence and convergence in norm, respectively.

## 2. Main Results

**Definition 2.1.** An element $x_\omega \in X$ ($\omega$ depends on $h$, $\alpha$ and $\varepsilon$) is called a regularized solution of (1.1) (see [**11**] for similar issues), if there exists an element $x_\omega^* \in X^*$ such that

$$\langle F_{1\alpha}^h(x_\omega) - x_\omega^*, x - x_\omega\rangle \;\geq\; -\varepsilon g_1(\|x_\omega\|)\|x - x_\omega\|, \qquad \forall\, x \in X, \tag{2.1}$$

$$\langle F_{2\alpha}^h(x_\omega^*) + x_\omega - f, x^* - x_\omega^*\rangle \;\geq\; -\varepsilon g_2(\|x_\omega^*\|)\|x^* - x_\omega^*\|, \qquad \forall\, x^* \in X^*, \tag{2.2}$$

$$\alpha > 0, \qquad \varepsilon \geq h,$$

where $F_{i\alpha}^h$ are defined as above for the case of monotonicity of $F_i^h$. And, we say that (2.1) – (2.2) have a solution $[x_\omega, x_\omega^*]$. We shall show that this system of variational inequalities has, in fact, solution. Indeed, by taking $x_\omega = x_\alpha$ and $x^*_\omega = F_{1\alpha}(x_\alpha)$, we have

$$\begin{aligned}\langle F^h_{1\alpha}(x_\alpha) - x^*_\omega, x - x_\alpha\rangle &= \langle F^h_1(x_\alpha) - F_1(x_\alpha), x - x_\alpha\rangle\\ &\geq -hg_1(\|x_\alpha\|)\|x - x_\alpha\|\\ &\geq -\varepsilon g_1(\|x_\alpha\|)\|x - x_\alpha\|,\end{aligned}$$

i.e. $x_\alpha$ and $x^*_\alpha$ satisfy inequality (2.1). On the other hand, from (1.2) it follows $x_\alpha - f = -F_{2\alpha}F_{1\alpha}(x_\alpha) = -F_{2\alpha}(x^*_\omega)$. Therefore,

$$\begin{aligned}\langle F^h_{2\alpha}(x^*_\omega) + x_\alpha - f, x^* - x^*_\omega\rangle &= \langle F^h_2(x^*_\omega) - F_2(x^*_\omega), x^* - x^*_\omega\rangle\\ &\geq -hg_2(\|x^*_\omega\|)\|x^* - x^*_\omega\|\\ &\geq -\varepsilon g_2(\|x^*_\omega\|)\|x^* - x^*_\omega\|.\end{aligned}$$

Thus, (2.1) – (2.2) have solution. Analogously, the system of following inequalities

$$\langle F^{hn}_{1\alpha}(x_{\omega n}) - x^*_{\omega n}, x_n - x_{\omega n}\rangle \geq -\varepsilon g_1(\|x_{\omega n}\|)\|x_n - x_{\omega n}\|, \quad \forall\, x_n \in X_n, \quad (2.3)$$

$$\langle F^{hn}_{2\alpha}(x^*_{\omega n}) + x_{\omega n} - f_n, x^*_n - x^*_{\omega n}\rangle \geq -\varepsilon g_2(\|x^*_{\omega n}\|)\|x^*_n - x^*_{\omega n}\|, \quad (2.4)$$

$$\forall\, x^*_n \in X^*_n, \qquad \varepsilon \geq h,$$

where $F^{hn}_{1\alpha} = P^*_n F^h_{1\alpha} P_n$, $F^{hn}_{2\alpha} = P_n F^h_{2\alpha} P^*_n$, also has solution $[x_{\omega n}, x^*_{\omega n}]$.

We have the results.

**Theorem 2.2.** *If $\varepsilon/\alpha \to 0$, then the set $\{x_\omega\}$ is bounded, and there exists a subsequence of $\{x_\omega\}$ converging to a solution of* (1.1). *Moreover, all its limit points are solutions of* (1.1).

**Proof.** From the monotone property of $F_i$, $i = 1, 2$, and (2.1) – (2.2) it implies that

$$\begin{aligned}(\varepsilon + h)g_1(\|x_\omega\|)\|x - x_\omega\| &+ \alpha\langle U_1(x), x - x_\omega\rangle + \langle F_1(x) - x^*_\omega, x - x_\omega\rangle\\ &+ (\varepsilon + h)g_2(\|x^*_\omega\|)\|x^* - x^*_\omega\| + \alpha\langle U_2(x^*), x^* - x^*_\omega\rangle\\ &+ \langle F_2(x^*) + x_\omega - f, x^* - x^*_\omega\rangle \geq 0, \quad \forall\, x \in X,\ \forall\, x^* \in X^*.\end{aligned} \quad (2.5)$$

When $x \in S_0$ this inequality is written in the form

$$\begin{aligned}\frac{2\varepsilon}{\alpha}g_1(\|x_\omega\|)\|x - x_\omega\| &+ \frac{2\varepsilon}{\alpha}g_2(\|x^*_\omega\|)\|x^* - x^*_\omega\| + \langle U_1(x), x - x_\omega\rangle\\ &+ \langle U_2(x^*), x^* - x^*_\omega\rangle \geq m_1\|x - x_\omega\|^{s_1}\\ &+ m_2\|x^* - x^*_\omega\|^{s_2}, \qquad \forall\, x \in S_0,\end{aligned} \quad (2.6)$$

where $x^* = F_1(x)$. Therefore, since $\varepsilon/\alpha \to 0$, $s_i \geq 2$, and $g_i(t) \leq M_i t + N_i$, we have the sets $\{x_\omega\}$ and $\{x^*_\omega\}$ are bounded. Without loss of generality, assume that

$$x_\omega \rightharpoonup x_1,\ x^*_\omega \rightharpoonup y^*_1, \quad \text{as} \quad \varepsilon/\alpha \to 0.$$

After passing $\varepsilon$, $\alpha \to 0$, in (2.5) we obtain

$$\langle F_1(x), x - x_1\rangle + \langle F_2(x^*) - f, x^* - y_1^*\rangle \geq \langle y_1^*, x\rangle + \langle x_1, -x^*\rangle, \qquad \forall\, x \in X,\ x^* \in X^*.$$

Consequently,

$$\langle F_1(x) - y_1^*, x - x_1\rangle + \langle F_2(x^*) + x_1 - f, x^* - y_1^*\rangle \geq 0, \qquad \forall\, x \in X,\ x^* \in X^*.$$

This inequality is equivalent to the system of following variational inequalities

$$\begin{aligned} \langle F_1(x) - y_1^*, x - x_1\rangle &\geq 0, \qquad \forall\, x \in X,\\ \langle F_2(x^*) + x_1 - f, x^* - y_1^*\rangle &\geq 0, \qquad \forall\, x^* \in X^*. \end{aligned}$$

Thus,

$$F_1(x_1) - y_1^* = 0, \quad F_2(y_1^*) + x_1 - f = 0$$

(see [**20**]). It means that $x_1$ is a solution of (1.1). Replacing $x$ by $x_1$ and $x^*$ by $y_1^*$ in (2.6) we can conclude that $x_\omega \to x_1$ (also $x_\omega^* \to y_1^*$), because $\varepsilon/\alpha$, $\alpha \to 0$. □

**Remark 2.3.** If equation (1.1) has a unique solution $x_0$, then $\{x_\omega\}$ converges to $x_0$.

Assume that the dual mappings $U_i$ of the spaces $X$ and $X^*$ satisfy the conditions:

$$\begin{aligned} \langle U_i(y_1^i) - U_i(y_2^i), y_1^i - y_2^i\rangle &\geq m_i\|y_1^i - y_2^i\|^{s_i}, \qquad m_i > 0,\ s_i \geq 2,\\ \|U_i(y_1^i) - U_i(y_2^i)\| &\leq c_i(R_i)\|y_1^i - y_2^i\|^{\nu_i}, \qquad 0 < \nu_i \leq 1, \end{aligned}$$

where $y_1^i$, $y_2^i \in X$ or $X^*$ on dependence of $i = 1$ or 2, respectively, and $c_i(R_i)$, $R_i > 0$, are the positive increasing functions on $R_i = \max\{\|y_1^i\|, \|y_2^i\|\}$ (see [**17**]).

Without loss of generality, suppose that $[x_\omega, x_\omega^*]$ is a solution of (2.1) – (2.2), and $x_\omega \to x_0 \in S_0$ as $\alpha$, $\varepsilon$ $(\varepsilon > h) \to 0$ $(\varepsilon/\alpha \to 0)$. The value $\|x_\omega - x_0\|$ is estimated by Theorem 2.4.

**Theorem 2.4.** *Let the following conditions hold:*

(i) *$F_1$ is Fréchet differentiable at some neighbourhood $\mathcal{U}_0$ of $x_0$ $s_1 - 1$–times if $s_1 = [s_1]$, the integer part of $s_1$, $[s_1]$–times if $s_1 \neq [s_1]$, and $F_2$ is Fréchet differentiable at some neighbourhood $\mathcal{V}_0$ of $x_0^*$ $s_2 - 1$–times, if $s_2 = [s_2]$, $[s_2]$–times if $s_2 \neq [s_2]$,*

(ii) *there exists a constant $L > 0$ such that*

$$\begin{aligned} \|F_1^{(k)}(x_0) - F_1^{(k)}(y)\| &\leq L\|x_0 - y\|, \qquad \forall\, y \in \mathcal{U}_0,\\ \|F_2^{(k)}(x_0^*) - F_2^{(k)}(y^*)\| &\leq L\|x_0^* - y^*\|, \qquad \forall\, y^* \in \mathcal{V}_0, \end{aligned}$$

*for $F_i^{(k)}$: $k = s_i - 1$ if $s_i = [s_i]$, $k = [s_i]$ if $s_i \neq [s_i]$, and if $[s_i] \geq 3$, then $F_1^{(2)}(x_0) = \ldots = F_1^{(k)}(x_0) = 0$, and $F_2^{(2)}(x_0^*) = \ldots = F_1^{(k)}(x_0^*) = 0$,*

(iii) *there exists an element $x^1 \in X$ such that*

$$\big(I + F_2'(x_0^*)^* F_1'(x_0)^*\big)x^1 = F_2'(x_0^*)^* U_1(x_0) - U_2(x_0^*),$$

*if $s_1 = [s_1]$ then $L\|x^1\| < m_1 s_1!$, and if $s_2 = [s_2]$ then $L\|F_1'(x_0)^* x^1 - U_1(x_0)\| < m_2 s_2!$.*

*Then, if* $\alpha$ *is chosen such that* $\alpha \sim \varepsilon^{\rho}$, $0 < \rho < 1$, *we have*

$$\begin{aligned} \|x_\omega - x_0\| &= O(\varepsilon^{\theta}), \\ \theta &= \min\left\{\theta_1, \frac{1-\rho+\theta_2}{s_1-1}\right\}, \\ \theta_i &= \min\left\{\frac{1-\rho}{s_i}, \frac{\rho}{s_i}\right\}, \qquad i = 1,2. \end{aligned}$$

**Proof.** Put

$$A = m_1\|x_\omega - x_0\|^{s_1} + m_2\|x^*_\omega - x^*_0\|^{s_2}.$$

Basing on the property (2.7) of $U_i$ and (2.1) – (2.2) it is easy to see

$$A \le \langle U_1(x_0), x_0 - x_\omega\rangle + \langle U_2(x^*_0), x^*_0 - x^*_\omega\rangle + \frac{1}{\alpha}\big[\varepsilon\big(g_1(\|x_\omega\|)\|x_\omega - x_0\|$$
$$+ \; g_2(\|x^*_\omega\|)\|x^*_\omega - x^*_0\|\big) + \langle F^h_1(x_\omega) - x^*_\omega, x_0 - x_\omega\rangle + \langle F^h_2(x^*_\omega) + x_\omega - f, x^*_0 - x^*_\omega\rangle\big].$$

Put $x^2 = U_1(x_0) - F'_1(x_0)^* x^1$. From condition (iii) it follows that $x^1$ and $x^2$ ($\in X^*$) satisfy the system of following equations

$$\begin{aligned} F'_1(x_0)^* x^1 + x^2 &= U_1(x_0), \\ F'_2(x^*_0)^* x^2 - x^1 &= U_2(x^*_0). \end{aligned}$$

Since

$$\begin{aligned} \langle F^h_1(x_\omega) - x^*_\omega, x_0 - x_\omega\rangle &= \langle F^h_1(x_\omega) - F_1(x_\omega), x_0 - x_\omega\rangle \\ &\quad + \langle F_1(x_\omega) - F_1(x_0) + x^*_0 - x^*_\omega, x_0 - x_\omega\rangle \\ &\le hg_1(\|x_\omega\|)\|x_\omega - x_0\| + \langle x^*_0 - x^*_\omega, x_0 - x_\omega\rangle, \\ \langle F^h_2(x^*_\omega) + x_\omega - f, x^*_0 - x^*_\omega\rangle &= \langle F^h_2(x^*_\omega) - F_2(x^*_\omega), x^*_0 - x^*_\omega\rangle \\ &\quad + \langle F_2(x^*_\omega) - F_2(x^*_0) - x_0 + x_\omega, x^*_0 - x^*_\omega\rangle \\ &\le hg_2(\|x^*_\omega\|)\|x^*_\omega - x^*_0\| - \langle x_0 - x_\omega, x^*_0 - x^*_\omega\rangle, \end{aligned}$$

we obtain

$$\begin{aligned} A &\le \frac{h+\varepsilon}{\alpha}\big(g_1(\|x_\omega\|)\|x_\omega - x_0\| + g_2(\|x^*_\omega\|)\|x^*_\omega - x^*_0\|\big) \\ &\quad + \langle U_1(x_0), x_0 - x_\omega\rangle + \langle U_2(x^*_0), x^*_0 - x^*_\omega\rangle \\ &\le \frac{2\varepsilon}{\alpha}\big(g_1(\|x_\omega\|)\|x_\omega - x_0\| + g_2(\|x^*_\omega\|)\|x^*_\omega - x^*_0\|\big) \\ &\quad + \langle x^1, F'_1(x_0)(x_0 - x_\omega)\rangle + \langle x^2, x_0 - x_\omega\rangle \\ &\quad + \langle x^2, F'_2(x^*_0)(x^*_0 - x^*_\omega)\rangle - \langle x^1, x^*_0 - x^*_\omega\rangle. \end{aligned} \tag{2.7}$$

First, consider the case $s_i = [s_i]$, $i = 1,2$. As

$$\begin{aligned} F'_1(x_0)(x_0 - x_\omega) &= F_1(x_0) - F_1(x_\omega) + r_\omega, \\ F'_2(x^*_0)(x^*_0 - x^*_\omega) &= F_1(x^*_0) - F_1(x^*_\omega) + \widetilde{r}_\omega, \end{aligned}$$

$$\|r_\omega\| \le \frac{L}{s_1!}\|x_\omega - x_0\|^{s_1}, \qquad \|\widetilde{r}_\omega\| \le \frac{L}{s_2!}\|x^*_\omega - x^*_0\|^{s_2},$$

inequality (2.7) has then the form

$$\begin{aligned}
A \;\le\;& \frac{2\varepsilon}{\alpha}\big(g_1(\|x_\omega\|)\|x_\omega - x_0\| + g_2(\|x^*_\omega\|)\|x^*_\omega - x^*_0\|\big)\\
&+ \langle x^1, F_1(x_0) - F_1(x_\omega)\rangle + \frac{L\|x^1\|}{s_1!}\,\|x_\omega - x_0\|^{s_1} + \langle x^2, F_2(x^*_0) - F_2(x^*_\omega)\rangle\\
&+ \frac{L\|x^2\|}{s_2!}\,\|x^*_\omega - x^*_0\|^{s_2} + \langle x^2, x_0 - x_\omega\rangle - \langle x^1, x^*_0 - x^*_\omega\rangle\\
\le\;& \frac{2\varepsilon}{\alpha}\big(g_1(\|x_\omega\|)\|x_\omega - x_0\| + g_2(\|x^*_\omega\|)\|x^*_\omega - x^*_0\|\big)\\
&+ hg_1(\|x_\omega\|)\|x^1\| + \langle x_1, F_1(x_0) - F_1^h(x_\omega)\rangle + \frac{L\|x^1\|}{s_1!}\,\|x_\omega - x_0\|^{s_1}\\
&+ hg_2(\|x^*_\omega\|)\|x^2\| + \langle x^2, F_2(x^*_0) - F_2^h(x^*_\omega)\rangle + \frac{L\|x^2\|}{s_2!}\,\|x^*_\omega - x^*_0\|^{s_2}\\
&+ \langle x^2, x_0 - x_\omega\rangle - \langle x^1, x^*_0 - x^*_\omega\rangle. \qquad (2.8)
\end{aligned}$$

Replacing $x$ in (2.1) by $x_\omega + x^1$ and $x^*$ in (2.2) by $x^*_\omega + x^2$, we can esimate

$$\begin{aligned}
\langle x^1, F_1(x_0) - F_1^h(x_\omega)\rangle \;=\;& \langle x^1, x^*_0 - x^*_\omega\rangle + \langle x^1, x^*_\omega - F_1^h(x_\omega)\rangle\\
\le\;& \langle x^1, x^*_0 - x^*_\omega\rangle + \varepsilon g_1(\|x_\omega\|)\|x^1\| + \alpha\langle x^1, U_1(x_\omega)\rangle\\
\le\;& \langle x^1, x^*_0 - x^*_\omega\rangle + \varepsilon g_1(\|x_\omega\|)\|x^1\| + \alpha\|x^1\|\,\|x_\omega\|,\\
\langle x^2, F_2(x^*_0) - F_2^h(x^*_\omega)\rangle \;=\;& \langle x^2, -x_0 + x_\omega\rangle + \langle x^2, -F_2^h(x^*_\omega) + f - x_\omega\rangle\\
\le\;& -\langle x^2, x_0 - x_\omega\rangle + \varepsilon g_2(\|x^*_\omega\|)\|x^2\| + \alpha\langle x^2, U_2(x^*_\omega)\rangle\\
\le\;& -\langle x^2, x_0 - x_\omega\rangle + \varepsilon g_2(\|x^*_\omega\|)\|x^2\| + \alpha\|x^2\|\,\|x^*_\omega\|.
\end{aligned}$$

Hence, from (2.8) it implies that

$$\begin{aligned}
A \;\le\;& \frac{2\varepsilon}{\alpha}\big(g_1(\|x_\omega\|)\|x_\omega - x_0\| + g_2(\|x^*_\omega\|)\|x^*_\omega - x^*_0\|\big)\\
&+ 2\varepsilon\big(g_1(\|x_\omega\|)\|x^1\| + g_2(\|x^*_\omega\|)\|x^2\|\big) + \alpha\big(\|x^1\|\,\|x_\omega\| + \|x^2\|\,\|x^*_\omega\|\big)\\
&+ \frac{L\|x^1\|}{s_1!}\,\|x_\omega - x_0\|^{s_1} + \frac{L\|x^2\|}{s_2!}\,\|x^*_\omega - x^*_0\|^{s_2}. \qquad (2.9)
\end{aligned}$$

In order to estimate convergence rates of the sequence $\{x_\omega\}$ we, first, find an estimate for $\|x_\omega^* - x_0^*\|$. From (2.9), the boundness of $g_i$, $\{x_\omega\}$, $\{x_\omega^*\}$ we get

$$m_2\left(1-\frac{L\|x^2\|}{m_2 s_2!}\right)\|x_\omega^*-x_0^*\|^{s_2} \le m_1\left(1-\frac{L\|x^1\|}{m_1 s_1!}\right)\|x_\omega-x_0\|^{s_1} + m_2\left(1-\frac{L\|x^2\|}{m_2 s_2!}\right)\|x_\omega^*-x_0^*\|^{s_2} \le O\left(\frac{\varepsilon}{\alpha}\right)\|x_\omega^*-x_0^*\| + O\left(\frac{\varepsilon}{\alpha}+\varepsilon+\alpha\right). \tag{2.10}$$

Applying the relation

$$a,\ b,\ c>0,\ p>q>0,\ a^p \le ba^q + c \Longrightarrow a^p = O(b^{p/(p-q)}+c)$$

in [**13**] to the inequality (2.10) we obtain

$$\|x_\omega^*-x_0^*\| \ = \ O\big(\varepsilon^{\theta_2}\big).$$

Now, from (2.9) and the last equality we also obtain

$$m_1\left(1-\frac{L\|x^1\|}{m_1 s_1!}\right)\|x_\omega-x_0\|^{s_1} \le O\left(\frac{\varepsilon}{\alpha}\right)\|x_\omega-x_0\| + O\left(\frac{\varepsilon^{1+\theta_2}}{\alpha}+\varepsilon+\alpha\right).$$

Again, applying the above relation to the last inequality we have got

$$\|x_\omega - x_0\| \ = \ O(\varepsilon^{\theta}).$$

If $s_i \neq [s_i]$ for one or both the two numbers $s_i$, for example $s_2 \neq [s_2]$, then

$$\|\widetilde{r}_\omega\| \ \le \ \frac{L}{([s_2]+1)!}\|x_\omega^*-x_0^*\|^{[s_2]+1}$$

and the left–hand side of (2.10) will be replaced by

$$m_2\left(1-\frac{L\|x^2\|}{m_2([s_2]+1)!}\ \|x_\omega^*-x_0^*\|^{[s_2]+1-s_2}\right)\|x_\omega^*-x_0^*\|^{s_2}.$$

Because $\|x_\omega^*-x_0^*\| \to 0$, and $[s_2]+1-s_2>0$

$$1-\frac{L\|x^2\|}{([s_2]+1)!}\ \|x_\omega^*-x_0^*\|^{[s_2]+1-s_2} \ \ge \ 1/2$$

for sufficiently small $\varepsilon$, $\alpha$. The case $s_1 \neq [s_1]$ and both of the two numbers $s_i$ are not integer is considered analogously. This remark completes the proof of the theorem. □

Now, we establish convergence and convergence rates for the sequence $\{x_{\omega n}\}$, as $h,\ \alpha,\ \varepsilon \to 0$, $n \to +\infty$ by estimating the value $\|x_{\omega n} - x_0\|$.

**Theorem 2.5.** *Assume that the conditions of Theorem* 2.4 *hold, and* $\alpha$ *is chosen such that* $\alpha \sim (\varepsilon+\gamma_n)^\rho$, $1<\rho<1$, *where*

$$\gamma_n \ = \ \max\{\|(I-P_n)x_0\|, \|(I-P_n)f\|, \|(I-P_n)x^1\|, \|(I^*-P_n^*)x_0^*\|, \|(I^*-P_n^*)x^2\|\},$$

*and $I^*$ denotes the identity operator in $X^*$. Then*

$$\begin{aligned}
\|x_{\omega n} - x_0\| &= O(\varepsilon^{\theta_1} + \gamma_n^{\theta_2}),\\
\theta_1 = \tau_1,\ \theta_2 &= \min\left\{\tau_2, \frac{\nu_1}{s_1 - 1}\,,\ \frac{\nu_2}{s_1}\right\},\\
\tau_i &= \min\left\{\frac{1-\rho}{s_i}\,,\ \frac{\rho}{s_i}\right\}, \qquad i = 1, 2.
\end{aligned}$$

**Proof.** Put

$$B \ = \ m_1\|x_{\omega n} - x_n\|^{s_1} + m_2\|x^*_{\omega n} - x^*_n\|^{s_2},$$

with $x \in S_0$, $x^* = F_1(x)$, $x_n = P_n x$, and $x^*_n = P^*_n x^*$. From (2.3) – (2.4) it follows

$$\begin{aligned}
B \le\ & \langle U_1(x_n), x_n - x_{\omega n}\rangle + \langle U_2(x^*_n), x^*_n - x^*_{\omega n}\rangle\\
&+ \frac{1}{\alpha}\big[\varepsilon\big(g_1(\|x_\omega\|)\|x_{\omega n} - x_n\| + g_2(\|x^*_\omega\|)\|x^*_{\omega n} - x^*_n\|\big)\\
&+ \langle F_1^{hn}(x_{\omega n}) - x^*_{\omega n}, x_n - x_{\omega n}\rangle + \langle F_2^{hn}(x^*_{\omega n}) + x_{\omega n} - f_n, x^*_n - x^*_{\omega n}\rangle\big]\\
\le\ & \langle U_1(x_n), x_n - x_{\omega n}\rangle + \langle U_2(x^*_n), x^*_n - x^*_{\omega n}\rangle\\
&+ \frac{1}{\alpha}\big[\varepsilon\big(g_1(\|x_\omega\|)\|x_{\omega n} - x_n\| + g_2(\|x^*_\omega\|)\|x^*_{\omega n} - x^*_n\|\big)\\
&+ \langle F_1^h(x_{\omega n}) - F_1(x_{\omega n}) + F_1(x_{\omega n}) - F_1(x_n) + F_1(x_n) - x^*_{\omega n}, x_n - x_{\omega n}\rangle\\
&+ \langle F_2^h(x^*_{\omega n}) - F_2(x^*_{\omega n}) + F_2(x^*_{\omega n}) - F_2(x^*_n) + F_2(x^*_n) + x_{\omega n} - f_n, x^*_n - x^*_{\omega n}\rangle\big].
\end{aligned}$$

Because of the monotonicity of $F_i$, $i = 1, 2$, the approximative property of $F_i^h$ and

$$\begin{aligned}
\langle F_1(x_n) - x^*_{\omega n}, x_n - x_{\omega n}\rangle &= \langle P^*_n F_1(x_n) - x^*_n, x_n - x_{\omega n}\rangle + \langle x^*_n - x^*_{\omega n}, x_n - x_{\omega n}\rangle\\
&= \langle F_1(x_n) - F_1(x), x_n - x_{\omega n}\rangle\\
&\quad + \langle x^*_n - x^*_{\omega n}, x_n - x_{\omega n}\rangle, \langle F_2(x^*_n) + x_{\omega n} - f_n, x^*_n - x^*_{\omega n}\rangle\\
&= \langle P_n\big(F_2(x^*_n) + x - f\big), x^*_n - x^*_{\omega n}\rangle\\
&\quad + \langle -x_n + x_{\omega n}, x^*_n - x^*_{\omega n}\rangle\\
&= \langle F_2(x^*_n) - F_2(x^*), x^*_n - x^*_{\omega n}\rangle - \langle x_n - x_{\omega n}, x^*_n - x^*_{\omega n}\rangle
\end{aligned}$$

we have

$$\begin{aligned}
B \ \le\ & \langle U_1(x_n), x_n - x_{\omega n}\rangle + \langle U_2(x^*_n), x^*_n - x^*_{\omega n}\rangle\\
&+ \frac{1}{\alpha}\big[2\varepsilon g_1(\|x_{\omega n}\|)\|F_1(x_n) - F_1(x)\|\ \|x_{\omega n} - x_n\|\\
&+ 2\varepsilon g_2(\|x^*_{\omega n}\|)\|F_2(x^*_n) - F_2(x^*)\|\ \|x^*_{\omega n} - x^*_n\|\big]. \qquad (2.11)
\end{aligned}$$

Thus, the sequences $\{x_{\omega n}\}$ and $\{x^*_{\omega n}\}$ are bounded as $\varepsilon/\alpha \to 0$, $n \to +\infty$. In the case $s_i = [s_i]$ we can write

$$\begin{aligned} F_1(x_n) - F_1(x) &= F_1'(x)(x_n - x) + r_n, \\ F_2(x_n^*) - F_2(x^*) &= F_2'(x^*)(x_n^* - x^*) + \widetilde{r}_n, \end{aligned}$$

where

$$\|r_n\| \le \frac{L}{s_1!}\|(P_n - I)x\|^{s_1}, \qquad \|\widetilde{r}_n\| \le \frac{L}{s_2!}\|(P_n^* - I^*)x^*\|^{s_2}.$$

On the other hand, from (2.7) with $x = x_0$ ($x_{0n} = P_n x_0$) it implies that

$$\langle U_1(x_{0n}), x_{0n} - x_{\omega n}\rangle \le c_1(R_1)\gamma_n^{\nu_1}\|x_{0n} - x_{\omega n}\| + \langle U_1(x_0), x_{0n} - x_{\omega n}\rangle.$$

The second term on the right-hand side of this inequality is estimated as follows:

$$\begin{aligned} \langle U_1(x_0), x_{0n} - x_{\omega n}\rangle &= \langle U_1(x_0), x_{0n} - x_0\rangle + \langle U_1(x_0), x_0 - x_{\omega n}\rangle \\ &\le O(\gamma_n) + \langle x^1, F_1(x_0) - F_1(x_{\omega n})\rangle + \langle x^2, x_0 - x_{\omega n}\rangle \\ &\quad + \frac{L\|x^1\|}{s_1!}\|x_0 - x_{\omega n}\|^{s_1}. \end{aligned}$$

In the similar way, we also have

$$\langle U_2(x^*_{0n}), x^*_{0n} - x^*_{\omega n}\rangle \le c_2(R_2)\gamma_n^{\nu_2}\|x^*_{0n} - x^*_{\omega n}\| + \langle U_2(x^*_0), x^*_{0n} - x^*_{\omega n}\rangle$$

with the estimate

$$\begin{aligned} \langle U_2(x^*_0), x^*_{0n} - x^*_{\omega n}\rangle &\le O(\gamma_n) + \langle x^2, F_2(x^*_0) - F_2(x^*_{\omega n})\rangle - \langle x^1, x^*_0 - x^*_{\omega n}\rangle \\ &\quad + \frac{L\|x^2\|}{s_2!}\|x^*_0 - x^*_{\omega n}\|^{s_2}, \qquad x^*_{0n} = P_n^* x^*_0 = P_n^* F_1(x_0). \end{aligned}$$

As $\{x_{\omega n}\}$ and $\{x^*_{\omega n}\}$ are bounded,

$$\begin{aligned} \langle x^1, F_1(x_0) - F_1(x_{\omega n})\rangle &= \langle x^1, x^*_0 - x^*_{\omega n}\rangle + \langle x^1, x^*_{\omega n} - F_1^{hn}(x_{\omega n})\rangle \\ &\quad - \langle (I - P_n)x^1, F_1^h(x_{\omega n})\rangle + \langle x^1, F_1^h(x_{\omega n}) - F_1(x_{\omega n})\rangle \\ &\le \gamma_n\|F_1^h(x_{\omega n})\| + \langle x^1, x^*_0 - x^*_{\omega n}\rangle + \|x^1\|hg_1(\|x_{\omega n}\|) \\ &\quad + \widetilde{c}\|x^1\|\varepsilon g_1(\|x_{\omega n}\|) \\ &\quad + \widetilde{c}\|x^1\|\alpha\|x^1 - x_{\omega n}\|, \langle x^2, F_2(x^*_0) - F_2(x^*_{\omega n})\rangle \\ &= \langle x^2, -x_0 + x_{\omega n}\rangle + \langle x^2, f_n - x_{\omega n} - F_2^{hn}(x^*_{\omega n})\rangle \\ &\quad + \langle x^2, f - f_n\rangle - \langle (I_n^* - P_n^*)x^2, F_2^h(x^*_{\omega n})\rangle \\ &\quad + \langle x^2, F_2^h(x^*_{\omega n}) - F_2(x^*_{\omega n})\rangle \\ &\le \gamma_n\|F_2^h(\|x^*_{\omega n})\| - \langle x^2, x_0 - x_{\omega n}\rangle + \|x^2\|hg_2(\|x^*_{\omega n}\|) \\ &\quad + \widetilde{c}\|x^2\|\varepsilon g_2(\|x^*_{\omega n}\|) + \widetilde{c}\|x^2\|\alpha\|x^2 - x^*_{\omega n}\|. \end{aligned}$$

Hence, from

$$\begin{aligned} \|x_0 - x_{\omega n}\|^{s_1} &\le O(\gamma_n) + \|x_{0n} - x_{\omega n}\|^{s_1}, \\ \|x^*_0 - x^*_{\omega n}\|^{s_2} &\le O(\gamma_n) + \|x^*_{0n} - x^*_{\omega n}\|^{s_2}, \end{aligned}$$

the boundeness of $g_i$ and (2.11) it follows

$$\begin{aligned} m_1\left(1-\frac{L\|x^1\|}{m_1 s_1!}\right)\|x_{\omega n}-x_{0n}\|^{s_1} &\le m_1\left(1-\frac{L\|x^1\|}{m_1 s_1!}\right)\|x_{\omega n}-x_{0n}\|^{s_1}\\ &\quad + m_2\left(1-\frac{L\|x^2\|}{m_2 s_2!}\right)\|x^*_{\omega n}-x^*_{0n}\|^{s_2}\\ &\le O\big((\varepsilon+\gamma_n)^{1-\rho}+\gamma_n^{\nu_1}\big)\|x_{\omega n}-x_{0n}\|\\ &\quad + O\big((\varepsilon+\gamma_n)^{1-\rho}+\gamma_n^{\nu_2}\big)\|x^*_{\omega n}-x^*_{0n}\|\\ &\quad + O\big(\varepsilon+\gamma_n+(\varepsilon+\gamma_n)^{\rho}\big)\\ &\le O\big((\varepsilon+\gamma_n)^{1-\rho}+\gamma_n^{\nu_1}\big)\|x_{\omega n}-x_{0n}\|\\ &\quad + O\big((\varepsilon+\gamma_n)^{1-\rho}+\gamma_n^{\nu_2}+(\varepsilon+\gamma_n)^{\rho}\big). \end{aligned}$$

Applying the relation in [**13**], again, to the last inequality we obtain

$$\|x_{\omega n}-x_{0n}\| = O(\varepsilon^{\theta_1}+\gamma_n^{\theta_2}).$$

Therefore,

$$\|x_{\omega n}-x_0\| = O(\varepsilon^{\theta_1}+\gamma_n^{\theta_2}).$$

The case $s_i \neq [s_i]$ is considered in the similar way as in the proof of Theorem 2.2. □

**Remarks.**

1. If $S_0$ contains more then one element, then $F_1$ and $F_2$ are affine on the sets $S_0$ and $F_1(S_0)$, respectively (see [**14**]). Therefore, the condition $F_1^{(2)}(x) = \ldots = F_1^{(k)}(x) = 0$, $x \in S_0$, and $F_2^{(2)}(x^*) = \ldots = F_2^{(k)}(x^*)$, $x \in F_1(S_0)$ is automatically satisfied. Moreover, $F_1'(x)$ and $F_2'(x^*)$ do not depend on $x$ and $x^*$, respectively. Hence, condition (iii) of Theorem 2.2, in fact, is an existence condition of the solution of a system of two linear operator equations.
2. When $X$ is the spaces of type $L_p(\Omega)$ or $W_p(\Omega)$, $1 < p < +\infty$ : If $p = 2$ $X$ is a Hilbert space and $U_i = I$, $s_i = 2$, $m_i = 1$, $\nu_i = 1$ and $c(R_i) \equiv 1$, and if $1 < p < 2$ we have $s_1 = 2$, $m_1 = p-1$, $c(R_1) = p2^{2p-1}e^pL^{p-1}$, $e = \max\{2^p, 2R_1\}$, $1 < L < 3.18$, $\nu_1 = p-1$, and $s_2 = q$, $m_2 = 2^{2-q}/q$, $c(R_2) = 2^qR_2^{q-2}\{q[q-1+max\{R_2, L\}]\}^{-1}$, $\nu_2 = 1$, $p^{-1}+q^{-1} = 1$. The case $p > 2$ is analogously considered (see [**17**]).

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