Abstract. Two weakenings of the anti-Specker property—a principle of some significance in constructive reverse mathematics—are introduced, examined, and in one case applied, within Bishop-style constructive mathematics. The weaker of these anti-Specker properties is shown to be equivalent to a very weak version of Brouwer’s fan theorem. This leads to a study of antitheses of various types of fan theorem—in particular, to new proofs of Diener’s theorem on the equivalence of some of these antitheses. In addition, the antithesis of the positivity principle for uniformly continuous functions on $[0,1]$ is shown to be equivalent to that of the fan theorem for detachable bars. Finally, a positivity principle for pointwise continuous functions is examined, partly in order to provide a neat application of the stronger of the two anti-Specker properties introduced early in the paper.

1. Introduction

A sequence $(z_n)_{n \geq 1}$ is said to be **eventually bounded away from the point** $x \in \mathbb{R}$ if there exist $N \in \mathbb{N}^+$ and $\delta > 0$ such that $|x - z_n| > \delta$ for all $n \geq N$. The **anti-Specker property** (for $[0,1]$),

\[
\text{AS}_{[0,1]}: \quad \text{For each sequence } z \text{ in } [0,1] \cup \{2\}, \text{ if } z \text{ is eventually bounded away from each point of } [0,1], \text{ then } z_n = 2 \text{ eventually,}
\]

has made several appearances in constructive reverse mathematics\(^1\) since its introduction in \[3\] (cf. also Proposition 7 of \[10\]). We shall examine some weakenings of $\text{AS}_{[0,1]}$ within the framework of Bishop-style constructive mathematics, BISH, which is, roughly, mathematics with intuitionistic logic and an appropriate set- or type-theoretic foundation such as those described in \[1, 12, 17, 18\]. For clarity, our proofs will be presented in the style used by the working mathematician rather than the formal one of the logician; we believe that it would be relatively straightforward, but uninformative, to translate them into a formal setting such as EL (see \[16\] and 3.6 of \[20\]).

Property $\text{AS}_{[0,1]}$ is the antithesis of the famous theorem of Specker in recursive analysis, a variation upon which states:

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\(^1\)For more information about constructive reverse mathematics, consult \[13\].
Speck: There exists a Specker sequence in \([0, 1]\)—that is, a sequence in \([0, 1]\) that is eventually bounded away from each point of \([0, 1]\) ([19]; [8], Chapter 3, Theorem (3.1)).

With classical logic one can easily show that \(\text{AS}_{[0,1]}\) is equivalent to the essentially nonconstructive property of sequential compactness for \([0, 1]\). Over BISH, the anti-Specker property is equivalent to the fan theorem \(\text{FT}_c\) for \(c\)-bars;\(^2\) it is therefore reasonable to regard it as a ‘quasi-constructive’ substitute for sequential compactness. The framework \(\text{BISH} + \text{AS}_{[0,1]}\) is worth exploring, as there are some indications that the application of \(\text{AS}_{[0,1]}\) to proofs in analysis may have the same slick quality as do those of sequential compactness in classical analysis (see, for example, the proof of Proposition 13 below).

In addition to anti-Specker properties, we shall be concerned with weak forms, and antitheses, of versions of Brouwer’s fan theorem. In particular, we shall give new, direct proofs of

- the equivalence, relative to BISH, of the antithesis of the full fan theorem, that of the fan theorem for \(c\)-bars, and the existence of Specker sequences in \([0, 1]\); and
- the equivalence, relative to BISH, of the antithesis of the fan theorem for detachable bars and that of the principle that every uniformly continuous, positive-valued function on \([0, 1]\) has positive infimum.

Several of these equivalences were established by Diener in Chapter 4 of [11]. However, our proofs are different, and in some cases more direct, than Diener’s and are, we believe, worthy of interest in their own right.

2. The Non-Specker Property

We begin by recalling three important principles that arise in constructive reverse mathematics:

- **LPO**: For each binary sequence \(a \equiv (a_n)_{n \geq 1}\),
  \[\forall_n (a_n = 0) \lor \exists_n (a_n = 1).\]

- **WLPO**: For each binary sequence \(a\),
  \[\forall_n (a_n = 0) \lor \neg\forall_n (a_n = 0).\]

- **MP (Markov’s principle)**: For each binary sequence \(a\),
  \[\neg\forall_n (a_n = 0) \Rightarrow \exists_n (a_n = 1).\]

The first two of these principles are false under the recursive interpretation of BISH, and so are incontrovertibly nonconstructive. Markov’s principle holds recursively, represents an unbounded search, and is not regarded as an intrinsic part of BISH. For more on these matters, see Chapters 1 and 3 of [8].

It is known that, relative to BISH, LPO implies both WLPO (trivially) and \(\text{AS}_{[0,1]}\) ([6], Theorem 3.1)—indeed, \(\text{AS}_{[0,1]}\) is classically equivalent to LPO. It is also known that in BISH + WLPO, we can derive Brouwer’s fan theorem \(\text{FT}_\Pi^0_1\).

\(^2\)This, and two other versions of the fan theorem, are described fully in Section 3.
for $\Pi_1^0$-bars,\(^3\) which implies (but is not known to be equivalent to) a version of the fan theorem that is equivalent to $\text{AS}_{[0,1]}$. Thus, voyaging via fan theorems, we obtain $\text{AS}_{[0,1]}$ as a consequence of $\text{WLPO}$. A direct route from $\text{WLPO}$ to $\text{AS}_{[0,1]}$, without digression through fan theorems, seems hard to find. However, as we show in this section, there is such a direct route leading to a weakened version of the anti-Specker property; namely, the non-Specker property:

$$\text{AS}_{[0,1]}^-$$: For each sequence $z$ in $[0,1]$, it is impossible that $z$ is eventually bounded away from each point of $[0,1]$.

—in other words, the denial of the Specker property.

We begin the discussion of $\text{AS}_{[0,1]}^-$ with a simple, useful, multiple application of $\text{WLPO}$.

**Lemma 1.** $\text{BISH} + \text{WLPO} \vdash$ For any decidable predicate $P$ on $\mathbb{N}^+ \times \mathbb{N}^+$,

$$\forall i \neg \exists n P(i,n) \lor \neg \exists i \forall n \neg P(i,n).$$

**Proof.** For each positive integer $i$ construct a binary sequence $\lambda^{(i)}$ such that

$$\lambda^{(i)}_n = 0 \Rightarrow \neg P(i,n), \quad \lambda^{(i)}_n = 1 \Rightarrow P(i,n).$$

By $\text{WLPO}$, either\(^4\) $\lambda^{(i)} = 0$ or $\neg (\lambda^{(i)} = 0)$. In the first case, set $\mu_i \equiv 1$, and in the second, $\mu_i \equiv 0$. Again applying $\text{WLPO}$, we obtain either $\mu = 0$ or else $\neg (\mu = 0)$. If the first alternative holds, then

$$\forall i \neg \forall n \neg P(i,n)$$

and therefore

$$\forall i \neg \exists n P(i,n).$$

If, on the other hand, $\neg (\mu = 0)$, then

$$\neg \forall i \neg \forall n \neg P(i,n)$$

and therefore

$$\neg \exists i \forall n \neg P(i,n).$$

The proof is complete. \(\Box\)

**Theorem 2.** $\text{BISH} + \text{WLPO} \vdash \text{AS}_{[0,1]}^-.$

**Proof.** Let $(z_n)_{n \geq 1}$ be a sequence in $[0,1]$ that is eventually bounded away from each point of $[0,1]$. We construct a sequence $(I_k)_{k \geq 1}$ of compact intervals, a strictly increasing sequence $(n_k)_{k \geq 1}$ of positive integers, and a sequence $(\delta_k)_{k \geq 1}$ of positive numbers such that for all $k$,

(i) $I_1 = [0,1] \supset I_2 \supset I_3 \supset \cdots$ and $|I_{k+1}| = 2^{-k}$;

(ii) $|z_n - \xi_k| \geq \delta_k$ for all $n \geq n_k$, where $\xi_k$ is the midpoint of $I_k$; and

(iii) $\forall i \neg \exists n (\# \{n_k \leq j \leq n : z_j \in I_k \} \geq i).$

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\(^3\)Since $\text{FT}^{\Pi_1^0}$ plays no significant role in this paper, we refrain from overloading the reader by defining what is meant by a ‘$\Pi_1^0$-bar’.

\(^4\)We denote by $\mathbf{0}$ the sequence with each term equal to $0$. 
First compute \( n_1 \) and \( \delta_1 > 0 \) such that \( |z_n - \frac{1}{2^n}| > \delta_1 \) for all \( n \geq n_1 \). Suppose we have found \( I_1, \ldots, I_k \) and the numbers \( n_k, \delta_k \) with the applicable properties. Let \( H_L \) and \( H_R \) be, respectively, the left and right closed halves of the interval \( I_k \). Set

\[
P(i, n) \equiv (\# \{ n_k \leq j \leq n_k + n : z_j \in H_R \} \geq i)\,.
\]

Note that, since property (ii) holds for the interval \( I_k \), we can determine for each \( j \geq n_k \) whether \( z_j \in H_L \) or \( z_j \in H_R \); hence, \( P(i, n) \) is decidable. Now by Lemma 1, we find that either

\[
\forall i \neg \exists n (\# \{ n_k \leq j \leq n_k + n : z_j \in H_R \} \geq i)\,.
\]

in which case we take \( I_{k+1} \equiv H_R \), or else

\[
\neg \exists n \forall i (\# \{ n_k \leq j \leq n_k + n : z_j \in H_R \} < i)\,.
\]

(1)

In the latter event, it is impossible that there be infinitely many \( n \) with \( z_n \in H_R \). If also there exists \( i \) such that

\[
\neg \exists n (\# \{ n_k \leq j \leq n : z_j \in H_L \} \geq i)\,, \tag{2}
\]

then there are at most \( i - 1 \) values of \( j \) with \( z_j \in H_L \), and hence there cannot be infinitely many terms of \( z_n \) in the interval \( I_k \). Since, by our induction hypothesis, (iii) holds for \( I_k \), we conclude that if (1) obtains, then (2) must be false—that is:

\[
\forall i \neg \exists n (\# \{ n_k \leq j \leq n : z_j \in H_L \} \geq i)\,.
\]

In that case we take \( I_{k+1} \equiv H_L \). Thus in either case we have

\[
\forall i \neg \exists n (\# \{ n_k \leq j \leq n : z_j \in I_{k+1} \} \geq i)\,.
\]

To complete the inductive construction, it remains to compute \( n_{k+1} \) and \( \delta_{k+1} > 0 \) such that \( |z_n - \xi_{k+1}| \geq \delta_{k+1} \) for all \( n \geq n_{k+1} \), where \( \xi_{k+1} \) is the midpoint of \( I_{k+1} \).

It follows from (i) that \( \bigcap_{k \geq 1} I_k \) consists of a single point \( \xi \). Pick \( N \) and \( \delta \) such that \( |z_n - \xi| \geq \delta \) for all \( n \geq N \). Again by (i), there exists \( \kappa \) such that \( |I_\kappa| < \delta/2 \). By (ii), for all \( j \geq n_\kappa \) we have

\[
|z_j - \xi_\kappa| > |z_j - \xi| - |\xi - \xi_\kappa| > \frac{\delta}{2}
\]

and therefore \( z_j \notin I_\kappa \). Hence

\[
\neg \exists n (\# \{ n_k \leq j \leq n : z_j \in I_\kappa \} \geq n_\kappa + 1)\,,
\]

which contradicts (iii) in the case \( i = n_\kappa + 1 \). \( \square \)

We extend Theorem 2 to cover an equivalent formulation of the non-Specker property:

**Corollary 3.** \( \text{BISH} + \text{WLPO} \vdash \) For each sequence \( z \) in \([0, 1] \cup \{2\} \), if \( z \) is eventually bounded away from each point of \([0, 1]\), then

\[
\forall i \neg \exists n (n > i \land z_n = 2)\,.
\]

While this property seems *prima facie* stronger than \( \text{AS}_{(0,1)}^- \), it is straightforward to show that the two are equivalent; hence, we omit the proof.
3. A Fan-Theoretic Equivalent of $\text{AS}_{[0,1]}$

Knowing that, over BISH, $\text{AS}_{[0,1]}$ is equivalent to a version of the fan theorem, we aim to produce a corresponding equivalence for $\text{AS}_{[0,1]}$. This requires us to put on paper some basic facts about bars.

Let $2^*$ denote the complete binary fan—the set of all finite sequences in $\{0, 1\}$—and $2^{\mathbb{N}^+}$ the set of all binary sequences. We denote by $|u|$ the length of an element of $2^*$. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a finite or infinite binary sequence. Then for each applicable $n \in \mathbb{N}$,

$$\pi(n) = (\alpha_1, \ldots, \alpha_n)$$

is called a restriction of $\alpha$. By a path in $2^*$ we mean a finite or infinite binary sequence. We say that a path $\alpha$ is blocked by a subset $B$ of $2^*$ if some restriction of $\alpha$ is in $B$. A subset $B$ of $2^*$ is called a bar for $2^*$ if each binary sequence is blocked by $B$.

A subset $D$ of a set $X$ is said to be detachable from $X$ if

$$\forall x \in X \ (x \in D \lor x \notin D).$$

A subset $B$ of $2^*$ is called a $c$-set if there exists a detachable subset $D$ of $2^*$ such that

$$B = \{u \in 2^* : \forall v \in 2^* \ (uv \in D)\},$$

where $uv$ is the concatenation of the strings $u$ and $v$. A $c$-bar for $2^*$ is a $c$-set that is also a bar.

We say that a bar $B$ for $2^*$ is

- uniform if there exists a positive integer $n$ such that each finite path of length $n$ is blocked by $B$;
- nonuniform if for each positive integer $n$ there exists a path of length $n$ in $2^*$ that misses $B$.\(^5\)

We shall be particularly concerned with antitheses of the following three versions of Brouwer’s fan theorem:

- the fan theorem for detachable bars, $\text{FT}_D$: every detachable bar of the complete binary fan is uniform;
- the fan theorem for $c$-bars, $\text{FT}_c$: every $c$-bar of the complete binary fan is uniform;
- the (full) fan theorem, $\text{FT}$: every bar of the complete binary fan is uniform.

The corresponding antithetical principles are these:

- anti-$\text{FT}_D$: There exists a nonuniform detachable bar for $2^*$.
- anti-$\text{FT}_c$: There exists a nonuniform $c$-bar for $2^*$.
- anti-$\text{FT}$: There exists a nonuniform bar for $2^*$.

These antithetical principles were studied in detail by Diener [11], who proved inter alia that anti-$\text{FT}_c$ and anti-$\text{FT}$ are equivalent over BISH. We shall connect Diener’s work with $\text{AS}_{[0,1]}$ by new proofs.

For each finite path $\zeta$ in $2^*$ and for each positive integer $k$, let $\zeta^k$ (resp., $\zeta^{0^k}$) denote the concatenation of $\zeta$ with a string of $k$ 1’s (resp., 0’s); and let $\zeta^{1^\omega}$ (resp.,

\(^5\)For a detachable bar, “nonuniform” and “not uniform” are equivalent.
\(\zeta(0^n)\) denote the concatenation of \(\zeta\) with a sequence of 1’s (reps., 0’s). For each binary sequence \(\alpha\), we define

\[F(\alpha) \equiv \sum_{n=1}^{\infty} \alpha_n 2^{-n}.\]

Then \(F\) maps the set of all binary sequences into \([0,1]\). For each finite path \(\zeta\) in \(2^*\) we define \(F(\zeta) \equiv F(\zeta(0^n))\). We observe that for any path \(\gamma\) of length \(N\) in \(2^*\), there exists a unique \(j \in \{0,1,2,\ldots,2^N-1\}\) such that \(F(\gamma) = j 2^{-N}\). Let \(\gamma'\) be another path of length \(N\) in \(2^*\), and let \(F(\gamma') = j'2^{-N}\). We say that the terminal node of \(\gamma'\) is \(m\) places to the left (resp., right) of that of \(\gamma\) if \(j - j' = m\) (resp., \(j' - j = m\)). We also say that the terminal node of \(\gamma'\) is at least \(m\) places away from that of \(\gamma\) if \(|j - j'| \geq m\).

**Lemma 4.** Let \(N\) be a positive integer, and let \(\alpha, \beta\) be binary sequences such that the terminal nodes of \(\overline{\alpha}(N)\) and \(\overline{\beta}(N)\) are at least two places away from each other. Then \(|F(\alpha) - F(\beta)| \geq 2^{-N}\).

**Proof.** Compute \(j, k\) such that \(F(\overline{\alpha}(N)) = j 2^{-N}\) and \(F(\overline{\beta}(N)) = k 2^{-N}\). If \(k \leq j - 2\), then

\[F(\beta) = \sum_{n=1}^{\infty} \beta_n 2^{-n} \leq \sum_{n=1}^{N} \beta_n 2^{-n} + \sum_{n=N+1}^{\infty} 2^{-n} = k 2^{-N} + 2^{-N} = (k + 1)2^{-N},\]

so \(F(\alpha) - F(\beta) \geq 2^{-N}\). The case where \(j \geq k + 2\) is handled similarly, and yields \(F(\beta) - F(\alpha) \geq 2^{-N}\). \(\square\)

**Proposition 5.** If there exists a nonuniform bar for \(2^*\), then Speck holds.

**Proof.** Suppose that such a bar \(B\) exists; we may assume that \(B\) is closed under extension. For each positive integer \(n\), let \(z_n = F(\zeta_n) \in [0,1]\). Given a binary sequence \(\alpha\), compute \(N\) such that \(\overline{\alpha}(N) \in B\), and then \(k\) such that \(F(\overline{\alpha}(N)) = k 2^{-N}\). To illustrate, take the case where \(\alpha_N = 1\). Let \(\beta_L\) be the path of length \(N\) whose terminal node is one place to the left of that of \(\overline{\alpha}(N)\); then \((\beta_L)_N = 0\). Since \(B\) is a bar and is closed under extension, there exists \(\nu_L > N\) such that the path \(\beta_L 1^{\nu_L-N}\) is in \(B\). For \(k \geq 1\), any path \(\delta\) with \(\overline{\delta}(N + k) = \beta_L 1^{k-1}\) has its node at level \(N + k\) at least two places to the left of the node of \(\alpha\) at that level; so, by Lemma 4,

\[|F(\alpha) - F(\delta)| \geq 2^{-N-k}.\]  \(3\)

On the other hand, if \(\beta_R\) is the path of length \(N\) in \(2^*\) whose terminal node is one place to the right of that of \(\overline{\alpha}(N)\); then \((\beta_R)_N = 0\). Since \(B\) is a bar and is closed under extension, there exists \(\nu_L > N\) such that the path \(\beta_L 0^{\nu_L-N}\) is in \(B\). A similar argument to the one for \(\beta_L\) now shows that for \(k \geq 1\), any path \(\delta\) with \(\overline{\delta}(N + k) = \beta_R 0^{k-1}\) has its node at level \(N + k\) at least two places to the right of the node of \(\alpha\) at that level; so (3) holds, again by Lemma 4.
Theorem 8. \textsc{BISH}

Now consider any \( n \geq \max \{ \nu_L, \nu_R \} \). Pick \( j \) such that \( F(\overline{\zeta}(N)) = j2^{-N} \). If \( j \leq k - 2 \), then, by Lemma 4, \(|F(\alpha) - z_n| \geq 2^{-N} \). If \( j = k - 1 \), then since \( |\zeta_n| \geq \nu_L \) and \( B \) is closed under extension, \( \overline{\zeta}(\nu_L) \notin B \). If \( \overline{\zeta}(N + k) \neq \beta_L1^{k-1}0 \) for each \( k \) with \( 1 \leq k \leq \nu_L - N \), then \( \overline{\zeta}(\nu_L) = \beta_L1^{\nu_L-N} \in B \), a contradiction. Thus we must have \( \overline{\zeta}(N + k) = \beta_L1^{k-1}0 \) for some \( k \) with \( 1 \leq k \leq \nu_L - N \). But then, the node of \( \zeta_n \) at level \( N + K \) is at least two places to the left of the node of \( \alpha \) at that level; hence,

\[ |F(\alpha) - z_n| \geq 2^{-N-k} \geq 2^{-\nu_L}. \]

In the case where \( j > k \), we similarly find that \( z_n - F(\alpha) \geq 2^{-\nu_R} \). Hence

\[ |F(\alpha) - z_n| \geq 2^{-\max\{\nu_L, \nu_R\}} \]

for all \( n \geq \max \{ \nu_L, \nu_R \} \). This completes the proof that the sequence \((z_n)_{n \geq 1}\) is eventually bounded away from \( F(\alpha) \). To complete the proof of the theorem, we need only apply Lemma 2 of [3].

We next aim for a converse of the preceding proposition.

**Proposition 6.** If \textsc{Speck} holds, then there exists a nonuniform \( c \)-bar for \( 2^{*} \).

This will require a simple preliminary.

**Lemma 7.** If there exists a Specker sequence \((z_n)_{n \geq 1}\) in \([0, 1]\), then there exists a sequence \((\alpha_n)_{n \geq 1}\) in \( 2^{N^*} \) such that

(i) \(|F(\alpha_n) - z_n| < 2^{-n}\) for each \( n \), and

(ii) \((F(\alpha_n))_{n \geq 1}\) is a Specker sequence in \([0, 1]\).

**Proof.** Since the dyadic rationals are dense in \([0, 1]\), for each positive integer \( n \) there exists \( \alpha_n \) such that \(|F(\alpha_n) - z_n| < 2^{-n}\). Given \( x \in [0, 1]\), pick a positive integer \( N \) such that \(|z_n - x| > 2^{-N+1}\) for all \( n \geq N \). Then for such \( n \), \(|F(\alpha_n) - x| > 2^{-N}\). Thus (ii) holds.

Now we have the proof of Proposition 6.

**Proof.** Suppose there exists a Specker sequence \((\zeta_n)_{n \geq 1}\) in \([0, 1]\). It is a straightforward exercise to show that there then exists a rational Specker sequence \( z \equiv (z_n)_{n \geq 1}\) in \([0, 1]\). In view of Lemma 7, we may assume that for each \( n \) there exists \( \alpha_n \in 2^{N^*} \) such that \( z_n = F(\alpha_n) \). Following the proof of [3] (Theorem 6), we see that

\[ D = \left\{ u \in 2^{*} \mid |F(u) - z_{|u|}| > 2^{-|u|+1} \right\} \]

is detachable (since the numbers \( z_n \) are rational), and that

\[ B = \{ u \in 2^{*} \mid \forall w (uw \in D) \} \]

is a \( c \)-bar for \( 2^{*} \). Given a positive integer \( n \), suppose that \( \overline{\alpha(n)} \in B \). Then

\[ |F(\overline{\alpha(n)}) - z_n| > 2^{-n+1} \geq 2^{-n} \geq |F(\overline{\alpha(n)}) - F(\alpha(n))| = |F(\overline{\alpha(n)}) - z_n|, \]

which is absurd. Since \( B \) is closed under extensions, we conclude that \( \overline{\alpha(n)} \) is a path of length \( n \) in \( 2^{*} \) that misses \( B \).

Putting Propositions 5 and 6 together, we obtain

**Theorem 8.** \textsc{BISH} \( \vdash \) The following are equivalent:
(i) There exists a Specker sequence in \([0, 1]\).
(ii) \text{anti-FT}_c.
(iii) \text{anti-FT}.

The equivalences in this theorem are found in Proposition 4.5.2 of [11]. However, we have taken a rather different route to establishing them.

By contraposition, we see that \(\text{AS}_{[0, 1]}\), as the negation of (i), is equivalent to \(\neg \text{anti-FT}\) and \(\neg \text{anti-FT}_c\), which are \text{weak fan theorems} of the form:

\[\text{For every } \bar{\varphi}\text{-bar } B \text{ for } 2^\ast, \text{it is impossible that } B \text{ be nonuniform.}\]

4. Fan and Anti-Fan for Detachable Bars

In our discussion of fan theorems, we have so far dealt mainly with \(\text{FT}_c\) and its antithesis \(\text{anti-FT}_c\); it is time to look at \(\text{FT}_D\) and \(\text{anti-FT}_D\). We begin by recalling the first major result in constructive reverse mathematics, due to Julian and Richman [15]:

\textbf{Theorem 9.} Let \(B\) be a detachable subset of \(2^\ast\). Then there exists a uniformly continuous function \(f : [0, 1] \rightarrow \mathbb{R}\) such that

(a) \(f(x) > 0\) for each \(x \in [0, 1]\) if and only if \(B\) is a bar for \(2^\ast\), and
(b) \(\inf f > 0\) if and only if \(B\) is a uniform bar for \(2^\ast\).

Conversely, if \(f : [0, 1] \rightarrow \mathbb{R}\) is a uniformly continuous function, then there exists a detachable subset \(B\) of \(2^\ast\) that satisfies (a) and (b).

We shall actually be more interested in the alternative proof of this theorem given in [4]. Modifying some of the arguments of that proof will enable us to supplement the Julian-Richman connection between \(\text{FT}_D\) and the \text{positivity principle},

\text{POS:} Every uniformly continuous, positive-valued function on \([0, 1]\) has positive infimum,

with one between \(\text{anti-FT}_D\) and

\text{anti-POS:} There exists a uniformly continuous, positive-valued function on \([0, 1]\) with infimum 0.

The latter connection is given by:

\textbf{Theorem 10.} \(\text{BISH} \vdash \text{anti-FT}_D\) is equivalent to \text{anti-POS}.

We establish this theorem as a consequence of the next two propositions.

\textbf{Proposition 11.} Let \(f : [0, 1] \rightarrow \mathbb{R}\) be uniformly continuous. Then there exists a detachable subset \(B\) of \(2^\ast\) such that

(a) if \(f(x) > 0\) for each \(x \in [0, 1]\), then \(B\) is a bar for \(2^\ast\), and
(b) if, in addition, \(\inf f = 0\), then \(B\) is a nonuniform bar for \(2^\ast\).

\textbf{Proof.} Let \(f : [0, 1] \rightarrow \mathbb{R}\) be uniformly continuous, and suppose that \(f(x) > 0\) for each \(x \in [0, 1]\). We first reprise a construction used in the proof of [4] (Theorem 1). Using countable choice, construct a strictly increasing sequence \((n_k)_{k \geq 1}\) of positive integers such that for all \(\alpha \in 2^{\mathbb{N}^+}\) and all \(k \in \mathbb{N}^+\),

\[|f(F(\alpha)) - f(F(\pi(n_k)))| < 2^{-k}.\]
Again using countable choice, construct a family $\lambda_u \in \{0, 1\}$ such that for each $u \in 2^*$,
\begin{align*}
\lambda_u = 0 & \Rightarrow \forall \alpha (|u| \neq n_k) \vee \exists \alpha (|u| = n_k \land f(F(u)) < 2^{-k+2}), \\
\lambda_u = 1 & \Rightarrow \exists \alpha (|u| = n_k \land f(F(u)) > 2^{-k+1}).
\end{align*}

Let
\[ B = \{ u \in 2^* : \lambda_u = 1 \}, \]
which is a detachable subset of $2^*$. Consider $\alpha \in 2N^+$ and $k \in N^+$. If $f(F(\pi(n_k))) < 2^{-k+3}$, then $f(F(\alpha)) < 2^{-k+4}$. Thus if $f(F(\alpha)) > 2^{-k+4}$, then $f(F(\pi(n_k))) > 2^{-k+2}$ and therefore $\lambda_{\pi(n_k)} \neq 0$; whence $\pi(n_k) \in B$. It follows that if $f(x) > 0$ for each $x \in [0, 1]$, then $B$ is a bar for $2^*$ (and, although we do not need this fact, that if $\inf f > 0$, then $B$ is a uniform bar for $2^*$). If also $\inf f = 0$, then, given a positive integer $k$, we can pick $\alpha \in 2^*$ such that $f(F(\alpha)) < 2^{-k}$. This gives
\[ f(F(\pi(n_k))) \leq f(F(\alpha)) + |f(F(\alpha)) - f(F(\pi(n_k)))| < 2^{-k} + 2^{-k} = 2^{-k+1}, \]
so $\lambda_{\pi(n_k)} \neq 1$ and therefore $\pi(n_k) \notin B$. Since $k$ is arbitrary and for each positive integer $n$ there exists $k$ such that $n_k > n$, it follows that $B$ is nonuniform. \qed

Our next proposition should be compared with Proposition 4.5.1 of [11].

**Proposition 12.** If there exists a nonuniform detachable bar for $2^*$, then there exists a uniformly continuous, positive-valued function on $[0, 1]$ with infimum 0.

**Proof.** Let $B$ be a detachable bar for $2^*$. According to Theorem 9, there exists a uniformly continuous function $f : [0, 1] \to \mathbb{R}^+$ such that $\inf f > 0$ if and only if $B$ is a uniform bar for $2^*$. So if $B$ is nonuniform, we must have $\inf f = 0$. \qed

Taken together, Propositions 11 and 12 provide our desired proof of Theorem 10.

Although, as was first shown by Diener [11] and as we have proved alternatively above, anti-$\text{FT}_c$ and anti-$\text{FT}$ are equivalent, and although, clearly, anti-$\text{FT}_D$ implies anti-$\text{FT}_c$, it is not known whether these three antithetical fan theorems are equivalent over BISH.\footnote{For more on this, see [11].} It is noteworthy that, in view of Theorem 10, to prove that equivalence, it would suffice to prove that the existence of a Specker sequence in $[0, 1]$ implies anti-$\text{POS}$. 

5. The Limited Anti-Specker Property

We turn now from fan theorems to a natural strengthening of the non-Specker property—namely, the **limited anti-Specker property**: 

\[ \text{AS}^{\text{std}}_{[0,1]} : \text{ For each sequence } z \text{ in } [0, 1] \cup \{2\}, \text{ if } z \text{ is eventually bounded away from each point of } [0, 1], \text{ then there exists } n \text{ such that } z_n = 2. \]

A simple induction argument shows that, under $\text{AS}^{\text{std}}_{[0,1]}$, if a sequence $z$ is eventually bounded away from each point of $[0, 1]$, then $z_n = 2$ infinitely often.

We apply the limited anti-Specker property to give a neat proof of the most significant case of a result (proved in [7] with the full anti-Specker property) about positive-valued, pointwise continuous functions.
Proposition 13. BISH + AS\textsuperscript{[td]}\([0,1]\) ⊬ If \(f : [0,1] \to \mathbb{R}^+\) is pointwise continuous and has an infimum \(\mu\), then \(\mu > 0\).

Proof. Let \(f : [0,1] \to \mathbb{R}^+\) be pointwise continuous and have an infimum \(\mu\). Replacing \(f\) by \(f(1)^{-1}f\), we may assume that \(f(1) = 1\). Construct an increasing binary sequence \((\lambda_n)_{n \geq 1}\) such that

\[
\lambda_n = 0 \Rightarrow \mu < 3^{-n}, \quad \lambda_n = 1 \Rightarrow \mu > 3^{-n-1}.
\]

We may further assume that \(\lambda_1 = 0\). If \(\lambda_n = 0\), then, by a weak constructive intermediate value theorem,\(^7\) we can find \(z_n \in [0,1]\) such that

\[
\max\{\mu, 2.3^{-n-1}\} < f(z_n) < 3^{-n}.
\]

If \(\lambda_n = 1\), set \(z_n = 2\). We show that the sequence \(z\) is eventually bounded away from each point \(x \in [0,1]\). Pick a positive integer \(N\) such that \(f(x) > 3^{-N}\), and then compute \(\delta \in (0,1)\) such that if \(y \in [0,1]\) and \(|x-y| < \delta\), then \(|f(x) - f(y)| < 2.3^{-N-1}\). If \(n > N\) and \(\lambda_n = 0\), then

\[
f(x) - f(z_n) > 3^{-N} - 3^{-n} \geq 3^{-N} - 3^{-N-1} = 2.3^{-N-1}
\]

and therefore \(|z_n - x| \geq \delta\), an inequality which holds trivially if \(\lambda_n = 1\). Thus \(z\) is eventually bounded away from \(x\). We now apply AS\textsuperscript{[td]}\([0,1]\), to compute \(\nu\) such that \(z_\nu = 2\) and therefore \(\lambda_\nu = 1\); whence \(\mu > 3^{-\nu-1}\). \(\square\)

Corollary 14. BISH + AS\textsuperscript{[td]}\([0,1]\) ⊬ POS.

Proof. In order to apply Proposition 13, it suffices to observe that, by Corollary 2.2.7 of \([9]\), the infimum of a uniformly continuous, real-valued function on \([0,1]\) exists. \(\square\)

Corollary 15. BISH + AS\textsuperscript{[td]}\([0,1]\) ⊬ FT\textsubscript{D}

Proof. Apply the Julian–Richman result (Theorem 9) and Corollary 14. \(\square\)

Note that in the proof of Proposition 13, we need the hypothesis that the infimum exists. To show this, we work within RUSS, the recursive ‘model’ of BISH: that is, BISH augmented by Markov’s principle and the Church-Markov-Turing thesis \([8]\) (Chapter 3).\(^8\)

Proposition 16. RUSS ⊬ The statement

\((*)\) Every pointwise continuous, positive-valued function on \([0,1]\) has an infimum is false.

\(^7\)Namely, if \(f : [0,1] \to \mathbb{R}\) is pointwise continuous and \(f(0)f(1) < 0\), then for each \(\varepsilon > 0\), there exists \(x \in (0,1)\) with \(|f(x)| < \varepsilon\). See Chapter 3 of \([8]\), where it is shown that sequential continuity is enough.

\(^8\)It seems to be folklore that uniform continuity is required for the existence of the infimum for general real-valued functions on \([0,1]\). If there is in the literature a proof that pointwise continuity is definitely insufficient, we are unaware of its existence. For that reason we have included Proposition 16 and its proof in this paper.
Proof. By Specker’s theorem ([19], Theorem (3.1)), there exists a strictly increasing sequence \((x_n)_{n \geq 1}\) in \((0,1)\) that is eventually bounded away from each point of \([0,1]\). Construct a sequence \((\delta_n)_{n \geq 1}\) of positive numbers such that \(\delta_1 < x_1\) and \(x_n + \delta_n < x_{n+1} - \delta_{n+1}\) for each \(n\). Let \((a_n)_{n \geq 1}\) be a binary sequence, and define a sequence \((f_n)_{n \geq 1}\) of uniformly continuous functions on \([0,1]\) such that

- if \(a_n = 0\), then \(f_n = 0\);
- if \(a_n = 1\), then \(f_n(x) = 0\) whenever \(|x - x_n| \geq \delta_n\), \(f_n(x_n) = -1\), and \(f_n\) is linear on each of the intervals \([x_n - \delta_n, x_n]\), \([x_n, x_n + \delta_n]\).

For each \(x \in [0,1]\), since \((x_n)_{n \geq 1}\) is eventually bounded away from \(x\), it is straightforward to show that there exist \(N\) and \(\delta > 0\) such that if \(x' \in [0,1]\) and \(|x - x'| < \delta\), then

\[
\sum_{n=1}^{\infty} f_n(x') = \sum_{n=1}^{N} f_n(x).
\]

Hence the function \(g = 1 + \frac{1}{2} \sum_{n=1}^{\infty} f_n\) is well defined and pointwise continuous on \([0,1]\). Clearly, it maps \([0,1]\) into \([0,1]\). Suppose that \(\mu = \inf g\) exists. Then either \(\mu > \frac{1}{2}\), in which case \(a_n = 0\) for all \(n\), or else \(\mu < 1\). In the latter case, computing \(x\) such that \(g(x) < 1\), we see that \(\sum_{n=1}^{\infty} f_n(x) < 0\); whence there exists \(\nu\) such that \(f_\nu(x) < 0\) and therefore \(a_1 = 1\). We now see that the statement (*) implies LPO. But LPO is provably false in RUSS ([8], Chapter 3, Corollary (1.4)).

Proposition 16 does not rule out the possibility that \(\text{AS}^{\text{td}}_{[0,1]}\) implies that

- (†) every pointwise continuous function on \([0,1]\) that is bounded below has an infimum.

But is that possibility a likelihood? Suppose that it obtains. Then, by proposition 13, every pointwise continuous, positive-valued function on \([0,1]\) has a positive infimum. But, as is shown in [5], this result is equivalent to the uniform continuity theorem for \([0,1]\), UCT. In turn, it follows from work in [2] that UCT implies \(\text{FT}_c\) and therefore (see [3]) \(\text{AS}_{[0,1]}\). Thus if \(\text{AS}^{\text{td}}_{[0,1]}\) implies (†), then \(\text{AS}^{\text{td}}_{[0,1]}\) is equivalent to \(\text{AS}_{[0,1]}\), and \(\text{FT}_c\) is equivalent to UCT. However, the equivalence, relative to BISH, of \(\text{AS}^{\text{td}}_{[0,1]}\) and \(\text{AS}_{[0,1]}\) seems \textit{prima facie} unlikely, as therefore does the implication from \(\text{AS}^{\text{td}}_{[0,1]}\) to (†).

If \(\text{AS}^{\text{td}}_{[0,1]}\) is, as it would appear to be and its name suggests, strictly weaker than \(\text{AS}^{\text{td}}_{[0,1]}\), then Corollary 15 shows that \(\text{FT}_D\) is implied by something that lies between it and \(\text{FT}_c\) (equivalent, we recall, to \(\text{AS}_{[0,1]}\)). This raises two open questions:

- Is \(\text{AS}^{\text{td}}_{[0,1]}\) equivalent to \(\text{AS}_{[0,1]}\) and hence to \(\text{FT}_c\)?
- Is \(\text{AS}^{\text{td}}_{[0,1]}\) equivalent to \(\text{FT}_D\)?

Our final two results explore the relationship between \(\text{AS}^{\text{td}}_{[0,1]}\) and \(\text{AS}_{[0,1]}\) in the presence of MP.

Proposition 17. \(\text{AS}^{\text{td}}_{[0,1]} + \text{MP} \vdash\) Let \(z\) be a sequence in \([0,1] \cup \{2\}\) that is eventually bounded away from each point of \([0,1]\), and let \((n_k)_{k \geq 1}\) be a strictly increasing sequence of positive integers. Then there exists \(K\) such that \(z_n = 2\) whenever \(n_K < n < n_{K+1}\).
Proof. Construct a binary sequence $\lambda$ such that
\[\lambda_k = 0 \Rightarrow \exists j \ (n_k < j < n_{k+1} \land z_j \leq 1),\]
\[\lambda_k = 1 \Rightarrow \forall j \ (n_k < j < n_{k+1} \Rightarrow z_j = 2).\]
Suppose that $\lambda_k = 0$ for all $k$. Then we can construct a strictly increasing sequence $(j_k)_{k \geq 1}$ of positive integers such that for each $k$, $n_k < j_k < n_{k+1}$ and $z_{j_k} \in [0, 1]$. Being a subsequence of $z$, the sequence $(z_{j_k})_{k \geq 1}$ is eventually bounded away from each point of $[0, 1]$; so, by $\text{AS}^{\text{std}}_{[0,1]}$, there exists $k$ such that $z_{j_k} = 2$, a contradiction. Hence, by $\text{MP}$, there exists $K$ with the desired properties. □

Corollary 18. $\text{AS}^{\text{std}}_{[0,1]} + \text{MP} \vdash$ Let $z$ be a sequence in $[0,1] \cup \{2\}$ that is eventually bounded away from each point of $[0,1]$. Then for each pair $(m,n)$ of positive integers, there exists $j > n$ such that $z_j = z_{j+1} = \cdots = z_{j+m} = 2$.

Proof. Construct a strictly increasing sequence $(n_k)_{k \geq 1}$ of positive integers such that $n_1 = n + 1$ and, for each $k$, $n_{k+1} = n_k + m + 2$. Applying Proposition 17, we obtain $k$ such that $z_n = 2$ whenever $n_k < n < n_{k+1}$; it remains to take $j = n_k + 1$. □

Thus $\text{AS}^{\text{std}}_{[0,1]} + \text{MP}$ enables us, when given a sequence $z$ in $[0,1] \cup \{2\}$ that is eventually bounded away from each point of $[0,1]$, to construct arbitrarily long intervals of terms equal to 2 arbitrarily far out in the sequence. This shows that, under the assumption of $\text{MP}$, $\text{AS}^{\text{std}}_{[0,1]}$ is in some sense close to full $\text{AS}_{[0,1]}$, in which all the terms are equal to 2 far enough out.

References


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