A ZERO-FREE REGION FOR THE FRACTIONAL DERIVATIVES OF THE RIEMANN ZETA FUNCTION

RICKY E. FARR, SEBASTIAN PAULI, AND FILIP SAIDAK
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Abstract. For any $\alpha \in \mathbb{R}$, we denote by $D^{\alpha}_\zeta(s)$ the $\alpha$-th Grünwald-Letnikov fractional derivative of the Riemann zeta function $\zeta(s)$. For these derivatives we show:

$$D^{\alpha}_\zeta(s) \neq 0$$

inside the region $|s - 1| < 1$. This result, the first of its kind, is proved by a careful analysis of integrals involving Bernoulli polynomials and bounds for fractional Stieltjes constants.

1. Introduction

The Riemann zeta function $\zeta(s)$ and its derivatives $\zeta^{(k)}(s)$ are given by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{and} \quad \zeta^{(k)}(s) := (-1)^k \sum_{n=2}^{\infty} \frac{(\log n)^k}{n^s},$$

for all $k \in \mathbb{N}$, everywhere in the half-plane $\Re(s) > 1$. Each of them can, by a process of analytic continuation, be extended to a meromorphic function with a single pole at $s = 1$.

In 2003, Skorokhodov [13] observed that discretely increasing $k$ moves the non-trivial zeros of $\zeta^{(k)}(s)$, in a one-to-one fashion, to the right. Investigating the zero-free regions of higher derivatives $\zeta^{(k)}(s)$, in [2] the authors proved this phenomenon for sufficiently large $k$, and have discovered that, for integers $k \geq 0$, all of these derivatives have identical zero counts in the region $\Re(s) > 1/2$. Unfortunately, due to increasing densities of the zeros of derivatives of $\zeta(s)$ in the vertical direction, this simple bijective idea is very difficult to state quantitatively (e.g. in terms of counting functions such as $N_k(T) := \sum_{\zeta^{(k)}(\rho) = 0, 0<\Im(\rho)\leq T} 1$). However, the existence of a visible “flow” of the zeros suggests that perhaps an indepth study of the fractional derivatives could provide a missing link needed to establish this fascinating but currently little-understood property. Despite an incredible amount of research concerning the theory of $\zeta(s)$ and its $k$-th derivatives (for integer values of $k$), the problems of fractional derivatives have been largely neglected so far ([10], [9], [3] being a few rare exceptions).

In this paper we will not try to prove the audacious one-to-one conjecture stated above, but instead we will take a first step towards it by establishing a new general zero-free region for (arbitrary) fractional derivatives of $\zeta(s)$, a result that should be of independent interest. In particular, we will show that no integral or fractional

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<table>
<thead>
<tr>
<th>Function</th>
<th>Zero</th>
<th>Distance from 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \zeta )</td>
<td>( s = -2 )</td>
<td>(</td>
</tr>
<tr>
<td>( \zeta' )</td>
<td>( s \approx -2.7173 )</td>
<td>(</td>
</tr>
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<td>( D_{1.4677}^{1.4677} \zeta(s) )</td>
<td>( s \approx -1.5249 + 2.6383i )</td>
<td>(</td>
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**Figure 1.** Zeros of selected derivatives of \( \zeta \) close to \( s = 1 \).

![Figure 1: Zeros of selected derivatives of \( \zeta \) close to \( s = 1 \).](image1)

**Figure 2.** Zeros of integral and fractional derivatives \( D_{\alpha}^{\alpha} [\zeta(s)] \) of the Riemann zeta function \( \zeta \) in the neighbourhood of our zero free region \( |s - 1| < 1 \). Selected zeros of \( D_{\alpha}^{\alpha} [\zeta(s)] \) are denoted by \( (\alpha) \)•.

![Figure 2: Zeros of integral and fractional derivatives of \( \zeta(s) \).](image2)

derivative of \( \zeta(s) \) has a zero inside the disk \( |s - 1| < 1 \). And while the result is not the sharpest possible, it is not too far removed from it. The Figure 1 above gives the list of closest zeros to the pole at \( s = 1 \), and the Figure 2 depicts the distribution (and the flow) of these zeros in the left half-plane. Here, as it was noted in [5], the same phenomenon of translation of zeros continues; however, the linear and periodic movement found in the right half-plane is deformed into curves that terminate in the “trivial” zeros of derivatives of \( \zeta(s) \) found on the negative real axis.

The structure of the remainder of the paper is as follows. In Section 2 we begin with the definition of the Grünwald-Letnikov fractional derivatives and state some of their properties, then in Section 3 we recall some basic results concerning fractional
Stieltjes constants needed in our proof. We derive bounds for these constants and state a couple of other useful auxiliary results in Section 4. Finally, in Section 5, we prove our main result.

2. Fractional Derivatives

Fractional derivative operators are natural generalizations of the standard differentiation operator $D^\alpha$ to arbitrary (integer, rational, or complex) values of $\alpha$. We have found that, among the multitude of existing definitions of fractional derivatives, the reverse Grünwald-Letnikov derivative works best for situations dealing with $\zeta(s)$ and its derivatives. In fact, in [6] we have applied it successfully in a proof of a conjecture by Kreminski [10]. Here we follow suit: we write $^{(\alpha)}_k = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$, and for any $\alpha \in \mathbb{C}$, recall the definition of the so-called “reverse $\alpha$th Grünwald-Letnikov derivative” of a function $f(z)$ (see [8]):

$$D^\alpha_z [f(z)] = \lim_{h \to 0^+} \frac{\Delta^\alpha_h f(z)}{h^\alpha} = \lim_{h \to 0^+} \frac{(-1)^k \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z + k h)}{h^\alpha},$$

whenever the limit exists. Thus defined, $D^\alpha_z [f(z)]$ coincide with the standard derivatives for all $\alpha \in \mathbb{N}$. Furthermore, we have $D^0_z [f(z)] = f(z)$ and $D^\alpha_z [D^\beta_z [f(z)]] = D^{\alpha+\beta}_z [f(z)]$.

In [11] it was shown that for $z \in \mathbb{C}$ one has $D^\alpha_z [e^{-az}] = (-1)^\alpha a^\alpha e^{-az}$, and that $D^\alpha_z [1] = 0$. For the Riemann zeta function this implies that, if $\alpha > 0$ and $\Re(s) > 1$, then we can write

$$D^\alpha_z [\zeta(s)] = (-1)^\alpha \sum_{n=1}^{\infty} \frac{\log^\alpha(n+1)}{n^s}. \quad (1)$$

Note that the Grünwald-Letnikov derivative of the Riemann zeta function is defined for all real $\alpha > 0$, and $D^\alpha_z [\zeta(s)]$ is analytic in $s$; and what matters to us most is that analytic continuation will yield the Grünwald-Letnikov derivative for all $s \in \mathbb{C}$ with $\Re(s) \leq 1$.

3. Fractional Stieltjes Constants

We start by recalling some basics. First, note that $\zeta(s)$ can be extended to a meromorphic function with a simple pole at $s = 1$, with residue 1, and has a Laurent series expansion:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!} (s-1)^n, \quad (2)$$

where $\gamma_n$ are the Stieltjes constants [14]. The *fractional Stieltjes constants* $\gamma_\alpha$, with $\alpha \in \mathbb{R}_{>0}$, were introduced by Kreminski [10] and can be defined as the coefficients of the Laurent expansion of the $\alpha$-th Grünwald-Letnikov fractional derivative of $\zeta(s) - 1$, for all $s \neq 1$:

$$D^\alpha_s [\zeta(s)] = (-1)^{-\alpha} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}}{n!} (s-1)^n. \quad (3)$$
In [6] we have proved the following generalization of a result of Williams and Zhang [15]: Let \([x]\) denote the integer part of \(x\). Then, for \(\alpha > 0\) and \(m \in \mathbb{N}\), we have

\[
\gamma_\alpha = \sum_{k=1}^{m} \frac{\log^\alpha(k + 1) - \log^\alpha(1 + k)}{\alpha + 1} - \frac{\log^\alpha(1 + m)}{2(m + 1)} + \int_{m}^{\infty} P_1(x)f'_\alpha(x)dx, \quad (4)
\]

where \(P_1(x) = x - [x] - \frac{1}{2}\) and \(f_\alpha(x) = \frac{\log^\alpha x + 1}{x + 1}\). Integrating (4) by parts \(m\) times yields

\[
\int_{m}^{\infty} P_1(x)f'_\alpha(x)dx = \sum_{j=1}^{v} \left[ P_j(x)f^{(j-1)}_\alpha(x) \right]_{x=m}^{\infty} + (-1)^{v-1} \int_{m}^{\infty} P_v(x)f^{(v)}_\alpha(x)dx
\]

\[
= - \sum_{j=1}^{v} P_j(m)f^{(j-1)}_\alpha(m) + (-1)^{v-1} \int_{m}^{\infty} P_v(x)f^{(v)}_\alpha(x)dx, \quad (5)
\]

where for \(k \in \mathbb{N}\), \(P_k(x) = \frac{B_k(x-[x])}{k!}\) is the \(k^{th}\) periodic Bernoulli polynomial and \(B_k\) is the \(k^{th}\) Bernoulli number. These ideas can also be used to obtain an upper bound for the fractional Stieltjes constants: For any integer \(n\), with \(1 \leq n < \alpha\), we have (see [7]):

\[
|\gamma_\alpha| \leq \frac{(3 + (-1)^{n+1})\Gamma(\alpha + 1)}{(2\pi)^{n+1}} \frac{2(n + 1)!}{(n + 1)!}. \quad (6)
\]

The expression (3) for the fractional derivatives of the Riemann zeta function will be the starting point of our proof of their zero-free regions. In order to establish the non-vanishing result, bounds on Stieltjes constants will be needed, plus a careful estimation of the behavior of the periodic Bernoulli polynomials \(P_k(x)\), defined in (5). This is done in the next section.

4. Four Auxiliary Lemmas

Lemma 4.1. Let \(0 < \alpha \leq 1\) and \(f_\alpha(x) = \frac{\log^\alpha(x+1)}{x+1}\). Then \(\left| \int_{1}^{\infty} P_3(x)f''_\alpha(x)dx \right| < 0.013\).

**Proof.** Ostrowski [12] observed that, for odd integers \(n > 1\), one always has: \(|P_n(x)| < \frac{2}{(2\pi)^n}\). Combining this with the triangle inequality (and the change of variables for the integral), we are able to write:

\[
\left| \int_{1}^{\infty} P_3(x)f''_\alpha(x)dx \right| < \frac{2}{(2\pi)^3} \sum_{i=0}^{3} |s(4, i + 1)(\alpha)| \int_{1}^{\infty} \frac{\log^{\alpha-1}(x+1)}{(x+1)^4}dx
\]

\[
< \frac{2}{(2\pi)^3} \sum_{i=0}^{3} \frac{|s(4, i + 1)(\alpha)|}{3^{\alpha-i+1}} \int_{3\log(2)}^{\infty} x^{\alpha-1}e^{-x}dx.
\]

In what follows, we will estimate each of the four summands on the right side of this inequality separately. We start with \(i = 0\). Since \(x^\alpha \leq x\) in the interval
\[ \frac{|s(4, 1)(\alpha)|}{3^{\alpha+1}} \int_{3 \log(2)}^{\infty} x^{\alpha} e^{-x} dx \leq \frac{6}{3^{\alpha+1}} \int_{3 \log(2)}^{\infty} xe^{-x} dx \]

\[ = \frac{13 \log(2) + 1}{4} \cdot \frac{1}{3^{\alpha}}. \]  

For \( i = 1 \), in the interval \([3 \log(2), \infty)\) we have \( x^{\alpha - 1} \leq 3^{\alpha - 1} \log^{\alpha - 1}(2) \), for all \( \alpha \leq 1 \); thus

\[ \frac{|s(4, 2)(\alpha)|}{3^{\alpha}} \int_{3 \log(2)}^{\infty} x^{\alpha - 1} e^{-x} dx \leq \frac{11 \alpha}{3^{\alpha}} \cdot 3^{\alpha - 1} \log^{\alpha - 1}(2) \int_{3 \log(2)}^{\infty} e^{-x} dx \]

\[ \leq \frac{11 \log^{\alpha - 1}(2)}{24}. \]  

Now, for the summand corresponding to \( i = 2 \) we have

\[ \frac{|s(4, 3)(\alpha)|}{3^{\alpha - 1}} \int_{3 \log(2)}^{\infty} x^{\alpha - 2} e^{-x} dx = \frac{6|\alpha(\alpha - 1)|}{3^{\alpha - 1}} \int_{3 \log(2)}^{\infty} x^{\alpha - 2} e^{-x} dx \]

\[ \leq \frac{3}{2} \cdot \frac{1}{3^{\alpha - 1}} \cdot 3^{\alpha - 2} \log^{\alpha - 2}(2) \int_{3 \log(2)}^{\infty} e^{-x} dx \]

\[ = \frac{\log^{\alpha - 2}(2)}{16}, \]  

since for \( 0 < \alpha \leq 1 \) we have \( |\alpha(\alpha - 1)| \leq \frac{1}{4} \) and for \( x \in [3 \log(2), \infty) \): \( x^{\alpha - 2} \leq 3^{\alpha - 2} \log^{\alpha - 2}(2) \).

Finally, for \( i = 3 \) we can write

\[ \frac{|s(4, 4)(\alpha)|}{3^{\alpha - 2}} \int_{3 \log(2)}^{\infty} x^{\alpha - 3} e^{-x} dx = \frac{|\alpha(\alpha - 1)(\alpha - 2)|}{3^{\alpha - 2}} \int_{3 \log(2)}^{\infty} x^{\alpha - 3} e^{-x} dx \]

\[ \leq \frac{2 \sqrt{3} \log^{\alpha - 3}(2)}{9} \int_{3 \log(2)}^{\infty} e^{-x} dx \]

\[ = \frac{\sqrt{3} \log^{\alpha - 3}(2)}{108}, \]  

since \( |\alpha(\alpha - 1)(\alpha - 2)| \leq \frac{3}{2} \sqrt{3} \) for \( \alpha \in (0, 1] \) and \( x^{\alpha - 3} \leq 3^{\alpha - 3} \log^{\alpha - 3}(2) \) for \( x \in [3 \log(2), \infty) \).

Combining these four bounds, we conclude:

\[ \left| \frac{\int_{1}^{\infty} P_{\alpha}(x) f^{\prime \prime \prime}_{\alpha}(x)}{P_{\alpha}(x) f^{\prime \prime \prime}_{\alpha}(x)} \right| < \frac{2}{(2\pi)^{2}} \left[ \frac{4 \cdot 3 \log(2) + 1}{3^{\alpha}} + \frac{11 \log^{\alpha - 1}(2)}{24} + \frac{\log^{\alpha - 2}(2)}{16} + \frac{\sqrt{3} \log^{\alpha - 3}(2)}{108} \right] < 0.013, \]  

as desired. \( \Box \)
Lemma 4.2. If $0 < \alpha < 1$, then $|\gamma_\alpha| < 0.436$.

Proof. First, note that letting $m = 1$ in (4), we get

$$
\gamma_\alpha = \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha + 1} + \int_1^\infty P_1(x) f'_\alpha(x) dx.
$$

Second, observe that from (5) we also know

$$
\gamma_\alpha = \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha + 1} - P_2(1) f'\alpha(1) + P_3(1) f''\alpha(1) + \int_1^\infty P_3(x) f'''\alpha(x) dx.
$$

Therefore, recalling that $P_2(1) = \frac{B_2}{2!} = \frac{1}{12}$ and $P_3(1) = \frac{B_3}{3!} = 0$ and also noting that $f'\alpha(x) = \alpha \log^\alpha(2) - \frac{\log^{\alpha+1}(2)}{\alpha + 1}$, we obtain

$$
\gamma_\alpha = \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha + 1} - \frac{1}{12} \left[ \alpha \log^\alpha(2) - \frac{\log^{\alpha+1}(2)}{\alpha + 1} \right] + \int_1^\infty P_3(x) f'''\alpha(x) dx.
$$

Here, the maxima of the sum of the first three terms is attained when $\alpha = 0$. Combining this with the bound on the integral in Lemma 4.1, we get the wanted bound: $|\gamma_\alpha| \leq 0.436$.

Lemma 4.3. For all $\alpha > 0$, we have

(i) $\frac{|\gamma_\alpha|}{\Gamma(\alpha + 1)} < 0.348$ and (ii) $\frac{|\gamma_{\alpha+1}|}{\Gamma(\alpha + 1)} \leq 0.323$.

Proof. Combining the bound for $|\gamma_\alpha|$ proved in Lemma 4.2 and the fact that $\Gamma(\alpha + 1) \geq \Gamma(3/2) = \sqrt{\pi}$, for $0 < \alpha \leq 1$, we deduce that $\frac{|\gamma_\alpha|}{\Gamma(\alpha + 1)} < \frac{0.436}{\sqrt{\pi}} < 0.348$ in the region $0 < \alpha \leq 1$.

Now, in the complementary region $\alpha > 1$, one can apply the bound (6), and for all natural numbers $n$ that satisfy $1 \leq n < \alpha$ one can compute

$$
\frac{|\gamma_\alpha|}{\Gamma(\alpha + 1)} \leq \frac{4}{(2\pi)^{n+1}(n+1)\alpha + 1} \left( \frac{2(n+1)}{(n+1)!} \right) = \frac{4\sqrt{2}}{(2\pi)^{n+1}(n+1)\alpha + 1} \left( \frac{4(n+1)}{e} \right) \leq \frac{4\sqrt{2}}{(2\pi)^{n+1}(n+1)\alpha + 1} \left( \frac{2}{\pi e} \right)^{n+1} \leq 0.311,
$$

which is an even sharper bound. Together, these two bounds prove (i) for all $\alpha > 0$. 

Lemma 4.4. For all \( n \in N \), with \( n = 1 \) yields
\[
\frac{|\gamma_{n+1}|}{\Gamma(n+1)} \leq \frac{4\Gamma(n+2)4!}{(2\pi)^2n^{n+2}} = \frac{12(n+1)}{(2\pi)^22^n}.
\]
As is easy to check, the maximum of \( g(\alpha) = \frac{\alpha+1}{2^n} \) is attained at \( \alpha = \frac{1}{\log(2)} - 1 \), and this immediately yields the result (ii).
\[
\square
\]

We need one more technical lemma before we can prove our main theorem.

Lemma 4.4. For all \( \alpha > 0 \) and \( n \in N \cup \{0\} \),
\[
\frac{\Gamma(n+3)}{\Gamma(n+1)(n+2)!2^n(n+3)^\alpha} < \frac{(\alpha+2)(\alpha+1)}{3^{\alpha+2}} < 1.036,
\]
where
\[
\alpha = \sqrt{5\log^2(3) + 4} \frac{2\log(3)}{1} + \frac{1}{\log(3)} - \frac{3}{2}.
\]

Proof. We proceed by induction on \( n \). For \( n = 0 \) we have
\[
\frac{\Gamma(n+3)}{\Gamma(n+1)(n+2)!2^n(n+3)^\alpha} = \frac{\alpha^2 + 3\alpha + 2}{3\alpha^2}.
\]
The maximum of \( g(\alpha) = \frac{\alpha^2 + 3\alpha + 2}{3\alpha^2} \) is at \( \alpha = \sqrt{5\log^2(3) + 4} \frac{2\log(3)}{1} + \frac{1}{\log(3)} - \frac{3}{2} \), with \( g(\alpha) = 1.036 \). Now, let us assume that, for all integers \( j \) with \( 1 \leq j \leq n \), we have
\[
\frac{\Gamma(n+3)}{\Gamma(n+1)(n+2)!2^n(n+3)^\alpha} \leq \frac{(\alpha+2)(\alpha+1)}{3^{\alpha+2}}.
\]
We will show the assertion is true for \( j = n+1 \). Applying the induction hypothesis gives
\[
\frac{\Gamma(n+3)}{\Gamma(n+1)(n+2)!2^n(n+3)^\alpha} = \frac{\Gamma(n+3)}{\Gamma(n+1)(n+2)!2^n(n+3)^\alpha} \leq \frac{1}{2} \frac{(n+3)^\alpha}{n+4} \frac{\alpha+n+3}{n+3} \frac{\Gamma(n+3)}{\Gamma(n+1)(n+2)!2^n(n+3)^\alpha} \leq \frac{1}{2} \frac{(n+3)^\alpha}{n+4} \frac{\alpha+n+3}{n+3} \frac{(\alpha+2)(\alpha+1)}{3^{\alpha+2}}.
\]

Hence, all we need to show is that \( \frac{1}{2} \left( \frac{n+3}{n+4} \right)^\alpha \frac{\alpha+n+3}{n+3} \leq 1 \). However, notice that the function \( g(\alpha) = \frac{1}{2} \left( \frac{n+3}{n+4} \right)^\alpha \frac{\alpha+n+3}{n+3} \) is positive for all \( \alpha > 0 \); and taking the logarithmic derivative we get
\[
\frac{g'(\alpha)}{g(\alpha)} = \log \left( \frac{n+3}{n+4} \right) + \frac{1}{\alpha+n+3} \leq -\frac{1}{n+4} - \frac{1}{2} \left( \frac{1}{n+4} \right)^2 + \frac{1}{\alpha+n+3},
\]
thus using Taylor expansion we know that \( \log(1-x) \leq -x - \frac{1}{2}x^2 \), in the range \( 0 \leq x < 1 \). Moreover, \( \frac{1}{\alpha+n+3} \leq \frac{1}{n+4} \), and since \( g(\alpha) > 0 \), we can conclude that \( g'(\alpha) < 0 \). Therefore \( g(\alpha) \) is decreasing in the interval \([1, \infty)\), with the maximum at \( g(1) = \frac{1}{2} \).
On the other hand, if 0 < α < 1, the maximum of \(\left(\frac{n+3}{n+4}\right)^\alpha\) is attained at \(\alpha = 0\). And since \(\frac{\alpha+n+3}{\alpha+n+4} < \frac{n+3}{n+4} = 1 + \frac{\alpha}{n+4} \leq \frac{\alpha}{2}\), we have \(g(\alpha) < \frac{\alpha}{2} = \frac{\alpha}{2}\), for \(\alpha \in (0, 1)\). Combining these two results in (18), we deduce the bound for \(j = n + 1\). This completes the inductive proof. □

5. A Zero-Free Region

Now we are ready to prove our main result.

**Theorem 5.1.** For all \(\alpha \geq 0\),

\[D_\alpha^s[\zeta(s)] \neq 0,\]

in the region \(|s - 1| < 1\).

For a discussion of the special case \(\alpha = 0\), see Berndt [1].

**Proof.** We will deduce the result by showing \(\frac{4(n-1)\alpha+1}{\Gamma(\alpha+1)} D_\alpha^s[\zeta(s)] \neq 0\), in the region \(|s - 1| < 1\). Employing the expression (3), we start by writing:

\[
\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)} \zeta^{(\alpha)}(s)\right| = \left|1 + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(s-1)^{\alpha+n+1}}{\Gamma(\alpha+1)n!}\right|
\]

\[\geq 1 - \frac{\gamma_{\alpha}}{\Gamma(\alpha+1)} - \frac{\gamma_{\alpha+1}}{\Gamma(\alpha+1)} - \sum_{n=2}^{\infty} \frac{\gamma_{\alpha+n}}{\Gamma(\alpha+1)n!},\]

and then, applying the bound from Lemma 4.3 we obtain:

\[
\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)} \zeta^{(\alpha)}(s)\right| > 1 - 0.492 - 0.323 - \sum_{n=2}^{\infty} \frac{\gamma_{\alpha+n}}{\Gamma(\alpha+1)n!}.
\] \hspace{1cm} (19)

Now it suffices to focus on finding an upper bound for \(\sum_{n=2}^{\infty} \frac{\gamma_{\alpha+n}}{\Gamma(\alpha+1)n!}\). Using (6) gives:

\[
\frac{\gamma_{\alpha+n}}{\Gamma(\alpha+1)n!} \leq \frac{4\Gamma(\alpha+n+1)(2n+1)!}{(2\pi)^{n+1}(n+1)\alpha+n+1}(n+1)\alpha+n+1(n+1)!\Gamma(\alpha+1),
\]

while from Stirling’s formula it follows that \(\frac{(2n)!}{n!} \leq \sqrt{2} \left(\frac{4n}{e}\right)^n\) for all integers \(n \geq 1\). Therefore

\[
\sum_{n=2}^{\infty} \frac{\gamma_{\alpha+n}}{\Gamma(\alpha+1)n!} \leq \sum_{n=2}^{\infty} \frac{4\Gamma(\alpha+n+1)}{(2\pi)^{n+1}(n+1)\alpha+n+1}(n+1)!\Gamma(\alpha+1) \sqrt{2} \left(\frac{4(n+1)}{e}\right)^{n+1}
\]

\[= \sum_{n=2}^{\infty} \frac{4\sqrt{2}\Gamma(\alpha+n+1)}{(2\pi)^{n+1}(n+1)\alpha+n+1}(n+1)!\Gamma(\alpha+1) \left(\frac{4}{e}\right)^{n+1}\]

\[= 4\sqrt{2} \left(\frac{2}{\pi e}\right)^3 \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+1)(n+2)!\alpha(n+3)^n} \left(\frac{4}{\pi e}\right)^n\]

\[\leq 4\sqrt{2} \left(\frac{2}{\pi e}\right)^3 \sum_{n=0}^{\infty} \frac{(a_1+2)(a_1+1)}{3^{a_1+2}} \left(\frac{4}{\pi e}\right)^n < 0.142,
\]
by Lemma 4.4. Inserting this upper bound back into the expression (19), we obtain
\[
\left| \frac{(s - 1)^{\alpha+1}}{\Gamma(\alpha + 1)} \zeta^{(\alpha)}(s) \right| > 1 - 0.492 - 0.323 - 0.142 > 0,
\]
which implies that \(D_s^\alpha[\zeta(s)] \neq 0\), for all \(\alpha > 0\), in the region \(|s - 1| < 1\). \( \Box \)

References