APPROXIMATION BY GENERALIZED FABER SERIES IN BERGMAN SPACES ON INFINITE DOMAINS WITH A QUASICONFORMAL BOUNDARY

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Abstract. Using an integral representation on infinite domains with a quasiconformal boundary the generalized Faber series for the functions in the Bergman space \( A^2(G) \) are defined and their approximative properties are investigated.

1. Introduction and New Results

Let \( G \) be a simple connected domain in the complex plane \( \mathbb{C} \) and let \( \omega \) be a weight function given on \( G \). For functions \( f \) analytic in \( G \) we set

\[
A^2(G, \omega) := \left\{ f : \iint_G |f(z)|^2 \omega(z) d\sigma_z < \infty \right\},
\]

where \( d\sigma_z \) denotes the Lebesgue measure in the complex plane \( \mathbb{C} \).

If \( \omega = 1 \), we denote \( A^2(G) := A^2(G, 1) \). The space \( A^2(G) \) is called the Bergman space on \( G \). We refer to the spaces \( A^2(G, \omega) \) as “weighted Bergman spaces”. It becomes a normed spaces if we define

\[
\|f\|_{A^2(G, \omega)} := \left( \iint_G |f(z)|^2 \omega(z) d\sigma_z \right)^{1/2}.
\]

Hereafter, we consider only the special weight \( \omega(z) := 1/|z|^4 \) in this work.

Now let \( L \) be a finite quasiconformal curve in the complex plane \( \mathbb{C} \). We recall that \( L \) is called a quasiconformal curve if there exists a quasiconformal homeomorphism of the complex plane onto itself that maps a circle onto \( L \). We denote by \( G_1 \) and \( G_2 \) the bounded and unbounded complements of \( \mathbb{C} \setminus L \), respectively. It is clear that if \( f \in A^2(G_2) \), then it has zero in \( \infty \) at least second order. As in the bounded case \([7, p. 5]\), \( A^2(G_2) \) is a Hilbert space with the inner product

\[
\langle f, g \rangle := \iint_{G_2} f(z)\overline{g(z)} d\sigma_z,
\]

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which can be easily verified. Moreover, the set of polynomials of $1/z$ are dense in $A^2(G_2)$ with respect to the norm

$$
\|f\|_{A^2(G_2)} := (\langle f, f \rangle)^{1/2}.
$$

Indeed, let $f \in A^2(G_2)$. If we substitute $z = 1/\zeta$ and define

$$
f(z) = f \left(\frac{1}{\zeta}\right) =: f_\ast(\zeta),
$$

then $G_2$ maps to a finite domain $G_\ast$, and $f_\ast \in A^2(G_\ast)$, because

$$
\iint_{G_\ast} |f_\ast(\zeta)|^2 \, d\sigma_\zeta = \iint_{G_2} |f(z)|^2 \frac{d\sigma_z}{|z|^2} \leq c \iint_{G_2} |f(z)|^2 \, d\sigma_z < \infty,
$$

with some constant $c > 0$. Since $f$ has zero in $\infty$ at least second order, the point $\zeta = 0$ is the zero of $f_\ast$ at least second order and

$$
\iint_{G_\ast} \left|\frac{f_\ast(\zeta)}{\zeta^2}\right|^2 \, d\sigma_\zeta = \iint_{G_2} |f(z)|^2 \, d\sigma_z < \infty.
$$

Hence $f_\ast(\zeta)/\zeta^2 \in A^2(G_\ast)$. If $P_n(\varsigma)$ is a polynomial of $\varsigma$, then we have

$$
\iint_{G_\ast} \left|\frac{P_n(\varsigma) - f_\ast(\varsigma)}{\varsigma^2}\right|^2 \, d\sigma_\zeta = \iint_{G_\ast} \left|P_n(\varsigma)\varsigma^2 - f_\ast(\varsigma)\right|^2 \frac{1}{|\varsigma|^4} \, d\sigma_\zeta

= \iint_{G_\ast} \left|P_n \left(\frac{1}{z}\right) \frac{1}{z^2} - f(z)\right|^2 \, d\sigma_z.
$$

This implies that the set of polynomials of $1/z$ are dense in $A^2(G_2)$, since the set of polynomials $P_n(\varsigma)$ are dense in $A^2(G_\ast)$ with respect to the norm

$$
\|f\|_{A^2(G_\ast)} := (\langle f, f \rangle)^{1/2}.
$$

(see, for example: [7, Ch. 1]). Also, for $n = 1, 2, \ldots$ there exists a polynomial $P_\ast^n(1/z)$ of $1/z$, of degree $\leq n$, such that $E_n(f, G_2) = \|f - P_\ast^n\|_{A^2(G_2)}$ (see for example, [6, p. 59, Theorem 1.1.]), where

$$
E_n(f, G_2) := \text{Inf} \left\{\|f - P\|_{A^2(G_2)} : P \text{ is a polynomial of } 1/z, \text{ of degree } \leq n\right\}
$$

denotes the minimal error of approximation of $f$ by polynomials of $1/z$ of degree at most $n$. The polynomial $P_\ast^n(1/z)$ is called the best approximant polynomial of $1/z$ to $f \in A^2(G_2)$.

Let $D$ be the open unit disc and $w = \varphi(z)$ be the conformal mapping of $G_1$ onto $C \setminus D := \mathbb{C} \setminus \overline{D}$, normalized by the conditions

$$
\varphi(0) = \infty \quad \text{and} \quad \lim_{z \to 0} z \varphi(z) > 0,
$$

and let $\psi$ be the inverse of $\varphi$. In the neighborhood of the origin we have the expansion

$$
\varphi(z) = \frac{\alpha}{z} + \alpha_0 + \alpha_1 z + \ldots.
$$
Raising this function to the power \( m \) we obtain

\[
[\varphi(z)]^m = F_m(1/z) + Q_m(z) \quad \text{for} \quad z \in G_1,
\]

(1.1)

where \( F_m(1/z) \) denotes the polynomial of negative powers of \( z \) and the term \( Q_m(z) \) contains non-negative powers of \( z \) and is analytic in the domain \( G_1 \). The polynomial \( F_m(1/z) \) of negative powers of \( z \) is called the generalized Faber polynomial for the domain \( G_2 \). If \( z \in G_2 \), then integrating in the positive direction along \( L \), we have

\[
F_m \left( \frac{1}{z} \right) = -\frac{1}{2\pi i} \int \frac{[\varphi(\zeta)]^m}{\zeta - z} d\zeta = -\frac{1}{2\pi i} \int \frac{w^m \varphi'(w)}{\varphi(w) - z} dw.
\]

This formula implies that the functions \( F_m(1/z), \; m = 1, 2, \ldots \) are the Laurent coefficients in the expansion of the function

\[
\frac{\varphi'(w)}{\varphi(w) - z} \quad z \in G_2, \; w \in \mathbb{C}^D
\]

in the neighborhood of the point \( w = \infty \), i.e. the following expansion holds

\[
\frac{\varphi'(w)}{\varphi(w) - z} = \sum_{m=1}^{\infty} F_m \left( \frac{1}{z} \right) \frac{1}{w^{m+1}} \quad z \in G_2, \; w \in \mathbb{C}^D,
\]

which converges absolutely and uniformly on compact subsets of \( G_2 \times \mathbb{C}^D \). Differentiation of this equality with respect to \( z \) gives

\[
\frac{\varphi''(w)}{(\varphi(w) - z)^2} = \sum_{m=1}^{\infty} F'_m \left( \frac{1}{z} \right) \left( -\frac{1}{z^2} \right) \frac{1}{w^{m+1}}
\]

or

\[
\frac{z^2 \varphi''(w)}{(\varphi(w) - z)^2} = \sum_{m=1}^{\infty} -F'_m \left( \frac{1}{z} \right) \frac{1}{w^{m+1}}
\]

(1.2)

for every \( (z, w) \in G_2 \times \mathbb{C}^D \), where the series converges absolutely and uniformly on compact subsets of \( G_2 \times \mathbb{C}^D \). More information for Faber and generalized Faber polynomials can be found in [12, p. 255] and [7, p. 42].

In this work, for the first time, we obtain (Section 2, Lemma 2.1) an integral representation on the infinite domain \( G_2 \) with a quasiconformal boundary for a function \( f \in A^2(G_2) \). By means of this integral representation in Section 2 we define a generalized Faber series of a function \( f \in A^2(G_2) \) to be of the form

\[
\sum_{m=1}^{\infty} a_m(f) F'_m \left( \frac{1}{z} \right),
\]

with the generalized Faber coefficients \( a_m(f), \; m = 1, 2, \ldots \).

Our main results are presented in the following theorems, which are proved in Section 3.

**Theorem 1.1.** Let \( f \in A^2(G_2) \). If

\[
\sum_{m=1}^{\infty} a_m(f) F'_m \left( \frac{1}{z} \right)
\]

(1.3)

is a generalized Faber series of \( f \), then this series converges uniformly to \( f \) on the compact subsets of \( G_2 \).
Corollary 1.2. Let $P_n(1/z)$ be a polynomial of degree $n$ of $1/z$ and $P_n(1/z) \in A^2(G_2)$. If $a_m(P_n)$ are its generalized Faber coefficients, then $a_m(P_n) = 0$ for all $m \geq n + 2$ and

$$P_n \left( \frac{1}{z} \right) = \sum_{m=1}^{n+1} a_m(P_n) F_m' \left( \frac{1}{z} \right).$$

A uniqueness theorem for the series

$$\sum_{m=1}^\infty a_m(f) F_m' \left( \frac{1}{z} \right),$$

which converges to $f \in A^2(G_2)$ with respect to the norm $\| \cdot \|_{A^2(G_2)}$ is the following.

Theorem 1.3. Let $\{a_m\}$ be a complex number sequence. If the series

$$\sum_{m=1}^\infty a_m F_m' \left( \frac{1}{z} \right)$$

converges to a function $f \in A^2(G_2)$ in the norm $\| \cdot \|_{A^2(G_2)}$, then the $a_m$, $m = 1, 2, \ldots$, are the generalized Faber coefficients of $f$.

The following theorem estimates the error of the approximation of $f \in A^2(G_2)$ by the partial sums of the series (1.3) in the weighted norm $\| \cdot \|_{A^2(G_2, \omega)}$ for the special weight $\omega(z) := 1/|z|^4$, regarding to the minimal error $E_n(f, G_2)$.

Theorem 1.4. If $f \in A^2(G_2)$, $\omega(z) := 1/|z|^4$ and

$$S_n \left( f, \frac{1}{z} \right) = \sum_{m=1}^{n+1} a_m F_m' \left( \frac{1}{z} \right)$$

is the $n$th partial sum of its generalized Faber series

$$\sum_{m=1}^\infty a_m F_m' \left( \frac{1}{z} \right),$$

then

$$\| f - S_n(f, \cdot) \|_{A^2(G_2, \omega)} \leq \frac{c}{1 - k^2} \sqrt{n} E_n(f, G_2),$$

for all natural numbers $n$ and with a constant $c$ independent of $n$.

Similar results for the bounded domains with a quasiconformal boundary were stated and proved in [8] and [5], respectively. These problems in the weighted cases were studied in [9] and [10].

We shall use $c, c_1, \ldots$, to denote constants depending only on parameters that are not important for the questions of interest.

2. Definitions and Some Auxiliary Results

In [4], V.I. Belyi gave the following integral representation for the functions $f$ analytic and bounded in the domain $G_1$:

$$f(z) = -\frac{1}{\pi} \int_{G_2} \int_{G_2} \frac{(f \circ y)(\zeta)}{(\zeta-z)^2} \frac{\eta_\zeta(\zeta)}{\zeta} d\sigma_\zeta, \quad z \in G_1.$$
Here \( y(z) \) is a \( K \)-quasiconformal reflection across the boundary \( L \), i.e., a sense-reversing \( K \)-quasiconformal involution of the extended complex plane keeping every point of \( L \) fixed, such that \( y(G_1) = G_2, \; y(G_2) = G_1, \; y(0) = \infty \) and \( y(\infty) = 0 \). Such a mapping of the plane does exist \([11, \; p. \; 99]\). As follows from Ahlfors theorem \([1, \; p. \; 80]\) the reflection \( y(z) \) can always be chosen canonical in the sense that it is differentiable on \( C \) almost everywhere, except possibly at the points of the curve \( L \), and for any sufficiently small fixed \( \delta > 0 \) it satisfies the relations

\[
|y(z)| + |y(\zeta)| \leq c_1, \quad \text{if } \zeta \in \{ \zeta \mid \delta < |\zeta| < 1/\delta, \quad \zeta \notin L \}
\]

\[
|y(z)| + |y(\zeta)| \leq c_2|\zeta|^{-2}, \quad \text{if } |\zeta| \geq 1/\delta \text{ or } |\zeta| \leq \delta,
\]

with some constants \( c_1 \) and \( c_2 \), independent of \( \zeta \).

Considering only the canonical quasiconformal reflections, I.M. Batchaev \([3]\) generalized the integral representation above to functions \( f \in A^2(G_1) \). The accurate proof of the Batchaev’s result is given in \([2, \; p. \; 110, \; Th. \; 4.4]\). A similar integral representation can also be obtained for functions \( f \in A^2(G_2) \). The following result holds.

**Lemma 2.1.** Let \( f \in A^2(G_2) \). If \( y(z) \) is a canonical quasiconformal reflection with respect to \( L \), then

\[
f(z) = -\frac{1}{\pi} \int_{G_1} \frac{(f \circ y_{\zeta})(z)^2}{(\zeta - z)^2 |y_{\zeta}(\zeta)|^2} J d\zeta, \quad z \in G_2. \tag{2.1}
\]

**Proof.** Let \( y(z) \) a canonical quasiconformal reflection and \( f \in A^2(G_2) \). If we substitute \( \zeta = 1/u \) for \( \zeta \in G_2 \) and define

\[
f(\zeta) = f(1/u) =: f_\ast(u),
\]

then \( G_2 \) maps to a finite domain \( G_u \) and \( f_\ast \in A^2(G_u) \). If \( y^\ast(t) \) is a canonical quasiconformal reflection with respect to \( \partial G_u \), then from the Batchaev’s result we have

\[
f_\ast(t) = -\frac{1}{\pi} \int_{G_u} \frac{(f_\ast \circ y^\ast)(u)}{(u - t)^2 |y^\ast(u)|^2} J d\sigma_t, \quad t \in G_u,
\]

where \( \overline{G_u} := \mathbb{C} \setminus \overline{G_u} \). Substituting \( u = 1/\zeta \) in this integral representation we get

\[
f(z) = f(1/t) = f_\ast(t) = -\frac{1}{\pi} \int_{G_1} \frac{(f_\ast \circ y^\ast)(1/\zeta)}{(1/\zeta - 1/z)^2 |y^\ast(1/\zeta)| J d\sigma_t}
\]

\[
= \frac{1}{\pi} \int_{G_1} \frac{f(1/y^\ast(1/\zeta))z^2}{(\zeta - z)^2 |y^\ast(1/\zeta)|^2} J d\zeta, \quad z \in G_2.
\]

If we define

\[
y(\zeta) := \frac{1}{y^\ast(1/\zeta)},
\]

then \( y(\zeta) \) becomes a canonical quasiconformal reflection with respect to \( L \). Consequently, for \( f \in A^2(G_2) \) we get

\[
f(z) = -\frac{1}{\pi} \int_{G_1} \frac{(f \circ y)(\zeta)^2}{(\zeta - z)^2 |y(\zeta)|^2} J d\zeta, \quad z \in G_2.
\]
Thus, if we define the coefficients $a_m(f)$, $m = 1, 2, \ldots$, by

$$a_m(f) := \frac{1}{\pi} \int_{C^D} \frac{f(y(\psi(w)))\psi'(w)}{[y(\psi(w))]^2} \cdot \frac{z^2 \psi'(w)}{(\psi(w) - z)^2} d\sigma_w, \quad z \in G_2. \quad (2.2)$$

then, by (1.2) and (2.2), we can associate a formal series $\sum_{m=1}^{\infty} a_m(f)F_m^2(1/z)$ with the function $f \in A^2(G_2)$, i.e.,

$$f(z) \sim \sum_{m=1}^{\infty} a_m(f)F_m^2(1/z).$$

We call this formal series a generalized Faber series of $f \in A^2(G_2)$, and the coefficients $a_m(f)$, $m = 1, 2, \ldots$, generalized Faber coefficients of $f$.

**Lemma 2.2.** Let $\{F_m(1/z)\}$, $m = 1, 2, \ldots$, be the generalized Faber polynomials of $1/z$ for $G_2$. Then

$$\sum_{m=1}^{n} \left \| \frac{F_m}{F_m^2} \right \|_{A^2(G_2)}^2 \leq n\pi.$$ 

**Proof.** Since $\overline{\psi}(\zeta)$ is a canonical $K$-quasiconformal mapping of the extended complex plane onto itself, we have $\left | \overline{\psi}_\zeta / \overline{\psi}_z \right | \leq k$ and $\left | \overline{\psi}_z \right |^2 - \left | \overline{\psi}_\zeta \right |^2 > 0$. Also, it is known that $\left | \overline{\psi}_\zeta \right | = |y_\zeta|$ and $\left | \overline{\psi}_z \right | = |y_z|$. Therefore, $|y_\zeta| / |y_z| \leq k$ and $|y_z|^2 - |y_\zeta|^2 > 0$. Hence

$$\int_{G_2} \left | (f \circ y)(\zeta)^2 \right | \left | \frac{y_z}{y_\zeta} \right |^2 d\sigma_\zeta$$

$$= \int_{G_2} \left | (f \circ y)(\zeta)^2 \left (1 - |y_\zeta|^2 / |y_z|^2 \right ) \right |^{-1} \left | y_z - y_\zeta \right |^2 d\sigma_\zeta$$

$$\leq \frac{1}{1 - k^2} \int_{G_2} \left | (f \circ y)(\zeta)^2 \right | \left | y_z - y_\zeta \right |^2 d\sigma_\zeta.$$

Since $\left (|y_z|^2 - |y_\zeta|^2 \right )$ is the Jacobian of $y(\zeta)$, substituting $\zeta$ for $y(\zeta)$ on the right side of the last inequality we get

$$\int_{G_2} \left | (f \circ y)(\zeta)^2 \right | \left | y_z \right |^2 d\sigma_\zeta \leq \frac{\| f \|_{A^2(G_2)}^2}{1 - k^2}.$$
3. Proofs of the New Results

**Proof of Theorem 1.1.** Let \( M \) be a compact subset of \( G_2 \) and \( y(z) \) a canonical \( K \)-quasiconformal reflection with respect to \( L \). Since by Lemma 2.1

\[
f(z) = -\frac{1}{\pi} \int \frac{(f \circ y)(\zeta)z^2}{(\zeta - z)(y(\zeta))^2} d\sigma_{\zeta}
\]

by means of (2.3), H"older's inequality and Lemma 4 we obtain

\[
\left| f(z) - \sum_{m=1}^{n} a_m(f) F_m'(1/z) \right| \leq \frac{c_3 \| f \|_{A^2(G_2)}}{\pi \sqrt{1 - k^2}} \left( \int \left[ \frac{z^2 \psi'(w)}{(\psi(w) - z)^2} \sum_{m=1}^{n} \frac{F_m'(1/z)}{w^{m+1}} \right]^2 d\sigma_w \right)^{1/2},
\]

for every \( z \in M \), where the constant \( c_3 \) depends only on \( L \).

Let \( 1 < r < R < \infty \). In view of (1.2)

\[
\int_{r < |w| < R} \left[ \frac{z^2 \psi'(w)}{(\psi(w) - z)^2} \sum_{m=1}^{n} \frac{F_m'(1/z)}{w^{m+1}} \right]^2 d\sigma_w
\]

by letting \( r \to 1^+ \) and \( R \to \infty \) we get

\[
\int_{C_D} \left[ \frac{z^2 \psi'(w)}{(\psi(w) - z)^2} + \sum_{m=1}^{n} \frac{F_m'(1/z)}{w^{m+1}} \right]^2 d\sigma_w \leq 4\pi \sum_{m=1}^{n} \frac{|F_m'(1/z)|^2}{m+1},
\]

Therefore, by (3.1), (3.2) and Lemma 3 we conclude that

\[
\sum_{m=1}^{\infty} a_m(f) F_m'(1/z)
\]

converges uniformly to \( f \) on \( M \).

**Proof of Corollary 1.2.** Let \( z \in G_2 \). By Theorem 1.1 we have

\[
P_n(1/z) = \sum_{m=1}^{\infty} a_m(P_n) F_m'(1/z).
\]
On the other hand, $P_n(1/z)$ can be written in the form

$$P_n(1/z) = \sum_{k=1}^{n+1} A_k F'_k(1/z),$$

with the specific coefficients $A_k$, $k = 1, 2, \ldots, n+1$. Let $y(z)$ be a canonical $K$-quasiconformal reflection relative to $L$. Since $y(z)$ is identical on $L$, by Green’s formulae we get

$$a_m(P_n) = \frac{1}{\pi} \int_{C\mathcal{D}} \frac{P_n \left[1/y(\psi(w))\right]}{w^{m+1}} \frac{\overline{\psi'(w)}}{|y(\psi(w))|^2} y'(\psi(w)) d\sigma_w$$

$$= \frac{1}{\pi} \int_{C\mathcal{D}} -\frac{\partial}{\partial \overline{w}} \left( F_k \left[1/y(\psi(w))\right] \right) \frac{1}{w^{m+1}} d\sigma_w$$

$$= \frac{1}{2\pi i} \int_{|w|=1} F_k \left[1/\psi(w)\right] \frac{1}{w^{m+1}} dw.$$

By (1.1)

$$F_m \left[1/\psi(w)\right] = w^m - Q_m(\psi(w)),$$

where $Q_m(\psi(w))$ is analytic in $C\mathcal{D}$, and then

$$\frac{1}{2\pi i} \int_{|w|=1} F_k \left[1/\psi(w)\right] \frac{1}{w^{m+1}} dw = QATOPD \begin{cases} 1, & \text{if } k = m, \\ 0, & \text{if } k \neq m, \end{cases}$$

(3.3)

which implies that $a_m(P_n) = A_m$, for $m = 1, \ldots, n+1$, and $a_m(P_n) = 0$ for all $m \geq n + 2$. Hence

$$P_n(1/z) = \sum_{m=1}^{n+1} a_m(P_n) F'_m(1/z).$$

**Proof of Theorem 1.3.** Let $y(z)$ be a canonical $K$-quasiconformal reflection relative to $L$ and

$$S_n(f, 1/z) := \sum_{m=1}^{n+1} a_m F'_m(1/z)$$

be the $n$th partial sum of

$$\sum_{m=1}^{\infty} a_m F'_m(1/z).$$

Using (3.3) it can be shown that

$$\lim_{n \to \infty} \frac{1}{\pi} \int_{C\mathcal{D}} S_n \left[1/y(\psi(w))\right] \frac{\overline{\psi'(w)}}{|y(\psi(w))|^2} y'(\psi(w)) d\sigma_w = a_m, \quad m = 1, 2, \ldots. \quad (3.4)$$
If $m$ and $n$ are natural numbers, then by using Hölder’s inequality and Lemma 4 we get

$$|a_m(f) - a_m| \leq \frac{1}{\pi} \left| \int_{C\mathcal{D}} \frac{f(y(\psi(w))) - S_n[1/y(\psi(w))] \overline{\psi'}(w)}{w^{m+1} |y(\psi(w))|^2} y(\psi(w)) d\sigma_w \right|$$

$$+ \frac{1}{\pi} \int_{C\mathcal{D}} \frac{S_n[1/y(\psi(w))] \overline{\psi'}(w)}{w^{m+1} |y(\psi(w))|^2} y(\psi(w)) d\sigma_w - a_m \right| \leq \frac{1}{\pi} \left( \int_{C\mathcal{D}} \frac{d\sigma_w}{|w|^{2m+2}} \right)^{1/2}$$

$$\times \left( \int_{C\mathcal{D}} \left| \frac{f(y(\psi(w)) - S_n[1/y(\psi(w))] |\psi'(w)|^2}{y(\psi(w))} \right|^2 d\sigma_w \right)^{1/2}$$

$$+ \frac{1}{\pi} \int_{C\mathcal{D}} \frac{S_n[1/y(\psi(w))] \overline{\psi'}(w)}{w^{m+1} |y(\psi(w))|^2} y(\psi(w)) d\sigma_w - a_m \right| \leq \frac{c_4}{\sqrt{m\pi}} \left( \int_{\mathcal{G}_1} |(f - S_n) \circ y(\zeta)|^2 |y(\zeta)|^2 d\sigma_\zeta \right)^{1/2}$$

$$+ \frac{1}{\pi} \int_{C\mathcal{D}} \frac{S_n[1/y(\psi(w))] \overline{\psi'}(w)}{w^{m+1} |y(\psi(w))|^2} y(\psi(w)) d\sigma_w - a_m \right| \leq \frac{c_4 \|f - S_n\|_{A^2(G_2)}}{\sqrt{m\pi(1 - K^2)}}$$

$$+ \frac{1}{\pi} \int_{C\mathcal{D}} \frac{S_n[1/y(\psi(w))] \overline{\psi'}(w)}{w^{m+1} |y(\psi(w))|^2} y(\psi(w)) d\sigma_w - a_m \right|.$$  

Since $\lim_{n \to \infty} \|f - S_n\|_{A^2(G_2)} = 0$, (3.4) and (3.5) show that $a_m(f) = a_m$, $m = 1, 2, \ldots$.

**Proof of Theorem 1.4.** Let $y(z)$ be a canonical $K$-quasiconformal reflection with respect to $L$, and $P_n^*(1/z)$ the best approximant polynomial to $f \in A^2(G_2)$ in the norm $\|\|_{A^2(G_2)}$. For $z \in G_2$, by means of Hölder’s inequality, Lemma 4 and
Corollary 1.2 we obtain
\[ |f(z) - S_n(f, 1/z)| \leq |f(z) - P_n^*(1/z)| + |P_n^*(1/z) - S_n(f, 1/z)| \]
\[ \leq |f(z) - P_n^*(1/z)| + \left| \sum_{m=1}^{n+1} (a_m(P_n^*) - a_m(f)) F_n'(1/z) \right| \]
\[ \leq |f(z) - P_n^*(1/z)| \]
\[ + \frac{1}{\pi} \left| \int_{G} \left( f \circ y - P_n^* \circ y \right)(\psi(w)) \overline{\psi'(w)} y \overline{\psi}(\psi(w)) \sum_{m=1}^{n+1} \frac{F_n'(1/z)}{w^{m+1}} \right| \]
\[ \leq |f(z) - P_n^*(1/z)| \]
\[ + \frac{1}{\pi} \left( \int_{G} \left| f \circ y - P_n^* \circ y \right| |\psi'(w)| |y \overline{\psi}(\psi(w))|^2 \right)^{1/2} \]
\[ \times \left( \int_{G} \sum_{m=1}^{n+1} \left| \frac{F_n'(1/z)}{w^{m+1}} \right|^2 d\sigma_w \right)^{1/2} \]
\[ \leq |f(z) - P_n^*(1/z)| + \frac{c_5}{\pi} \left( \int_{G} \left| f \circ y - P_n^* \circ y \right| |\psi'(w)| \right)^{1/2} \]
\[ \times \left( \sum_{m=1}^{n+1} \left| \frac{F_n'(1/z)}{m} \right|^2 \right)^{1/2} \]
\[ \leq |f(z) - P_n^*(1/z)| + \frac{c_5}{\sqrt{\pi(1 - k^2)}} \left\| f - P_n^* \right\|_{A^2(G_2)} \left( \sum_{m=1}^{n+1} \left| \frac{F_n'(1/z)}{m} \right|^2 \right)^{1/2} \]
\[ = |f(z) - P_n^*(1/z)| + \frac{c_5}{\sqrt{\pi(1 - k^2)}} E_n(f, G_2) \left( \sum_{m=1}^{n+1} \frac{F_n'(1/z)^2}{m} \right)^{1/2} \]
for all natural numbers $n$. This shows that

\[ |f(z) - S_n(f, 1/z)|^2 \leq 2 |f(z) - P_n^*(1/z)|^2 + \frac{2c_5}{\pi(1 - k^2)} E_n^2(f, G_2) \sum_{m=1}^{n+1} \frac{F_n'(1/z)^2}{m}. \]
Multiplying both sides by $1/|z|^4$ and take into account that $1/|z|^4 \leq c_6$ for $z \in G_2$ and with a constant $c_6$, we get

$$|f(z) - S_n(f,1/z)|^2 \frac{1}{|z|^4} \leq c_7 |f(z) - P_n^*(1/z)|^2 + \frac{c_8}{\pi(1-k^2)} E_n^2(f,G_2) \sum_{m=1}^{n+1} \frac{|F_{m,z}^*(1/z)|^2}{m}.$$ 

Now, by integrating both sides over $G_2$ and by virtue of Lemma 2 we get

$$\|f(z) - S_n(f,\cdot)\|_{A^2(G_2,\omega)}^2 \leq c_7 E_n^2(f,G_2) + \frac{c_8}{\pi(1-k^2)} E_n^2(f,G_2) \sum_{m=1}^{n+1} \frac{\|F_{m,z}^*(1/z)\|_{A^2(G_2)}^2}{m}$$

i.e.,

$$\|f(z) - S_n(f,\cdot)\|_{A^2(G_2,\omega)} \leq \left( c_7 + \frac{c_8(n+1)}{1-k^2} \right) E_n^2(f,G_2)$$

for all natural numbers $n$.

References

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