SOME SMALL DEVIATION THEOREMS FOR THE SEQUENCE OF BINARY RANDOM TRUNCATED FUNCTIONS ON A HOMOGENEOUS TREE

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Abstract. In this paper, a class of small-deviation theorems for the sequence of random truncated functions on arbitrary random field indexed by the homogeneous tree are established by utilizing asymptotic logarithmic likelihood ratio on the homogeneous tree. As corollaries, some strong limit theorems for the arbitrary random field on the homogeneous tree are obtained.

1. Introduction.

Let $T$ be a homogeneous tree on which each vertex has $N+1$ neighboring vertices. We first fix any vertex as the "root" and label it by 0. Let $\sigma, \tau$ be vertices of a tree. Write $\tau \preceq \sigma$ if $\tau$ is on the unique path connecting 0 to $\sigma$, $|\sigma|$ for the number of edges on this path. For any two vertices $\sigma, \tau$, denote $\sigma \wedge \tau$ the vertex farthest from 0 satisfying

$$\sigma \wedge \tau \leq \sigma, \text{ and } \sigma \wedge \tau \leq \tau.$$ 

If $\sigma \neq 0$, then we let $\bar{\sigma}$ stand for the vertex satisfying $\bar{\sigma} \leq \sigma$ and $|\bar{\sigma}| = |\sigma| - 1$ (we refer to $\sigma$ as a son of $\bar{\sigma}$). It is easy to see that the root has $N+1$ sons and all other vertices have $N$ sons. The homogeneous tree $T$ is also called Bethe tree $T_{B,N}$. For example, we give the following Fig 1 $T_{B,2}$.

![Bethe tree $T_{B,2}$](image)

Fig 1. Bethe tree $T_{B,2}$

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**Definition 1** (see [6]). Let $T$ be a homogeneous tree, $G = \{s_0, s_1, s_2, \cdots \}$ be a countable alphabet-set space, $\{X_\sigma, \sigma \in T\}$ be a collection of $G$-valued random variables defined on the measurable space $(\Omega, \mathcal{F})$. Let
\[
q = \{q(x), x \in G\}
\]
be a distribution on $G$, and
\[
Q_n = (Q_n(y|x)), \ x, y \in G, \ n \geq 1.
\]
be a series of stochastic matrices on $G^2$. If for any vertices $\sigma, \tau$,
\[
Q(X_\sigma = y|X_\sigma = x, \ and \ X_\tau \ for \ \sigma \wedge \tau \leq \sigma) = Q(X_\sigma = y|X_\sigma = x) = Q_n(y|x) \ \forall x, y \in G, \ n \geq 1.
\]
and
\[
Q(X_0 = x) = q(x), \ \forall x \in G.
\]
$\{X_\sigma, \sigma \in T\}$ will be called $G$-valued nonhomogeneous Markov chains indexed by a homogeneous tree with the initial distribution (1) and transition matrices (2), or called tree-indexed nonhomogeneous Markov chains.

Two special finite tree-indexed Markov chains are introduced in Kemeny et al. (1976 [10]), Spitzer (1975 [11]), and there the finite transition matrix is assumed to be positive and reversible to its stationary distribution, and this tree-indexed Markov chains ensure that the cylinder probabilities are independent of the direction we travel along a path. In this paper, we have no such assumption.

If $|\sigma| = n$, it is said to be on the $n$th level on a tree $T$. We denote by $T^{(n)}$ the subtree of $T$ containing the vertices from level 0 (the root) to level $n$, and $L_n$ the set of all vertices on the level $n$. Let $B$ be a subgraph of $T$. Denote $X^B = \{X_\sigma, \sigma \in B\}$, and denote by $|B|$ the number of vertices of $B$. Let $S(\sigma)$ be the set of all sons of vertices $\sigma$. It is easy to see that $|S(0)| = N + 1$ and $|S(\sigma)| = N$, where $\sigma \neq 0$. In particular,
\[
|T^{(n)}| = 1 + N + 1 + (N + 1)N + \cdots (N + 1)N^{n-1} = 1 + \frac{(N + 1)(N^n - 1)}{N - 1}.
\]
Let $\Omega = G^T$, $\omega = \omega(\cdot) \in \Omega$, where $\omega(\cdot)$ is a function defined on $T$ and taking values in $G$, and $\mathcal{F}$ be the smallest Borel field containing all cylinder sets in $\Omega$, $\mu$ be the probability measure on $(\Omega, \mathcal{F})$. Let $X = \{X_\sigma, \sigma \in T\}$ be the coordinate stochastic process defined on the measurable space $(\Omega, \mathcal{F})$; that is, for any $\omega = \{\omega(t), t \in T\}$, define
\[
X_t(\omega) = \omega(t), \ t \in T^{(n)}.
\]
\[
X^{T^{(n)}} = \{X_t, t \in T^{(n)}\}, \ \mu(X^{T^{(n)}} = x^{T^{(n)})} = \mu(x^{T^{(n)})}.
\]
Now we give a definition of Markov chain fields on the tree $T$ by using the cylinder distribution directly, which is a natural extension of the classical definition of Markov chains (see[9]).

**Definition 2.** Let $\{Q_n = (Q_n(j|i)), i, j \in G, \ n \geq 1\}$ and $q = (q(s_0), q(s_1), q(s_2), \cdots)$ be defined as before, $\mu_Q$ be another probability measure on $(\Omega, \mathcal{F})$. If
\[
\mu_Q(x_0) = q(x_0),
\]
\[
\mu_Q(x^{T^{(n)}}) = q(x_0) \prod_{k=0}^{n-1} \prod_{\sigma \in L_k} \prod_{\tau \in S(\sigma)} Q_{k+1}(x_\tau|x_\sigma), \ n \geq 1,
\]
then $X = \{X_\sigma, \sigma \in T\}$ will be called a nonhomogeneous Markov chains field indexed by the homogeneous tree $T$ determined by the initial distribution $q$ and the stochastic matrices $Q_n$.

The tree models have drawn increasing interest from specialists in physics, probability and information theory. Benjamini and Peres have given the notion of the tree-indexed homogeneous Markov chains and studied the recurrence and ray-recurrence for them (see [1]). Berger and Ye have studied the existence of entropy rate for some stationary random fields on a homogeneous tree (see [2]). Pemantle proved a mixing property and a weak law of large numbers for a PPG-invariant and ergodic random field on a homogeneous tree (see [3]). Ye and Berger, by using Pemantle’s result and a combinatorial approach, have studied the asymptotic equipartition property (AEP) in the sense of convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree (see [4-5]). Yang and Liu have studied a strong law of large numbers and the limit properties for the states frequency of Markov chains field on trees (see [8]). Recently, Liu and Yang have studied some limit theorems for binary functions of countable homogeneous Markov chains indexed by a homogeneous tree and strong law of large numbers and the asymptotic equipartition property for finite homogeneous Markov chains indexed by a homogeneous tree (see [7] and [13]). Yang and Liu have investigated all kinks of applications of Markov processes in the economic management and optimization controls (see [14-39]). But their results only concern the case of strong limit theorems for Markov chains field, they do not discuss the limit properties for arbitrary random fields.

In this paper, our aim is to establish some strong limit theorems represented by inequalities which are also called strong deviation theorems for binary truncated function sequence of dependent random field on the homogeneous tree by use of analytical techniques, the tools of asymptotic logarithmic likelihood ratio defined by (15) and establishment of nonnegative sup-martingale. As corollaries, some strong laws of large numbers for the arbitrary random field on the homogeneous tree are obtained.

2. Main Results and its Proof.

**Lemma 1.** Let $\mu_1$ and $\mu_2$ be two probability measures on $(\Omega, \mathcal{F})$, $D \in \mathcal{F}$, denote $\alpha > 0$. Let $\{\sigma_n, n \geq 0\}$ be a nonnegative stochastic sequence such that

$$
\liminf_n \frac{\sigma_n}{\alpha^\alpha} > 0, \ \mu_1 - a.s. \ \omega \in D,
$$

then

$$
\limsup_{n \to \infty} \frac{1}{\sigma_n} \log \frac{\mu_2(X^{T(n)})}{\mu_1(X^{T(n)})} \leq 0, \ \mu_1 - a.s. \ \omega \in D.
$$

In particular, if $\sigma_n = |T^{(n)}|$, we have

$$
\limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \log \frac{\mu_2(X^{T^{(n)}})}{\mu_1(X^{T^{(n)}})} \leq 0, \ \mu_1 - a.s.
$$
Proof. Let $Z_n = \mu_2(X^{T(n)})/\mu_1(X^{T(n)})$. It is easy to see that $E_{\mu_1}(Z_n) \leq 1$ (see[9]), where $E_{\mu_1}$ denotes the expectation under the measure $\mu_1$. Hence for all $\varepsilon > 0$, we get by Markov’s inequality that
\[
\sum_{n=1}^{\infty} \mu_1 \left\{ \frac{1}{n^\alpha} \log \frac{\mu_2(X^{T(n)})}{\mu_1(X^{T(n)})} \geq \varepsilon \right\} \\
= \sum_{n=1}^{\infty} \mu_1 \left\{ \frac{1}{n^\alpha} \log Z_n \geq \varepsilon \right\} = \sum_{n=1}^{\infty} \mu_1(\log Z_n \geq \varepsilon n^\alpha) \\
= \sum_{n=1}^{\infty} \mu_1(Z_n \geq \exp(\varepsilon n^\alpha)) \leq \sum_{n=1}^{\infty} \frac{E_{\mu_2}(Z_n)}{\exp(\varepsilon n^\alpha)} \\
\leq \sum_{n=1}^{\infty} \exp(-\varepsilon n^\alpha) = \sum_{n=1}^{\infty} \frac{1}{\varepsilon n^\alpha} < \infty. \tag{12}
\]
Since $\varepsilon$ is arbitrary, by Borel-Cantelli lemma it follows from (12) that
\[
\limsup_{n \to \infty} \frac{1}{n^\alpha} \log \frac{\mu_2(X^{T(n)})}{\mu_1(X^{T(n)})} \leq 0, \mu_1 - a.s. \tag{13}
\]
Obviously (9) and (13) imply that (10) holds.

If $\sigma_n = |T(n)|$, we obtain by (5) that
\[
\liminf_{n \to \infty} \frac{|T(n)|}{n^\alpha} = \liminf_{n \to \infty} \frac{(N+1)(N^n - 1)}{(N-1)n^\alpha} = (N+1) \liminf_{n \to \infty} \frac{N^n}{n^\alpha} = \infty > 0, \tag{14}
\]
which together with (13) gives (11).

Let
\[
\gamma(\omega) = \limsup_{n \to \infty} \frac{1}{\sigma_n(\omega)} \log \frac{\mu(X^{T(n)})}{\mu_Q(X^{T(n)})}. \tag{15}
\]
$\gamma(\omega)$ is called the generalized sample relative entropy rate with respect to $\mu$ and $\mu_Q$. $\gamma(\omega)$ is also called generalized asymptotic logarithmic likelihood ratio. Letting $\mu_1 = \mu, \mu_2 = \mu_Q$, by (10) and (15) we have
\[
\gamma(\omega) \geq \liminf_{n \to \infty} \frac{1}{\sigma_n(\omega)} \log \frac{\mu(X^{T(n)})}{\mu_Q(X^{T(n)})} \\
\geq 0, \mu - a.s. \omega \in D. \tag{16}
\]
Hence $\gamma(\omega)$ can be look on as a type of measure of the deviation between the arbitrary random field and the nonhomogeneous Markov chains field on the homogeneous tree.

Although $\gamma(\omega)$ is not a proper metric between the probability measures, we nevertheless think of it as a measure of "dissimilarity" between their true measure $\mu$ and nonhomogeneous Markov measure $\mu_Q$. Obviously, $\gamma(\omega) = 0$ if and only if $\mu = \mu_Q$. It has been shown in (16) that $\gamma(\omega) \geq 0, \mu - a.s. \omega \in D$. Hence, $\gamma(\omega)$ can be used as a random measure of the deviation between the true joint distribution $\mu(x^{T(n)})$ and the nonhomogeneous Markov distribution $\mu_Q(x^{T(n)})$. Roughly speaking, this deviation may be regarded as the one between $x^{T(n)}$ and the nonhomogeneous Markov case. The smaller $\gamma(\omega)$ is, the smaller the deviation is.
\textbf{Theorem 1.} Let $X = \{X_\sigma, \sigma \in T\}$ be an arbitrary random field indexed by the homogeneous tree defined by (6). \{\{f_n(x, y), n \geq 0\}\} be a real-valued function sequence defined on $G^2$; \{\{\sigma_n(\omega), n \geq 0\}\}, $\gamma(\omega)$ be defined as above. Let \{\{a_n(\omega), n \geq 0\}\} be a positive-valued increasing stochastic sequence, \{\{g_n(x), n \geq 0\}\} be a series of continuous positive-valued even functions defined on $(-\infty, +\infty)$ which satisfy that when $|x|$ increases,

$$g_n(x) \uparrow, \quad \frac{g_n(x)}{x^2} \downarrow.$$  \hfill (17)

Denote

$$D(\omega) = \{\omega : \lim_{n \to \infty} \frac{1}{\sigma_n(\omega)} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} E_Q[g_k(f_k(X_\sigma, X_\tau)|X_\sigma)] = \sigma(\omega) < \infty\},$$  \hfill (18)

$$D(\alpha) = \{\omega : \lim_{n \to \infty} \frac{\sigma_n(\omega)}{n^\alpha} > 0, \quad \gamma(\omega) < \infty\}. \hfill (19)$$

Then

$$\limsup_{n \to \infty} \frac{1}{\sigma_n(\omega)} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{f_k(X_\sigma, X_\tau) - E_Q[f_k(X_\sigma, X_\tau)|X_\sigma]}{a_k(\omega)} \leq \alpha(\gamma(\omega), \sigma(\omega)), \quad \mu - \text{a.s. } \omega \in D(\alpha) \cap D(\omega). \hfill (21)$$

$$\liminf_{n \to \infty} \frac{1}{\sigma_n(\omega)} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{f_k(X_\sigma, X_\tau) - E_Q[f_k(X_\sigma, X_\tau)|X_\sigma]}{a_k(\omega)} \geq \beta(\gamma(\omega), \sigma(\omega)), \quad \mu - \text{a.s. } \omega \in D(\alpha) \cap D(\omega). \hfill (22)$$

Where

$$\alpha(\gamma(\omega), \sigma(\omega)) = \inf_{\lambda} \left\{ \frac{\gamma(\omega)}{\lambda} + \frac{1}{2} \lambda e^{2\lambda} \sigma(\omega), \quad \lambda > 0 \right\}, \quad 0 \leq \gamma(\omega), \sigma(\omega) < \infty. \hfill (23)$$

$$\beta(\gamma(\omega), \sigma(\omega)) = \sup_{\lambda} \left\{ \frac{\gamma(\omega)}{\lambda} + \frac{1}{2} \lambda e^{-2\lambda} \sigma(\omega), \quad \lambda < 0 \right\}, \quad 0 \leq \gamma(\omega), \sigma(\omega) < \infty. \hfill (24)$$

$$\alpha(0, \sigma(\omega)) = \alpha(\gamma(\omega), 0) = 0, \quad 0 \leq \gamma(\omega), \sigma(\omega) < \infty. \hfill (25)$$

$$\beta(0, \sigma(\omega)) = \beta(\gamma(\omega), 0) = 0, \quad 0 \leq \gamma(\omega), \sigma(\omega) < \infty. \hfill (26)$$

$$\lim_{x \to 0^+} \beta(x, y) = 0, \quad \lim_{x \to 0^+} \alpha(x, y) = 0. \hfill (27)$$

\(E_Q\) represents the expectation relative to the measure \(\mu_Q\).

\textbf{Proof.} We consider the probability measure space \((\Omega, \mathcal{F}, \mu)\), let \(\lambda\) be a constant, denote

$$M_k(\lambda; x_\sigma) = E_Q\{\exp[\lambda \frac{f_k(X_\sigma, X_\tau)}{a_k} |X_\sigma = x_\sigma] \}, \quad x_\tau \in G$$

$$= \sum_{x_\tau \in G} Q_{k+1}(x_\tau | x_\sigma) \exp\{\lambda \frac{f_k(x_\sigma, x_\tau)}{a_k} - E_Q(f_k(x_\sigma, x_\tau)|x_\sigma)\}. \hfill (30)$$
\[
m_k(\lambda; x_\sigma) = \frac{Q_{k+1}(x_\sigma | x_\sigma)}{M_k(\lambda; x_\sigma)} \exp\left\{ \lambda\left(\tilde{f}_k(x_\sigma, x_\sigma) - E_Q(\tilde{f}_k(x_\sigma, X_\tau | x_\sigma))\right) \right\}.
\]

(31)

\[
\mu_Q(\lambda; x_{T^{(n-1)}}) = q(x_0) \prod_{\ell=0}^{n-1} \prod_{\sigma \in L_\ell} \prod_{\tau \in S(\sigma)} m_k(\lambda; x_\sigma).
\]

(32)

It follows from (30)-(32) that

\[
\sum_{x^n \in G} \mu_Q(\lambda; x_{T^{(n)}}) = \sum_{x^n \in G} q(x_0) \prod_{\ell=0}^{n-1} \prod_{\sigma \in L_\ell} \prod_{\tau \in S(\sigma)} m_k(\lambda; x_\sigma)
\]

\[= \sum_{x^n \in G} q(x_0) \prod_{\ell=0}^{n-1} \prod_{\sigma \in L_\ell} \prod_{\tau \in S(\sigma)} \frac{Q_{k+1}(x_\sigma | x_\sigma)}{M_k(\lambda; x_\sigma)} \times \exp\left\{ \lambda\left(\tilde{f}_k(x_\sigma, x_\tau) - E_Q(\tilde{f}_k(x_\sigma, x_\tau | x_\tau))\right) \right\} \]

\[= \mu_Q(\lambda; x_{T^{(n-1)}}) \sum_{x^n \in G, \sigma \in L_{n-1}} \prod_{\ell=0}^{n-1} \prod_{\tau \in S(\sigma)} \frac{Q_{n}(x_\tau | x_\sigma)}{M_{n-1}(\lambda; x_\sigma)} \times \exp\left\{ \lambda\left(\tilde{f}_{n-1}(x_\sigma, x_\tau) - E_Q(\tilde{f}_{n-1}(x_\sigma, x_\tau | x_\tau))\right) \right\} \]

\[= \mu_Q(\lambda; x_{T^{(n-1)}}) \prod_{\sigma \in L_{n-1}} \prod_{\tau \in S(\sigma)} \sum_{x_\tau \in G} \frac{Q_{n}(x_\tau | x_\sigma)}{M_{n-1}(\lambda; x_\sigma)} \times \exp\left\{ \lambda\left(\tilde{f}_{n-1}(x_\sigma, x_\tau) - E_Q(\tilde{f}_{n-1}(x_\sigma, x_\tau | x_\tau))\right) \right\} \]

\[= \mu_Q(\lambda; x_{T^{(n-1)}}) \prod_{\sigma \in L_{n-1}} \prod_{\tau \in S(\sigma)} \frac{E^\left\{ e^{\lambda(\tilde{f}_{n-1}(X_\sigma, X_\tau) - E_Q(\tilde{f}_{n-1}(X_\sigma, X_\tau | X_\tau)))/a_{n-1}|X_\sigma = x_\sigma} \right\}}{E^\left\{ e^{\lambda(\tilde{f}_{n-1}(X_\sigma, X_\tau) - E_Q(\tilde{f}_{n-1}(X_\sigma, X_\tau | X_\tau)))/a_{n-1}|X_\sigma = x_\sigma} \right\}} \]

\[= \mu_Q(x_{T^{(n-1)}}; \lambda).
\]

(33)

Therefore \(\mu_Q(x_{T^{(n)}}; \lambda), n = 1, 2, \ldots\) are a family of consistent distribution functions on \(G_{T^{(n)}}\). Let

\[
U_n(\lambda, \omega) = \frac{\mu_Q(X_{T^{(n)}}; \lambda)}{\mu(X_{T^{(n)}})}.
\]

(34)

It is easy to see that \(\{U_n(\lambda, \omega), F_n, n \geq 1\}\) (where \(F_n = \sigma(X_{T^{(n)}})\)) is a nonnegative sup-martingale from Doob’s martingale convergence theorem (see [12]). Moreover,

\[
\lim_{n \to \infty} U_n(\lambda, \omega) = U_\infty(\lambda, \omega) < \infty, \mu - a.s.
\]

(35)

By the first inequality of (19), (34) and Lemma 1, we get

\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \log U_n(\lambda, \omega) \leq 0, \mu - a.s. \omega \in D(\alpha).
\]

(36)
By \((30-32), (34)\) and \((36)\), we have
\[
\frac{1}{\sigma_n} \log U_n(\lambda, \omega) = \frac{\lambda}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \tilde{f}_k(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_k(X_{\sigma}, X_{\tau})|X_{\sigma}) \leq \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \log M_k(\lambda; x_{\sigma})
\]
\[
- \frac{1}{\sigma_n} \log \left[ \mu(X^{T(n)}) \right] \leq \frac{1}{\sigma_n} \log \left[ \frac{\mu(X^{T(n)})}{q(x_0)} \prod_{k=0}^{n-1} \prod_{\sigma \in L_k} \prod_{\tau \in S(\sigma)} Q_{k+1}(x_{\tau}|x_{\sigma}) \right].
\]
\[
\mu - a.s. \omega \in D(\alpha)
\]
By \((15), (36)\) and \((37)\), we have
\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \tilde{f}_k(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_k(X_{\sigma}, X_{\tau})|X_{\sigma}) \leq \limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \log M_k(\lambda; x_{\sigma}) + \gamma(\omega). \mu - a.s. \omega \in D(\alpha).
\]
By \((20)\) and the property of the conditional expectation we obtain
\[
E_Q \left[ \frac{\lambda(\tilde{f}_n(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_n(X_{\sigma}, X_{\tau})|X_{\sigma}))}{a_n} \right] = 0. \tag{39}
\]
\[
\left| \frac{\tilde{f}_n(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_n(X_{\sigma}, X_{\tau})|X_{\sigma})}{a_n} \right| \leq \frac{\tilde{f}_n(X_{\sigma}, X_{\tau})}{a_n} + \frac{E_Q \left( \left| \frac{\tilde{f}_n(X_{\sigma}, X_{\tau})}{a_n} \right| X_{\sigma} \right)}{a_n} \leq 2. \tag{40}
\]
By the inequality \(e^x - 1 - x \leq (1/2)x^2e^{|x|}\) and \((30), (39)\) and \((40)\), we arrive at
\[
0 \leq M_n(\lambda; x_{\sigma}) - 1
\]
\[
= E_Q \left\{ \exp \left[ \frac{\lambda(\tilde{f}_n(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_n(X_{\sigma}, X_{\tau})|X_{\sigma}))}{a_n} \right] - 1 \right\}
\]
\[
\leq \frac{1}{2} \lambda^2 E_Q \left\{ e^{\lambda \left| \tilde{f}_n(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_n(X_{\sigma}, X_{\tau})|X_{\sigma}) \right|} \times \left( \frac{\tilde{f}_n(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_n(X_{\sigma}, X_{\tau})|X_{\sigma})}{a_n} \right)^2 \right\}
\]
\[
\leq \frac{1}{2} \lambda^2 e^{2\lambda |X_{\tau}|} E_Q \left\{ \left( \frac{\tilde{f}_n(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_n(X_{\sigma}, X_{\tau})|X_{\sigma})}{a_n} \right)^2 \right\} \left| X_{\sigma} \right|
\]
\[
= \frac{1}{2} \lambda^2 e^{2\lambda |X_{\tau}|} E_Q(\tilde{f}_n(X_{\sigma}, X_{\tau})^2|X_{\sigma}) - (E_Q(\tilde{f}_n(X_{\sigma}, X_{\tau})|X_{\sigma}))^2
\]
On the other hand, \( g_n(x)/x^2 \) decreases as \(|x|\) increases, so we obtain
\[
\frac{x}{a_n}^2 \leq \frac{g_n(x)}{g_n(a_n)}, \ |x| \leq a_n.
\]

According to the assumption, \( g_n(x)/x^2 \) decreases as \(|x|\) increases, so we obtain
\[
\frac{x}{a_n}^2 \leq \frac{g_n(x)}{g_n(a_n)}, \ |x| \leq a_n.
\]

On the other hand, \( g_n(x) \uparrow \) as \(|x|\) increases, therefore it follows from (42) that
\[
\left( \frac{\bar{f}_n(X_\sigma, X_\tau)}{a_n} \right)^2 \leq \frac{g_n(\bar{f}_n(X_\sigma, X_\tau))}{g_n(a_n)} \leq \frac{g_n(f_n(X_\sigma, X_\tau))}{g_n(a_n)}.
\]

By (41) and (43), we get
\[
0 \leq M_n(\lambda; x_\sigma) - 1 \leq \frac{1}{2} \lambda^2 e^{2|\lambda|} E_Q \left\{ \frac{g_n(f_n(X_\sigma, X_\tau))}{g_n(a_n)} X_\sigma \right\}.
\]

It follows from (44) and (18) that
\[
0 \leq \limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} (M_k(\lambda; x_\sigma) - 1)
\leq \frac{1}{2} \lambda^2 e^{2|\lambda|} \limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} E_Q \left( \frac{g_k(\bar{f}_k(X_\sigma, X_\tau))}{g_k(a_k)} \right) X_\sigma
\leq \frac{1}{2} \lambda^2 e^{2|\lambda|} \sigma(\omega), \ \mu - a.s. \omega \in D(\alpha) \cap D.
\]

By use of the inequality \( 0 \leq \ln x \leq x - 1 \) \((x > 1)\), we have from (45) that
\[
0 \leq \limsup_{n \to \infty} \frac{1}{\sigma_k} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \log M_k(\lambda; x_\sigma) \leq \frac{1}{2} \lambda^2 e^{2|\lambda|} \sigma(\omega), \ \mu - a.s. \omega \in D(\alpha) \cap D.
\]

By applying (38) and (46), we obtain
\[
\limsup_{n \to \infty} \frac{\lambda}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{\bar{f}_k(X_\sigma, X_\tau) - E_Q(\bar{f}_k(X_\sigma, X_\tau)|X_\sigma)}{a_k} \leq \frac{1}{2} \lambda^2 e^{2|\lambda|} \sigma(\omega) + \gamma(\omega),
\]
\[
\mu - a.s. \omega \in D(\alpha) \cap D.
\]

Denote the set of the positive rational numbers by \( Q^* \), letting \( \lambda \in Q^* \), dividing two sides of (47) by \( \lambda \), we have
\[
\limsup_{n \to \infty} \frac{\lambda}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{\bar{f}_k(X_\sigma, X_\tau) - E_Q(\bar{f}_k(X_\sigma, X_\tau)|X_\sigma)}{a_k} \leq \frac{1}{2} \lambda e^{2\lambda} \sigma(\omega) + \frac{\gamma(\omega)}{\lambda},
\]
\[
\mu - a.s. \omega \in D(\alpha) \cap D.
\]

Noticing that \( h(\lambda, \gamma(\omega), \sigma(\omega)) = (1/2) \lambda e^{2\lambda} \sigma(\omega) + \gamma(\omega)/\lambda \) is continuous with respect to \( \lambda \), there exist \( \lambda_n(\omega) \in Q^* \), \( n = 1, 2, \cdots \) such that
\[
\lim_{n \to \infty} h(\lambda_n(\omega), \gamma(\omega), \sigma(\omega)) = \alpha(\gamma(\omega), \sigma(\omega)).
\]
By (23), (48) we obtain
\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{\hat{f}_k(X_\sigma, X_\tau) - E_Q(\hat{f}_k(X_\sigma, X_\tau)|X_\sigma)}{a_k} \leq \alpha(\gamma(\omega), \sigma(\omega)),
\]
\[
\mu - \text{a.s. } \omega \in D(\alpha) \cap D.
\]  
(50)

Denote the set of the negative rational numbers by \( Q_* \), letting \( \lambda \in Q_* \), dividing two sides of (47) by \( \lambda \), we have
\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{\hat{f}_k(X_\sigma, X_\tau) - E_Q(\hat{f}_k(X_\sigma, X_\tau)|X_\sigma)}{a_k} \geq \frac{\lambda}{\lambda} \gamma(\omega) + \frac{1}{2} \lambda e^{-2\lambda(\omega)},
\]
\[
\mu - \text{a.s. } \omega \in D(\alpha) \cap D.
\]  
(51)

Analogously, by (24) we know that there exist \( \lambda_n(\omega) \in Q_* \), \( n = 1, 2, \ldots \) such that
\[
\lim_{n \to \infty} \left[ \frac{1}{2} \lambda_n(\omega) e^{-2\lambda(\omega)} \sigma(\omega) + \frac{\gamma(\omega)}{\lambda_n(\omega)} \right] = \beta(\gamma(\omega), \sigma(\omega)),
\]
which together with (51) gives
\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{\hat{f}_k(X_\sigma, X_\tau) - E_Q(\hat{f}_k(X_\sigma, X_\tau)|X_\sigma)}{a_k} \geq \beta(\gamma(\omega), \sigma(\omega)),
\]
\[
\mu - \text{a.s. } \omega \in D(\alpha) \cap D.
\]  
(52)

Hence (21) and (22) hold. According to the definition of \( \alpha(\gamma(\omega), \sigma(\omega)) \) and \( \beta(\gamma(\omega), \sigma(\omega)) \), (25-28) hold obviously.

**Lemma 2** ([9]). Let \( \{a_n, n \geq 0\} \), \( \{\sigma_n, n \geq 0\} \) be increasing positive-valued sequences, \( \{x_n, n \geq 0\} \) be a real number sequence. If either of \( \{a_n, n \geq 0\} \) and \( \{\sigma_n, n \geq 0\} \) is unbounded, and
\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \frac{x_k}{a_k} \leq b, \quad \liminf_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \frac{x_k}{a_k} \geq a,
\]  
(53)

where \( a \leq 0, b \geq 0 \), then
\[
\limsup_{n \to \infty} \frac{1}{a_n \sigma_n} \sum_{k=1}^{n} x_k \leq b - a, \quad \liminf_{n \to \infty} \frac{1}{a_n \sigma_n} \sum_{k=1}^{n} x_k \geq a - b.
\]  
(54)

**Corollary 1.** Under the assumption of Theorem 1, we have
\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{\hat{f}_k(X_\sigma, X_\tau) - E_Q(\hat{f}_k(X_\sigma, X_\tau)|X_\sigma)}{a_k} \leq \sqrt{\gamma(\omega)}\left[1 + \frac{1}{2} e^{2\sqrt{\gamma(\omega)} \sigma(\omega)}\right],
\]
\[
\mu - \text{a.s. } \omega \in D(\alpha) \cap D.
\]  
(55)
\[
\lim_{n \to \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \left( \tilde{f}_k(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_k(X_{\sigma}, X_{\tau}) | X_\sigma) \right) a_k \geq -\sqrt{\gamma(\omega)[1 + \frac{1}{2}e^{2\sqrt{\gamma(\omega)}\sigma(\omega)}]}.
\]

\[
\mu - \text{a.s. } \omega \in D(\alpha) \cap D. \tag{56}
\]

**Proof.** Letting \( \lambda = \sqrt{\gamma(\omega)} \) in Theorem 1, from (23) we obtain

\[
\alpha(\gamma(\omega), \sigma(\omega)) \leq \sqrt{\gamma(\omega)} + (1/2)\sqrt{\gamma(\omega)}e^{2\sqrt{\gamma(\omega)}\sigma(\omega)}, \ \gamma(\omega) \geq 0. \tag{57}
\]

It follows from (21) and (57) that (55) holds. Letting \( \lambda = -\sqrt{\gamma(\omega)} \) in Theorem 1, from (24) we obtain

\[
\beta(\gamma(\omega), \sigma(\omega)) \geq -\sqrt{\gamma(\omega)} - (1/2)\sqrt{\gamma(\omega)}e^{2\sqrt{\gamma(\omega)}\sigma(\omega)}, \ \gamma(\omega) \geq 0. \tag{58}
\]

(56) follows from (22) and (58).

**Corollary 2.** Under the assumption of Theorem 1, we have

\[
\limsup_{n \to \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \left( \tilde{f}_k(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_k(X_{\sigma}, X_{\tau}) | X_\sigma) \right) \leq \sqrt{\gamma(\omega)[2 + e^{2\sqrt{\gamma(\omega)}\sigma(\omega)}]},
\]

\[
\mu - \text{a.s. } \omega \in D(\alpha) \cap D. \tag{59}
\]

\[
\liminf_{n \to \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \left( \tilde{f}_k(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_k(X_{\sigma}, X_{\tau}) | X_\sigma) \right) \geq -\sqrt{\gamma(\omega)[2 + e^{2\sqrt{\gamma(\omega)}\sigma(\omega)}]},
\]

\[
\mu - \text{a.s. } \omega \in D(\alpha) \cap D. \tag{60}
\]

**Proof.** (59), (60) follow from (55), (56) and Lemma 2.

**Corollary 3.** Under the assumption of Theorem 1, if \( \sigma(\omega) = 0 \) or \( \gamma(\omega) = 0 \), a.s., we have

\[
\lim_{n \to \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \left( \tilde{f}_k(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_k(X_{\sigma}, X_{\tau}) | X_\sigma) \right) a_k = 0,
\]

\[
\mu - \text{a.s. } \omega \in D(\alpha) \cap D. \tag{61}
\]

\[
\lim_{n \to \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \{ \tilde{f}_k(X_{\sigma}, X_{\tau}) - E_Q(\tilde{f}_k(X_{\sigma}, X_{\tau}) | X_\sigma) \} = 0.
\]

\[
\mu - \text{a.s. } \omega \in D(\alpha) \cap D. \tag{62}
\]

**Proof.** When \( \sigma(\omega) = 0 \), a.s., \( \alpha(\gamma(\omega), 0) = \beta(\gamma(\omega), 0) = 0 \). (21) and (22) imply that (61) holds. Analogously, when \( \gamma(\omega) = 0 \), a.s., \( \alpha(0, \sigma(\omega)) = \beta(0, \sigma(\omega)) = 0 \). (61) also follows from (21), (22). According to (61) and Lemma 2, (62) holds.
3. Strong Laws of Large Numbers for Arbitrary Random Field Indexed by the Homogeneous Tree.

**Theorem 2.** Let \(\{X_\sigma, \sigma \in T\}\) be the random field on the homogeneous tree which satisfies the assumption of Theorem 1, we replace (17) with the condition as follows: as \(|x|\) increases,

\[
g_n(x) \quad \frac{g_n(x)}{|x|} \uparrow, \quad \frac{g_n(x)}{x^2} \downarrow.
\]

If for each \(\tau \in T^{(n)}\), \(E_Q(X_\tau | X_\sigma) = 0\) holds, and

\[
\sum_{k=0}^{\infty} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \sum_{x : |x| > a_k} Q_{k+1}(x_\tau | x_\sigma) < \infty,
\]

then

\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{X_\tau}{a_k} \leq \alpha(\gamma(\omega), \sigma(\omega)) + \sigma(\omega), \mu - \text{a.s.} \omega \in D(\alpha) \cap H(\omega).
\]

\[
\liminf_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} X_\tau \geq \beta(\gamma(\omega), \sigma(\omega)) - \sigma(\omega), \mu - \text{a.s.} \omega \in D(\alpha) \cap H(\omega).
\]

\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} X_\tau \leq \alpha(\gamma(\omega), \sigma(\omega)) - \beta(\gamma(\omega), \sigma(\omega)) + 2\sigma(\omega),
\]

\[
\mu - \text{a.s.} \omega \in D(\alpha) \cap H(\omega).
\]

\[
\liminf_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} X_\tau \geq \beta(\gamma(\omega), \sigma(\omega)) - \alpha(\gamma(\omega), \sigma(\omega)) - 2\sigma(\omega),
\]

\[
\mu - \text{a.s.} \omega \in D(\alpha) \cap H(\omega),
\]

where

\[
H(\omega) = \{\omega : \limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{E_Q[g_k(X_\tau)]}{g_k(a_k)} = \sigma(\omega) < \infty\}.
\]

**Proof.** Letting \(f_k(X_\sigma, X_\tau) = X_\tau\), denote \(Y_\tau = X_\tau I_{|X_\sigma| \leq a_k}\). Noticing that \(g_n(x)/|x| \uparrow\) implies \(g_n(x) \uparrow\), hence under the assumption of Theorem 2, (21) and (22) still hold. By (64) we easily know that \(\{X_\tau, \tau \in T^{(n)}\}\) is equivalent to \(\{Y_\tau, \tau \in T^{(n)}\}\). Hence by virtue of \(\sigma_n \to \infty\), a.s. and \(a_n \uparrow\), we obtain

\[
\lim_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{Y_\tau - X_\tau}{a_k} = 0, \text{ a.s.}
\]

For \(g_n(x)/|x| \uparrow\) as \(|x|\) increases, we have

\[
\frac{|x|}{a_n} \leq \frac{g_n(x)}{g_n(a_n)}, \text{ } |x| > a_n.
\]
Denote by $F_{X_{\tau}|X_{\sigma}}(x|x_{\sigma}) = P(X_{\tau} \leq x|X_{\sigma} = x_{\sigma})$ the conditional distribution function of $X_{\tau}$ relative to $X_{\sigma} = x_{\sigma}$. Noticing $E_Q(X_{\tau}|X_{\sigma}) = 0,$

\[
\frac{|E_Q(Y_{\tau}|X_{\sigma})|}{a_n} = \frac{|E_Q(X_{\tau}I_{X_{\tau} \leq a_n}|X_{\sigma})|}{a_n}
\]

\[
\leq \int_{|x| > a_n} \frac{|x|}{a_n} dF_{X_{\tau}|X_{\sigma}}(x|x_{\sigma}) \leq \int_{|x| > a_n} \frac{g_n(x)}{g_n(a_n)} dF_{X_{\tau}|X_{\sigma}}(x|x_{\sigma})
\]

\[
\leq E_Q \left[ \frac{g_n(X_{\tau})}{g_n(a_n)} \right]_{X_{\sigma}}.
\]  

(72)

Owing to (69) and (72),

\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{|E_Q(X_{\tau}I_{X_{\tau} \leq a_n}|X_{\sigma})|}{a_k} \leq \sigma(\omega), \mu - a.s. \omega \in D(\alpha) \cap H,
\]  

which implies that

\[
\limsup_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} E_Q(Y_{\tau}|X_{\sigma}) \leq \sigma(\omega), \mu - a.s. \omega \in D(\alpha) \cap H,
\]  

(74)

\[
\liminf_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} E_Q(Y_{\tau}|X_{\sigma}) \geq -\sigma(\omega), \mu - a.s. \omega \in D(\alpha) \cap H.
\]  

(75)

Noticing

\[
\sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{X_{\tau}}{a_k} = \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{(X_{\tau} - Y_{\tau}) + (Y_{\tau} - E_Q(Y_{\tau}|X_{\sigma})) + E_Q(Y_{\tau}|X_{\sigma})}{a_k},
\]  

by use of (21), (70) and (74), we have (65). By applying (22), (70) and (75), we acquire (66). By utilizing Lemma 2, (67) and (68) follow from (65) and (66).

**Corollary 4.** Let $X = \{X_{\sigma}, \sigma \in T\}$ be an arbitrary random field indexed by the homogeneous tree defined by (6), $E_Q(X_{\tau}|X_{\sigma}) = 0$, $\tau \in S(\sigma)$, $\sigma \in L_k$, $k \geq 0$. Let $\{\sigma_n, n \geq 0\}$ and $\{a_n, n \geq 0\}$ be increasing positive-valued stochastic sequences, $\{g_n(x), n \geq 0\}$ be defined as Theorem 2, denote

\[
J(\omega) = \{\omega : \sum_{k=0}^{\infty} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{E_Q[g_k(X_{\tau})|X_{\sigma}]}{\sigma_k g_k(a_k)} < \infty\},
\]  

(77)

then

\[
\lim_{n \to \infty} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \frac{X_{\tau}}{a_k} = 0, \mu - a.s. \omega \in D(\alpha) \cap J(\omega),
\]  

(78)

\[
\lim_{n \to \infty} \frac{1}{a_n \sigma_n} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} X_{\tau} = 0, \mu - a.s. \omega \in D(\alpha) \cap J(\omega).
\]  

(79)
Hence $D(\alpha) \cap J(\omega) \subseteq D(\alpha) \cap H(\omega)$. Obviously at the moment $\sigma(\omega) = 0$, by (27), (28) in Theorem 1, we get $\gamma(\omega) = 0 = \beta(\gamma(\omega)) = 0$. Therefore, (78) follows from (65), (66). (79) holds from (67), (68).

Corollary 5. Let $X = \{X_\sigma, \sigma \in T\}$ be an arbitrary random field indexed by the homogeneous tree defined by $(6)$. $\{f_n(x,y), n \geq 0\}, \{\alpha_n(\omega), n \geq 0\}$, $\{\gamma(\omega), \sigma(\omega)\}$ and $\beta(\gamma(\omega), \sigma(\omega))$ be all defined as before. Denote $0 \leq p_n \leq 2$, $n \geq 0$,

$L(\omega) = \{\omega : \limsup_{n \to \infty} \frac{1}{|T(n)|} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k \cap S(\sigma)} \frac{f_k(X_\sigma, X_\tau) - E_Q[f_k(X_\sigma, X_\tau)|X_\sigma]}{a_k(\omega)} \leq \alpha(\gamma(\omega), \sigma(\omega)), \mu - a.s. \omega \in L(\omega), \}$

$\liminf_{n \to \infty} \frac{1}{|T(n)|} \sum_{k=0}^{n-1} \sum_{\sigma \in L_k \cap S(\sigma)} \frac{f_k(X_\sigma, X_\tau) - E_Q[f_k(X_\sigma, X_\tau)|X_\sigma]}{a_k(\omega)} \geq \beta(\gamma(\omega), \sigma(\omega)), \mu - a.s. \omega \in L(\omega), \}$

Proof. Letting $\sigma_n(\omega) = |T(n)|$, $g_n(x) = |x|^{p_n}$. Obviously $\{|x|^{p_n}, n \geq 0\}$ are a series of continuous positive-valued even functions defined on $(-\infty, +\infty)$ and satisfy (17). By (14), (18) and (19) we attain that $L(\omega) = D(\alpha) \cap D(\omega)$. Corollary 5 follows from Theorem 1.

Corollary 6. Let $\{X_n, n \geq 0\}$ be an arbitrary stochastic sequence with the joint distribution as follows:

$P(X_0 = x_0, \cdots, X_n = x_n) = p(x_0, \cdots, x_n), \ x_i \in S, 0 \leq i \leq n. \ $ (84)

Let $\{f_n(x,y), n \geq 0\}, \{a_n, n \geq 0\}$ and $\{g_n(x), n \geq 0\}$ be all defined as Theorem 1, denote

$A(\omega) = \{\omega : \gamma(\omega) < \infty, \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} E_Q[g_k(f_k(X_k, X_{k+1}))]|X_k] = \sigma(\omega) < \infty, \}$

then

$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{f_k(X_k, X_{k+1}) - E_Q[f_k(X_k, X_{k+1})]|X_k]}{a_k} \leq \alpha(\gamma(\omega), \sigma(\omega)), \ $ (85)

$\mu - a.s. \omega \in A(\omega), \ $ (86)
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \tilde{f}_k(X_k, X_{k+1}) - E_Q[\tilde{f}_k(X_k, X_{k+1}) | X_k] \geq \beta(\gamma(\omega), \sigma(\omega)).
\]

\[P - \text{a.s. } \omega \in A(\omega), \quad (87)\]

where \(\alpha(\gamma(\omega), \sigma(\omega)), \beta(\gamma(\omega), \sigma(\omega))\) are defined by (23) and (24).

**Proof.** Letting \(\sigma_n(\omega) = n, |S(\sigma)| \equiv 1, \sigma \in T^{(n)}, 0 < \alpha < 1\) in Theorem 1, the tree model is changed into the chain model. At the moment the arbitrary random field on the homogeneous tree turns into the arbitrary stochastic sequence, and \(Q_{k+1}(x_\tau | x_\sigma) = Q_{k+1}(x_{k+1} | x_k), \forall \sigma \in L_k, \tau \in S(\sigma), k \geq 0, |T^{(n)}| = n + 1, \tilde{f}_k(X_\sigma, X_\tau) = \tilde{f}_k(X_k, X_{k+1}).\)

\[
\liminf_{n \to \infty} \frac{\sigma_n(\omega)}{n^\alpha} = \liminf_{n \to \infty} \frac{n}{n^\alpha} = \liminf_{n \to \infty} (n^{1-\alpha}) = \infty > 0, \quad (88)
\]

which implies \(A(\omega) = D(\alpha) \cap D(\omega).\) Hence (86), (87) follow from (21), (22) immediately.

**References**


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