

ON THE DERIVATIVES OF COMPOSITE FUNCTIONS

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Abstract. Let g be a non-constant polynomial and let f be transcendental and meromorphic of sub-exponential growth in the plane. Then if $k \geq 2$ and Q is a polynomial the function $(f \circ g)^{(k)} - Q$ has infinitely many zeros. The same conclusion holds for $k \geq 0$ and with Q a rational function if f has finitely many poles. We also show by example that this result is sharp.

1. Introduction

This paper will use standard notation of value distribution theory [7], including $\rho(f)$ for the order of growth of a meromorphic function f in the plane. In [4], the second author proved the following result, for application in a theorem concerning normal families.

Theorem 1.1. *Let $k \in \mathbb{N}$. Let f be a transcendental entire function with $\rho(f) < 1/2$. Let g and Q be polynomials, with g non-constant. Let F and H be defined by*

$$F = f \circ g, \quad H = F^{(k)} - Q. \quad (1.1)$$

Then H has infinitely many zeros.

The hypothesis $\rho(f) < 1/2$ was needed for the proof of Theorem 1.1 in [4], which made use of the celebrated $\cos \pi \rho$ minimum modulus theorem [8, Chapter 6]. In this paper, we show that Theorem 1.1 can be extended to transcendental functions of sub-exponential growth, that is, functions of at most order 1, minimal type. We state the result as follows.

Theorem 1.2. *Let k be a non-negative integer. Let g be a non-constant polynomial and let f be a transcendental meromorphic function in the plane with*

$$\lim_{r \rightarrow \infty} \frac{T(r, f)}{r} = 0. \quad (1.2)$$

Let F and H be defined by (1.1), with Q a rational function. Then the following conclusions hold.

- (a) If f has finitely many poles then the exponent of convergence of the zeros of H is equal to the order $\rho(F)$ of F .*
- (b) If $k \geq 2$ and Q is a polynomial, or if $k = 1$ and $Q \equiv 0$, then H has infinitely many zeros.*

Theorem 1.2 is sharp in the following sense. If $f(w) = e^w + P(w)$ and P and g are polynomials with g non-constant, then $F = f \circ g$ satisfies $F^{(k)} = Re^g + S$ with R and S polynomials so that the equation $F^{(k)}(z) = S(z)$ has finitely many solutions in the plane.

The case $k = 0$ of Theorem 1.2 may be compared with results on the frequency of fixpoints of $f \circ g$. Bergweiler proved that if g is a transcendental entire function and f is transcendental and meromorphic in the plane then $f(g) - Q$ has infinitely many zeros, for any non-constant rational function Q [3] (see also [1, 2]). It seems plausible that if f is a transcendental entire function satisfying (1.2) and g is any non-constant entire function then the function H defined by (1.1) has infinitely many zeros, for any $k \geq 1$ and any rational function Q . Some partial results are proved in [4] and [11, Theorem 6].

2. Proof of Theorem 1.2

We need the following result by the first author [10].

Lemma 2.1. *Suppose that G is meromorphic of finite order in the plane, and that $G^{(k)}$ has finitely many zeros, for some $k \geq 2$. Then G has finitely many poles.*

We now prove Theorem 1.2. We first note that since f has finite order and g is a polynomial it follows that

$$\rho(H) = \rho(F^{(k)}) = \rho(F) < \infty. \quad (2.1)$$

Next, we observe that it suffices to prove part (a). To see this, note first that the case where $k = 1$ and $Q \equiv 0$ is handled by the argument of [11, p.137], and is based on the fact that (1.2) implies that f' has infinitely many zeros, by a result of Eremenko, Langley and Rossi [5], from which it follows since g is a non-constant polynomial that so has $F' = f'(g)g'$. Suppose next that $k \geq 2$, that Q is a polynomial, and that H has finitely many zeros. Choose a polynomial Q_1 with $Q_1^{(k)} = Q$ and set $G = F - Q_1$. Then $H = G^{(k)}$ and it follows from Lemma 2.1 that G has finitely many poles and, again since g is a polynomial, so has f . Hence H is a transcendental meromorphic function of finite order with finitely many zeros and poles so that $\rho(F) = \rho(H) \geq 1$, using (2.1), and a contradiction arises from part (a).

To prove part (a), assume that f has finitely many poles but that the exponent of convergence of the zeros of H is less than $\rho(F)$. It follows using (2.1) that there exist a positive integer n , a meromorphic function Π with finitely many poles and a polynomial P , such that

$$H = F^{(k)} - Q = \Pi e^P, \quad \rho(\Pi) < n, \quad \deg P = \rho(H) = n. \quad (2.2)$$

Then we may write

$$g(z) = a_m z^m + \dots + a_0, \quad P(z) = b_n z^n + \dots + b_0, \quad b_n \in (0, \infty). \quad (2.3)$$

Here the assumption that b_n is real and positive involves no loss of generality, since otherwise we may apply a rotation of the independent variable z .

Since f has finitely many poles and satisfies (1.2), a standard application of the Poisson-Jensen formula [7, p.1] gives (compare [7, Theorem 1.6, p.18])

$$\log M(r, f) \leq 3m(2r, f) + O(\log r) = o(r) \quad \text{as } r \rightarrow \infty. \quad (2.4)$$

Denote positive constants by c, M , not necessarily the same at each occurrence. Combining (2.1), (2.2), (2.3) and (2.4) and using the fact that f and F have finitely many poles, it follows that

$$\begin{aligned} cr^n &\leq T(r, e^P) \\ &\leq (1 + o(1))T(r, F^{(k)}) \\ &\leq (1 + o(1))T(r, F) \\ &\leq (1 + o(1)) \log M(r, F) \\ &\leq (1 + o(1)) \log M(M(r, g), f) \\ &= o(M(r, g)) \end{aligned}$$

as $r \rightarrow \infty$. Using (2.3) again we deduce at once that

$$m > n. \quad (2.5)$$

Let δ and η be positive constants, with δ and η/δ small. If r_1 is large then (2.2) and (2.3) give

$$F^{(k)}(z) = O(|z|^M) \quad \text{for } |z| \geq r_1, \quad \frac{\pi}{2n} + \delta \leq \arg z \leq \frac{3\pi}{2n} - \delta.$$

Hence repeated integration, starting from the point $r_1 e^{i\pi/n}$, leads to

$$f(g(z)) = F(z) = O(|z|^{M+k}) \quad \text{for } |z| \geq r_1, \quad \frac{\pi}{2n} + \delta \leq \arg z \leq \frac{3\pi}{2n} - \delta. \quad (2.6)$$

Let u_1, u_2, \dots be the zeros of Π , repeated according to multiplicity. Since $\rho(\Pi) < n$ it follows that

$$\sum_{u_j \neq 0} |u_j|^{-n} < \infty$$

and a standard application of the Poisson-Jensen formula [7, p.1] gives

$$\log |\Pi(z)| = o(|z|^n) \quad \text{as } z \rightarrow \infty, \quad z \notin E = \bigcup_{u_j \neq 0} B(u_j, |u_j|^{-n}). \quad (2.7)$$

Moreover, standard estimates based on the differentiated Poisson-Jensen formula [7, p.22] (see [6] and [9, p.89]) give

$$\frac{F^{(k)}(z)}{F(z)} = O(|z|^M) \quad (2.8)$$

for large z outside a union E' of discs, having finite sum of radii. Let r be large and positive, such that the circle $|z| = r$ does not meet the exceptional sets E, E' corresponding to (2.7) and (2.8). Then (2.2), (2.3), (2.7) and (2.8) yield

$$\log |f(g(z))| = \log |F(z)| > cr^n \quad \text{for } |z| = r, \quad |\arg z| \leq \frac{\pi}{2n} - \delta. \quad (2.9)$$

With this same value of r let

$$\Omega_1 = \left\{ re^{i\theta} : -\frac{\pi}{2n} + 2\delta \leq \theta \leq \frac{\pi}{2n} - 2\delta \right\}, \quad \Omega_2 = \left\{ re^{i\theta} : \frac{\pi}{2n} + 2\delta \leq \theta \leq \frac{3\pi}{2n} - 2\delta \right\}. \quad (2.10)$$

Then (2.10) implies that for $j = 1, 2$ the image of the arc Ω_j under the mapping

$$\zeta = h(z) = a_m z^m \quad (2.11)$$

is the set $\{\zeta = |a_m|r^m e^{i\phi} : \phi \in I_j\}$, where I_j is an interval of length

$$m \left(\frac{\pi}{n} - 4\delta \right) > \pi + \delta,$$

since δ is small by assumption and $m > n$ by (2.5). Hence there exist

$$z_1 \in \Omega_1, \quad z_2 \in \Omega_2 \quad \text{such that} \quad w = h(z_1) = h(z_2). \quad (2.12)$$

For $|z - z_2| = \eta r$ we have using (2.3), (2.11) and (2.12), since η is small and r is large,

$$|h(z) - h(z_1)| = |h(z) - h(z_2)| \geq cr^m, \quad g(z) - h(z) = o(r^m), \quad h(z_1) - g(z_1) = o(r^m).$$

We then write

$$g(z) - g(z_1) = h(z) - h(z_1) + g(z) - h(z) + h(z_1) - g(z_1)$$

and apply Rouché's theorem, which shows using (2.12) again that g takes the value $g(z_1)$ at some $z_3 \in B(z_2, \eta r)$, provided r is large enough. Moreover,

$$\frac{\pi}{2n} + \delta \leq \arg z_3 \leq \frac{3\pi}{2n} - \delta,$$

using (2.10) and (2.12), since η is small compared to δ .

But now (2.6) gives

$$|f(g(z_1))| = |f(g(z_3))| \leq c|z_3|^{M+k} \leq c(|z_2| + \eta r)^{M+k} = O(r^{M+k}),$$

whereas (2.9), (2.10) and (2.12) give

$$\log |f(g(z_1))| > cr^n.$$

These estimates are obviously incompatible since r is large, and this contradiction completes the proof of part (a) and of Theorem 1.2.

References

- [1] W. Bergweiler, *Proof of a conjecture of Gross concerning fixpoints*, Math. Zeit. **204** (1990), 381–390.
- [2] W. Bergweiler, *Periodic points of entire functions, proof of a conjecture of Baker*, Complex Var. Theory Appl. **17** (1991), 57–72.
- [3] W. Bergweiler, *On the composition of transcendental entire and meromorphic functions*, Proc. Amer. Math. Soc. **123** (1995), 2151–2153.
- [4] E.F. Clifford, *Normal families and value distribution in connection with composite functions*, J. Math. Anal. Appl. **312** Pt. 1 (2005), 195–204.
- [5] A. Eremenko, J.K. Langley and J. Rossi, *On the zeros of meromorphic functions of the form $\sum_{k=1}^{\infty} \frac{a_k}{z-z_k}$* , J. d'Analyse Math. **62** (1994), 271–286.
- [6] G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc. (2) **37** (1988), 88–104.
- [7] W.K. Hayman, *Meromorphic Functions*, Oxford at the Clarendon Press, 1964.
- [8] W.K. Hayman, *Subharmonic Functions Vol. 2*, Academic Press, London, 1989.
- [9] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter Studies in Math. 15, Walter de Gruyter, Berlin/New York 1993.
- [10] J.K. Langley, *The second derivative of a meromorphic function of finite order*, Bulletin London Math. Soc. **35** (2003), 97–108.

- [11] J.K. Langley and J.H. Zheng, *On the fixpoints, multipliers and value distribution of certain classes of meromorphic functions*, Ann. Acad. Sci. Fenn. **23** (1998), 133–150.

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