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DEFORMATIONS OF SURFACES IN 4–DIMENSIONAL SPACE

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Abstract. The present paper describes geometric properties of liftabilities of immersed orientable closed 3–manifolds in 4–space into 5–space. We also discuss about tracks between generic surfaces in 3–space, and we construct non–liftable immersed 3–manifolds in 4–space without quadruple points using these tracks.

1. Introduction

We assume that all manifolds and maps are smooth. Let X be a manifold. Then let $T(X)$ denote the tangent bundle and let $T_x(X)$ denote the tangent space at $x \in X$. We denote the interval $[0,1]$ by I. Let M^m and N^n be manifolds with the dimensions m and n $(m < n)$ respectively. A map $f : M^m \to N^n$ is said to be *liftable* if there exists an embedding $\widetilde{f} : M^m \to N^n \times \boldsymbol{R}$ such that $f = p \circ \widetilde{f}$, where $p : N^n \times \boldsymbol{R} \to N^n$ is the canonical projection. Otherwise, it is said to be *non–liftable*. We call $\widetilde{f}$ an *embedded lift*. A map $f : M^m \to N^n$ $(m < n)$ is called an *immersion* if the differential map $df_x : T_x M^m \to T_{f(x)} N^n$ has full rank for each $x \in M^m$. In this paper, we assume that an immersion has normal crossings (see [**4**]). Poenaru [**14**], Giller [**5**], Carter, Saito [**3**] and others have studied about liftabilities of immersions between manifolds. Poenaru defined a set of invariants of the crossing set of an immersion $f : M^m \to N^n$ $(m < n)$ between orientable manifolds, and gave a necessary and sufficient condition for f to be liftable [**14**]. Giller constructed a non–liftable immersed sphere in 3–space, which is derived from the double cover of Boy's surface [**5**]. Carter and Saito gave geometric conditions for an immersed surface to be liftable [**3**] (Lemma 1.2). We will give similar conditions for an immersed orientable closed 3–manifold in 4–space to be liftable into 5–space (Theorem 1.3). An immersed 3–manifold in 4–space can be described as a series of immersed surfaces and singular surfaces in 3–space. Lifts of these immersed surfaces give a deformation of surfaces in 4–space. Using this idea, we will construct a non–liftable immersed 3–manifold in 4–space without quadruple points, (Theorem 1.6). For a set X, the cardinality of the set X will be denoted by $\#(X)$. A *crossing set* of an immersion $f : M^m \to N^n$, $(m < n)$ is the set:

$$C(f) \;=\; \left\{x \in M^m \mid \#\big(f^{-1}\big(f(x)\big)\big) > 1\right\}. \tag{1.1}$$

For a point $x \in C(f)$ with $\#\big(f^{-1}\big(f(x)\big)\big) = k$, $f(x)$ is called a *k–tuple point*. If M^m is compact, then $C(f)$ is stratified into a finite number of sets:

$$C_k(f) \;=\; \{x \in M^m \mid \#\big(f^{-1}\big(f(x)\big)\big) = k\}, \quad (k = 2, 3, \ldots, \ell < \infty)\,. \tag{1.2}$$

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Assume that $C(f) \neq \varnothing$. Then we denote the collection of immersed $(2m-n)$–manifolds $(n \leq 2m)$ in $C(f)$ by $\mathcal{C}(f)$:

$$C(f) = \bigcup_{C \in \mathcal{C}(f)} C. \tag{1.3}$$

We call each $C \in \mathcal{C}$ a *component*. Assume that the components of $\mathcal{C}$ are coloured by two distinct colours, a and b. If a point lies in $C_k(f)$, then the crossing point has a set of colours $(z_1, \ldots, z_{k-1})$, where $z_i \in \{a, b\}$ for $i = 1, \ldots, k-1$. Note that we do not distinguish the order of the colours

Remark 1.1. During a regular homotopy $\{H_t : M^m \to N^n, t \in I\}$ there is a finite number of points $t_1, t_2, \ldots, t_q \in I$ such that H_{t_i} does not have transversal crossings.

Lemma 1.2 (Carter and Saito [**3**])**.** *Let F^2 be a closed surface and let $f : F^2 \to \mathbb{R}^3$ be an immersion. Then f is liftable if and only if the crossing set $C(f)$ satisfies the following two conditions:*

- $(CS1)$ *The components of $\mathcal{C}(f)$ can be divided into two families $\{C_1^a, \ldots, C_n^a\}$ and $\{C_1^b, \ldots, C_n^b\}$ (called a–curves and b–curves respectively), such that $f(C_i^a) = f(C_i^b)$ for each i, and*
- $(CS2)$ *for points in $\{p_1, p_2, p_3\} \subset C_3(f)$ with $f(p_1) = f(p_2) = f(p_3)$ have colours $\{(a,a), (a,b), (b,b)\}$.*

We call these conditions CS–conditions and we call the second condition $(CS2)$ the $(3,2)$–*colouring condition*. We will introduce similar conditions for the case of immersions from orientable closed 3–manifolds into $\mathbb{R}^4$. Let M^3 be an orientable closed 3–manifold and let $f : M^3 \to \boldsymbol{R}^4$ be an immersion. Then f may have double surfaces, triple arcs and quadruple points as the crossing sets. We provide the following conditions:

- (Z1) $\mathcal{C}(f)$ is divided into two families of coloured immersed surfaces $\{F_i^a\}$ and $\{F_i^b\}$, $(i = 1, \ldots, n)$, called a–surfaces and b–surfaces respectively, such that $f(F_i^a) = f(F_i^b)$ and
- (Z2) for every set $\{p_1, p_2, p_3\} \subset C_3(f)$ with $f(p_1) = f(p_2) = f(p_3)$, there are neighbourhoods $U(p_i)$ of p_i in M^3, such that three pairs of coloured discs in $U(p_i) \cap C(f)$ $(i = 1, 2, 3)$, have colours, $\{(a,a),\ (a,b),\ (b,b)\}$.

The following is a generalisation of Lemma 1.2, (see also [**14**]).

Theorem 1.3. *Let M^3 be an orientable closed 3–manifold and let $f : M^3 \to \mathbb{R}^4$ be an immersion. Then f has an embedded lift into $\mathbb{R}^5$ via $p : \mathbb{R}^5 \to \mathbb{R}^4$ if and only if $C(f)$ satisfies $(Z1)$ and $(Z2)$.*

The proof will be given in Section 3. Two immersions $f, g : M^m \to N^n$ are said to be *regularly homotopic* if there is a homotopy

$$H : M^m \times I \to N^n \tag{1.4}$$

such that setting $H_t(x) = H(x,t)$ for all $t \in I$ and for all $x \in M^m$, then $H_0(x) = f(x)$, $H_1(x) = g(x)$ and H_t is an immersion for each $t \in I$. The homotopy H is called a *regular homotopy* from f to g. Regular homotopy is an equivalence relation. For a regular homotopy $H : M^m \times I \to N^n$, it induces a *track*

$$\widehat{H} : I \times M^m \to I \times N^n \tag{1.5}$$

defined by $\widehat{H}(t,x) = \big(t, H_t(x)\big)$, which is an immersion from $I \times M^m$ into $I \times N^n$. It is proved that for an orientable surface, every regular homotopy class of an immersion $f : F^2 \to \boldsymbol{R}^3$ is represented by a immersed surface constructed with an embedding $g : F^2 \to \boldsymbol{R}^3$ and an immersed circle $\gamma : S^1 \to F^2$ (see Section 5) [**11**] [**22**]. We denote the constructed immersion by $g_\gamma : F^2 \to \boldsymbol{R}^3$. The immersed circle γ can be modified by smoothing operations (see Section 7.3.1) at the crossing points so that we obtain a set δ of disjoint embedded circles on the surface F^2. Embeddings g and δ determine an immersed surface $g_\delta(F^2)$ representing the regular homotopy class of g_γ. Then we have the following theorem.

Theorem 1.4. *Let γ be an oriented immersed circle on an orientable closed surface F^2 and let δ be the set of disjoint oriented simple circles on F^2 derived from γ by smoothings. Then g_γ and g_δ have embedded lifts $\widetilde{g}_\gamma$ and $\widetilde{g}_\delta$ into $\boldsymbol{R}^4$ respectively such that there is a regular homotopy from g_γ to g_δ and this regular homotopy is covered by an isotopy from the lift $\widetilde{g}_\gamma$ to the lift $\widetilde{g}_\delta$.*

Example 1.5. Let $i : S^2 \to \mathbb{R}^3$ be the inclusion. Let $\bigcirc_+ : S^1 \to S^2$ be an oriented embedding on the sphere and let $\bigcirc_-$ be a reversed oriented embedding on the sphere. Let $8 : S^1 \to S^2$ be the eight–immersion (see Figure 1).

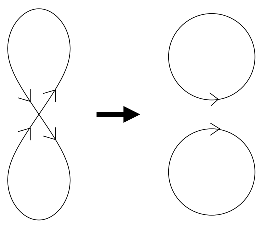

FIGURE 1

Then we obtain two immersions i_8 and $i_{\{\bigcirc_+,\bigcirc_-\}}$, where $\{\bigcirc_+,\bigcirc_-\}$ is obtained by the smoothing of the 8–immersion (see Section 5). Then there exists a regular homotopy from i_8 to $i_{\{\bigcirc_+\cup\bigcirc_-\}}$ such that the regular homotopy is covered by an isotopy from $\widetilde{i}_8$ to $\widetilde{i}_{\{\bigcirc_+\cup\bigcirc_-\}}$.

It is known that an immersed 3–sphere in 4–space obtained from a sphere eversion track has odd quadruple points [**12**], and which does not satisfy (Z1) in Theorem 1.3 [**1**] thus it is not liftable. In Section 8 we will construct a non–liftable immersed 3–sphere in 4–space without quadruple points.

Theorem 1.6. *There is a non–liftable immersed 3–sphere in $\boldsymbol{R}^4$ without quadruple points.*

It is proved that for every orientable closed 3–manifold M^3, there is an immersion from M^3 into $\mathbb{R}^4$ without quadruple points such that it has an embedded lift into $\mathbb{R}^5$ [**22**]. Thus the connected sum of the above non–liftable immersed 3–sphere and a liftable immersed 3–manifold in $\boldsymbol{R}^4$ is a non–liftable immersed orientable closed 3–manifolds in $\mathbb{R}^4$ without quadruple points. Thus the following holds.

Corollary 1.7. *For every orientable closed 3–manifold M^3, there is a non-liftable immersion from M^3 into $\mathbb{R}^4$ without quadruple points.*

We will prove Theorem 1.3 in Section 3, Theorem 1.4 in Section 7 and Theorem 1.6 in Section 8.1. This research was partially supported by the Marsden Fund NZ. The author is supported by the Japan Society for the Promotion of Science postdoctoral fellowship.

2. Liftabilities of Immersed 3–Manifolds

Let M^3 be an orientable closed 3–Manifold. Then there exists an immersion $f : M^3 \to \mathbb{R}^4$ [**9**]. We will provide some lemmas to discuss about liftabilities of immersions from M^3 into $\mathbb{R}^4$ and colouring conditions of their crossing sets.

2.1. Models of k–tuple points.

We will provide some models of multiple points of immersed surfaces in $\boldsymbol{R}^3$, which will be used later. For $x = (x_1, \ldots, x_m) \in \boldsymbol{R}^m$, $\|x\|$ is defined by $\sqrt{x_1^2 + \cdots + x_m^2}$. Let A be a subset of $\boldsymbol{R}^m$. Then we denote the interior of A by $\mathrm{Int}(A)$ and the closure of A by $\mathrm{Cl}(A)$. Define the following subset of $\boldsymbol{R}^3$:

$$B^3(0; r) \ = \ \{x = (x_1, x_2, x_3) \in \mathbb{R}^| \|x\| < r\}. \tag{2.1}$$

A model of a set of k–tuple points of planes in $\mathbb{R}^3$ denoted by P_k is given as followings:

$$P_2 \ = \ \{(x_1, x_2, x_3) \mid x_2 x_3 = 0, \quad \|x\| < 1\}. \tag{2.2}$$

P_2 consists of two discs intersecting at the segment $\{(x_1, 0, 0) \mid |x_1| < 1\}$. We call the segment a *double segment*.

$$P_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 x_2 x_3 = 0, \qquad \|x\| < 1\}. \tag{2.3}$$

P_3 consists of three discs containing a *triple point* at the origin.

2.2. Crossing sets for immersed 3–manifolds.

Let M^3 be an orientable closed 3–manifold and let $f : M^3 \to \boldsymbol{R}^4$ be an immersion with normal crossings. Then the crossing set is

$$C(f) \ = \ \bigcup_{i=2}^{4} C_i(f), \tag{2.4}$$

where $C_2(f)$ is a set of 2–dimensional open submanifolds, $C_3(f)$ is a set of 1–dimensional open submanifolds and $C_4(f)$ is a set of discrete points of M^3 respectively. We call $f\big(C_2(f)\big)$, $f\big(C_3(f)\big)$ and $f\big(C_4(f)\big)$ *double surfaces*, *triple arcs* and *quadruple points* respectively. The set $\mathcal{C}(f)$ consists of a set of immersed surfaces in M^3.

Note 2.1. The set $f\big(C(f)\big)$ is a union of immersed surfaces in $f(M^3)$.

2.2.1. *Triple curves and quadruple points.*

Let $f : M^3 \to \mathbb{R}^4$ be an immersion. Then there may be a set of triple arcs in $f(M^3) \subset \mathbb{R}^4$. Let τ be a triple arc and let $y \in \tau$. Then there are three distinct points p_1, p_2, $p_3 \in C_3(f) \subset M^3$ such that $f(p_1) = f(p_2) = f(p_3) = y$. Let $U(p_i)$ be neighbourhoods of p_i $(i = 1, 2, 3)$ in M^3 and let

$$D_i = U(p_i) \cap C(f), \qquad (i = 1, 2, 3). \tag{2.5}$$

Then D_i is diffeomorphic to P_2 for each i. Assume that f has quadruple points. We denote a quadruple point by $\rho \in f(M^3)$. Then there are four points $q_1, \ldots,$

$q_4 \in M^3$ with $f(q_i) = \rho$, $(i = 1, \ldots, 4)$ and there are small neighbourhoods $U(q_i)$ of q_i in M^3 $(i = 1, \ldots, 4)$ such that for each i, $U(q_i) \cap C(f)$ is diffeomorphic to the model P_3 (see (2.3)) denoted by T_i. Each T_i consists of three discs forming a triple point. If we can colour these discs of each T_i with two colours a and b such that their combinations are $\{(a,a,a),(a,a,b),(a,b,b),(b,b,b)\}$, then we will call the colouring the $(4,3)$–*condition*.

Lemma 2.2. *Let $f : M^3 \to \mathbb{R}^4$ be an immersion with a quadruple point $\rho \in f(M^3)$. Let $U(q_i)$ be neighbourhoods of q_i in M^3, $(i = 1, \ldots, 4)$ with $f(q_i) = \rho$ for all i. For $\{T_i \subset U(q_i) \mid i = 1, \ldots, 4\}$, there are four triples of pairs of two colours. Then every triple of them satisfies the $(3,2)$–condition if and only if the set $\{T_i | i = 1, \ldots, 4\}$ satisfies $(4,3)$–condition.*

Proof. Each quadruple point ρ is formed as an intersection of four triple curves in the image. Take a ball neighbourhood $V(\rho) \cong B^4$ of the quadruple point ρ in $\boldsymbol{R}^4$. Note that $\partial V(\rho) \cap f\big(C(f)\big)$ is a one–dimensional complex, in which we can view a vertex as a triple curve and view an edge as a double disc. We can find a complete graph with four vertices: every pair of vertices is joined with an edge. We replace each edge with a pair of edges since each edge corresponds to a double surface. Put two colours a and b on each pair of edges of the above graph. We denote the graph by G_Q. Similarly, we can obtain a graph for every T_i in which the graph with three vertices and three edges. We denote the graph by G_{T_i} $(i = 1, \ldots, 4)$. Each edge of G_{T_i} is coloured by a or b. Assume that all triple curves around the quadruple point ρ satisfy $(3,2)$–condition. Then this is interpreted as a relation among G_{T_i}, $(i = 1, \ldots, 4)$ in G_Q. Let τ be a triple curve. For each triple point $z \in \tau$, there are three points p_1, p_2 and $p_3 \in M^3$ such that $p_i \in D_i$, $f(p_i) = z$ for $i = 1, 2, 3$. Colourings of these discs in D_i satisfy the $(3,2)$–condition. T_1, T_2, T_3 and T_4 contain pre–images of triple curves so that for every triple curve, the colourings of D_1, D_2 and D_3 determine colourings of T_1, T_2, T_3 and T_4. Therefore, we can find four complete 3–graphs G_{T_i} $(i = 1, \ldots, 4)$ as subgraphs of G_Q such that every vertex of G_Q is contained in three of these subgraphs; that is, if we fix a vertex v of G_Q, then there are three above subgraphs, each of which contains v. These three pairs of colours of edges around v are $\{(a,a),(a,b),(b,b)\}$. Such subgraphs can be found (see Figure 2). Furthermore, we can find such four subgraphs, each of which

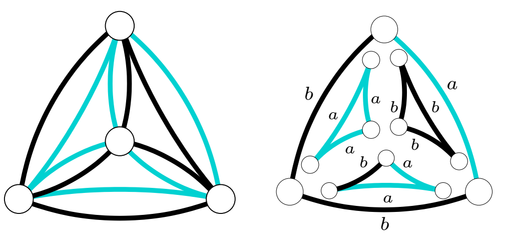

FIGURE 2. The graph G_Q and its four cycles

has three vertices so that each pair of subgraphs has different colourings:

$$\{(a,a,a),(a,a,b),(a,b,b),(b,b,b)\}. \tag{2.6}$$

This colouring gives $\{G_{T_i} \mid i = 1, \ldots, 4\}$ and it is equivalent to the $(4,3)$–condition.

Conversely, if the set $\{T_1, \dots, T_4\}$ satisfies the $(4,3)$–condition, then each T_i induces a 3–complete graph where each vertex corresponds to the double curves of T_i and each edge corresponds to a disc of T_i. We denote these graphs by $G_{T_i}, \dots, G_{T_4}$. They form the graph G_Q so that each pair of edges between vertices has distinct colours a and b. Thus each vertex of G_Q satisfies $(3,2)$–condition. □

The above lemma implies that the colouring conditions around a quadruple point is determined by colouring conditions of all triple curves connecting to the quadruple point. Let M^m and N^n be orientable manifolds with dimensions m and n, $(m < n)$. First, we define a subset of $M^m \times M^m$ by

$$\widetilde{C}_2 \;=\; \{(x_1, x_2) \mid x_1, x_2 \in M^m, f(x_1) = f(x_2) \quad \text{for} \quad x_1 \neq x_2\}. \tag{2.7}$$

The symmetric group Σ_2 of order two acts on $\widetilde{C}_2$ by exchanging factors. We denote the quotient space $\widetilde{C}_2/\Sigma_2$ by $T(2)$. The quotient map $q : \widetilde{C}_2 \to T(2)$ is a double covering map. Then the following holds [**14**] [**2**].

Lemma 2.3. *If an immersion $f : M^m \to N^n$ has an embedded lift $\widetilde{f} : M^m \to N^n \times \boldsymbol{R}$, then $\widetilde{C}_2 \cong T(2) \times \{0, 1\}$.*

3. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3.

Proof of Theorem 1.3. Let $f : M^3 \to \boldsymbol{R}^4$ be an immersion. We assume that there is an embedded lift, $\widetilde{f} : M^3 \to \boldsymbol{R}^5$ over the immersion f via the projection $\boldsymbol{R}^4 \times \boldsymbol{R} \to \boldsymbol{R}^4$. The crossing set $C(f)$ of f is a union of immersed surfaces. Form Lemma 2.3, the associated covering $q : \widetilde{C}_2 \to T(2)$ is a trivial double covering. Thus there is a pair of immersed surfaces F_a and F_b in $C(f)$ such that $f(F_a) = f(F_b)$. This implies that the immersion f satisfies the condition $(Z1)$. Let $p : \boldsymbol{R}^5 \to \boldsymbol{R}^4$ be the projection defined by $p(x_1, \dots, x_5) = (x_1, \dots, x_4)$. Let $p_1 : \boldsymbol{R}^4 \to \boldsymbol{R}$ be the projection defined by $p_1(x_1, \dots, x_4) = x_1$. We can assume that $h = p_1 \circ f$ is a Morse function. Let $q_0, \dots, q_k \in M^3$ be critical points for h. We assume that $h(q_0) = c_0 < h(q_1) = c_1 < \dots < h(q_k) = c_k$ and we also assume that

$$C(f) \cap \{q_0, \dots, q_k\} \;=\; \varnothing. \tag{3.1}$$

Take an interval $[c, d] \subset \mathbb{R}$ which does not contain any critical values of h. Then it is not difficult to see that $f\big(h^{-1}([c, d])\big)$ is a regular homotopy track on some surfaces. This means that for $y \in [c, d]$, $h^{-1}(y)$ is a set of disjoint closed surfaces in M^3 and each surface is immersed into $\{y\} \times \mathbb{R}^3$ by f. We denote the collection of these surfaces by $\mathcal{F}_y$ for $y \in [c, d]$. The restriction $f \mid_{F_y}$ is an immersion into $\{y\} \times \mathbb{R}^3$ and $f \mid_{F_y}$ has an embedded lift into $p^{-1}\big(p_1^{-1}(y)\big) \cong \mathbb{R}^4$. Thus the restricted immersion $f \mid_{F_y}$ satisfies the CS–conditions. Let τ_y be a triple point of $f \mid_{F_y} (F_y)$ in $p_1^{-1}(y) \cong \mathbb{R}^3$ and let $V(\tau_y)$ be a neighbourhood of τ_y in $p_1^{-1}(y) \cong \mathbb{R}^3$. Then for each y, $V(\tau_y) \cap f(F_y)$ is diffeomorphic to P_3 and we will denote this by $P_3(\tau_y)$. For each τ_y, each disc of $P_3(\tau_y)$ has two proper arcs crossing at the middle point of each arc. Each pair of arcs has a colour a or b and the colourings on arcs are $\{a, a\}$, $\{a, b\}$ and $\{b, b\}$ from the CS–conditions for $f_{F_y} : F_y \to p_1^{-1}(y) \cong \mathbb{R}^3$. Thus the triple arc τ is coloured with $\{(a, a), (a, b), (b, b)\}$. Then we assume that $p_1(\tau_y)$ is a critical value with respect to h on $C_3(f)$. Without loss of generality, we can assume that τ_y is a local maximum on τ with respect to to p_1. Then the colourings of double curves in $P_3(\tau_y)$ are determined by the ones of $P_3(\tau_y)$, where $p_1(\tau_y) < p_1(\tau_{y_0})$. Thus every triple arc in $p_1^{-1}([c,d])$ satisfies the $(3,2)$–colouring condition. Next, we assume that the interval $[c,d]$ contains a critical value c_i of h in its interior $(i = 0,\ldots,k)$. Then $h^{-1}(c_i)$ contains a singular surface in M^3. Since h satisfies the condition (3.1), $h^{-1}(y-\varepsilon)$ or $h^{-1}(y+\varepsilon)$ satisfies CS–conditions for any $\varepsilon > 0$. Thus from the above arguments, triple arcs in $f(M^3)\cap p_1^{-1}\big((c,d)\big)$ satisfy the $(3,2)$–condition. Therefore, $C(f)$ satisfies conditions (Z1) and (Z2).

Conversely, assume that $f : M^3 \to \boldsymbol{R}^4$ satisfies the conditions (Z1) and (Z2). We define a height function $\varphi : M^3 \to \boldsymbol{R}$ so that a lift $\widetilde{f} : M^3 \to \boldsymbol{R}^5$, defined by $\widetilde{f}(x) = \big(f(x), \varphi(x)\big)$, is an embedding. In the following, we will define a height function φ and we will partially deform the function to give the embedded lift. We use the same notation φ and $\widetilde{f}$ for the partially modified height function and the lift respectively. Since f satisfies the condition (Z1), there are two families of immersed surfaces in $C(f)$, $\{F_1^a,\ldots,F_n^a\}$ and $\{F_1^b,\ldots,F_n^b\}$ such that $f(F_i^a) = f(F_i^b)$ for $i = 1,\ldots,n$. Let $F^b = \bigcup_{i=1}^n F_i^b$. F^b is a compact subset in M^3. Thus there is a small neighbourhood $U(F^b)$ of F^b in M^3 such that a function $\varphi : M^3 \to \boldsymbol{R}$ is defined as $\varphi(x) = 0$ if $x \in M^3 \setminus U(F^b)$, $0 < \varphi(x) < 1$ if $x \in U(F^b) \setminus F^b$ and $\varphi(x) = 1$ if $x \in F^b$. We define a lift $\widetilde{f}$ of f via the projection $p : \boldsymbol{R}^5 \to \boldsymbol{R}^4$ by $\widetilde{f}(x) = \big(f(x), \varphi(x)\big)$. This lift may have non–empty crossings in the image. We will modify this height function φ so that $\widetilde{f}$ will be an embedding as follows. ¿From (Z2), there are three kinds of double curves in $C(f)$ coloured by (a,a), (a,b) and (b,b) for each triple curve component in $f(M^3)$. We change the height function along curves in $C_3(f)$ so that the height of the (b,b) curve is bit higher than those of others. This makes the lift $\widetilde{f}$ an embedding on M^3 except the points in $C_4(f)$. If there is no quadruple points in $f(M^3)$, then we obtain an embedded lift. Assume that there are some quadruple points. Let $\{\rho^1,\ldots,\rho^k\}$ be the quadruple points in $f(M^3)$. Let $\{q_i^j : i = 1,\ldots,4, j = 1,\ldots,k\}$ be the set of points in $C_4(f)$ with $f(q_i^j) = \rho^j$ $(i = 1,\ldots,4)$. For each q_i^j, there are four neighbourhoods $U(q_i^j)$ of q_i^j in M^3 and there is a 4–ball $B^4(\rho^j;\varepsilon)$ in $\boldsymbol{R}^4$ such that each $U(q_i^j)$ is embedded in $B^4(\rho^j;\varepsilon)\cap f(M^3)$ by f. We remove $\bigcup_{j=1}^k\Big(\bigcup_{i=1}^4 U(q_i^j)\Big)$ from M^3 and denote the resulting manifold by $M^{3\prime}$. As we modified the height function along $C_3(f)$, the restricted lift $\widetilde{f}|_{M^{3\prime}}$ is an embedding. Around each quadruple point ρ^j, the triple arcs satisfy (Z2). ¿From Lemma 2.2, $\{U(q_i^j)\cap C_3(f) \mid i = 1,\ldots,4\}$ has colours $\{(a,a,a), (a,a,b), (a,b,b), (b,b,b)\}$. Assume that q_1^j, q_2^j, q_3^j and q_4^j have colours (a,a,a), (a,a,b), (a,b,b), (b,b,b) respectively. Then we can modify the height function φ on $U(q_i^j)$ so that $\varphi(q_1^j) < \varphi(q_2^j) < \varphi(q_3^j) < \varphi(q_4^j)$. This implies that $\widetilde{f}\,|_{M^{3\prime}}$ extends to $M^{3\prime}\cup\big(\cup_{i=1}^4 U(q_i^j)\big)$ as an embedding. This is a local modification of the height function. Therefore, we can modify the function for all quadruple points and hence we obtain an embedded lift. □

4. Elementary Deformations

A homotopy on an immersed closed surface in a 3–manifold can be expressed by a sequence of six types of local deformations shown in Figure 3 and Figure 4 (see [**7**], [**8**]). The deformation from the left column of the arrow to the right column is called type $*^+$ h–move. Otherwise, called type $*^-$ h–move, $(* = \mathrm{I}, \ldots, \mathrm{V})$. Note that type VI h–move is not distinguished by the directions of arrows.

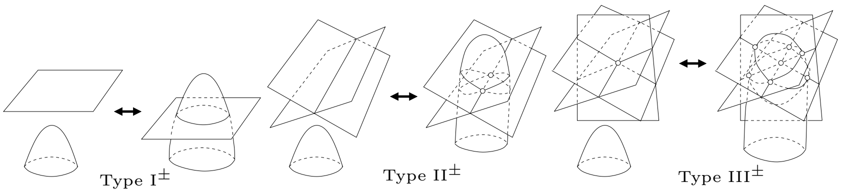

FIGURE 3. Type I$^\pm$, II$^\pm$ and III$^\pm$ h–moves

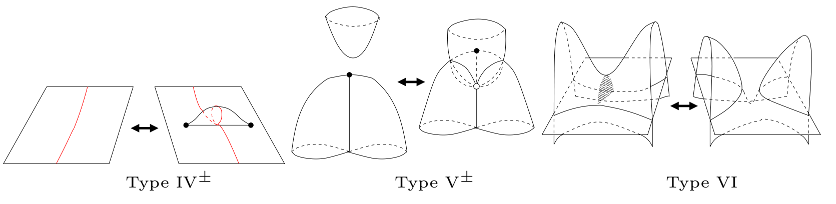

FIGURE 4. Type IV$^\pm$, V$^\pm$ and VI h–moves

Here we describe two types of h–moves; type IV$^\pm$ and type VI. Type IV$^\pm$ h–moves consist of only one disc and they create or eliminate a pair of branch points, and a double segment between the pair of branch points. The type VI h–move is described as a pair of discs in $\boldsymbol{R}^3$. It consists of two discs D_1 in $\boldsymbol{R}^3$ forming a saddle shape with respect to the z–axis and a horizontal disc D_2, which have a pair of proper double arcs, say α from a_1 to a_2 and β from b_1 to b_2, where a_i and b_i, $(i = 1, 2)$ are boundary points of D_2. We assume that these boundary points are ordered as $\{a_1, a_2, b_1, b_2\}$ with respect to the orientation of the boundary. As one of discs moves, these two segments move closer and touch at the middle point of each segment, then new segments γ from a_1 to b_2 and δ from a_2 to b_1 will appear (see the right picture of Type VI in Figure 4). Assume that the type VI h–move is applied to the pair of discs D_1 and D_2 and we use the same notations as the above. Let a be the middle point of the double curve α, and let b be the middle point of the double curve β. Then we can find a disc E bounded by a pair of arcs terminated by a and b; one of them is on D_1 and the other one is on D_2 such that the interior of E does not meet the immersed surface. This disc E is called a *descendent disc* [**10**]. On the other hand, if we find a descendent disc, then we can apply the type VI h–move. The following lemmas are proved in [**7**], [**8**] and [**13**].

Lemma 4.1 (Homma and Nagase [**7, 8**]). *Let $f, g : F^2 \to N^3$ be two immersions from a closed surface into a 3–manifold. If f is homotopic to g, then there is a sequence of h–moves realising this homotopy.*

Lemma 4.2 (Nagase [**13**]). *Let $f, g : F^2 \to N^3$ be two immersions from a closed orientable surface into a 3–manifold N^3. Then there is a sequence of h–moves which consist of type* I, II, III *and* VI *that deform the immersed surface $f(F^2)$ to $g(F^2)$ if and only if f is regularly homotopic to g.*

5. Constructing Immersed Surfaces

In this section, we assume that all surfaces are closed and orientable. Let $f : F^2 \to \mathbb{R}^3$ be an immersion from a surface into $\mathbb{R}^3$. It is proved that for the immersion f, there are an embedding $g : F^2 \to \mathbb{R}^3$ and an immersion $\gamma : S^1 \to F^2$ such that f is regularly homotopic to an immersion obtained from γ and g [**11**][**22**]. In this section, we describe a construction of such classes of immersed surfaces.

5.1. Bug constructions.

5.1.1. *Bugs.*
Let F^2 be a closed surface and let $g : F^2 \to \mathbb{R}^3$ be an embedding. Take a point $x \in F^2$ and set $y = f(x)$. Let $V(y)$ be a small neighbourhood of y in $\mathbb{R}^3$ and assume that $V(y) \cap f(F^2)$ is a disc. Applying type IV^+ h–move on the disc, we obtain a singular disc with a pair of branch points (see Type $\mathrm{IV}^\pm$ in Figure 4). We call this resulting part a *bug* (see [**23**]). Take an oriented immersion $\gamma : S^1 \to F^2$. Create a bug on γ, where branch points are on $g \circ \gamma(S^1) \subset F^2$ so that the double segment of the bug lies along γ, also it does not meet the crossing points of γ. We call one of the branch points a *head* and the other a *tail*. Then lengthen the bug along $g \circ \gamma$ on $g(F^2)$. The head can pass through a part of the bug itself creating a pair of triple points (see Figure 7). Finally, the head will reach the tail on $g \circ \gamma(S^1)$. Then we can eliminate the pair of branch points with a combination of type VI and type IV^- h–moves (see [**7**], [**8**]). This modification gives an immersed surface. We denote the immersion by g_γ. For every immersion $f : F^2 \to \mathbb{R}^3$, there is an embedding $g : F^2 \to \boldsymbol{R}^3$ and an immersion $\gamma : S^1 \to F^2$ such that f is regularly homotopic to g_γ [**6**] [**22**] (see Example 1.5).

6. Pipe Swappings

We introduce deformations of a pair of immersed discs in $\boldsymbol{R}^3$.

6.0.2. *Pipes.*
We will provide some notations to describe a modification of an immersed surface. Let J be the closed interval $[-1, 1]$. Let J^2 be a disc defined by $J \times J$. An immersed disc, which has an embedded proper double arc in the image, is called a *pipe* (Figure 5). Let $g : J^2 \to \boldsymbol{R}^3$ be an embedding defined by $g(x, y) = (x, y, 0) \in \boldsymbol{R}^3$,

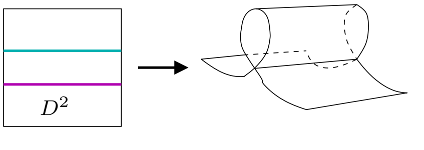

FIGURE 5. A pipe

$(x, y) \in J^2$. A pipe can be created on $g(J^2)$ such that the double segment of $g(J^2)$ lies along a proper line segment σ. We denote the pipe by $g_\sigma(J^2)$. Let $J_1^2 = J \times J$ and let $J_2^2 = J \times J$. Define $g_1 : J_1^2 \to J^2 \times \{0\}$ by $g_1(x, y) = (x, y, 0)$ for $(x, y) \in J_1^2$ and define $g_2 : J_2^2 \to J \times \{0\} \times J$ by $g_2(x, y) = (x, 0, y)$ for $(x, y) \in J_2^2$. Let σ_1 be the proper line segment $\{1/2\} \times J \times \{0\}$ in $g_1(J^2)$ and let σ_2 be the proper line segment $\{-1/2\} \times \{0\} \times J$ in $g_2(J^2)$. Then we have two pipes $g_{1\sigma_1}(J^2)$ and $g_{2\sigma_2}(J^2)$. Let

$N_1 = J^2 \times (-1/2, 1/2)$ and let $N_2 = J \times (-1/2, 1/2) \times J$. Then we assume that $g_{1\sigma_1}(J^2)$ and $g_{2\sigma_2}(J^2)$ are properly embedded in N_1 and N_2 respectively. Obviously, for each $i = 1, 2$, $g_{i\sigma_i}$ has an embedded lift, $\widetilde{g}_{i\sigma_i}$ defined by

$$\widetilde{g}_{i\sigma_1}(x, y) \;=\; \big(g_{i\sigma_i}(x, y), \varphi_1(x, y)\big), \tag{6.1}$$

where φ_i is a height function on J_i^2. Set $B_2 = {g_{2\sigma_2}}^{-1}\big(N_1 \cap g_{2\sigma_2}(J^2)\big)$. We can modify the function φ_2 on B_2 so that pipes can be swapped in the projection and the deformation track is liftable. A regular homotopy $H : (J_1^2 \cup J_2^2) \times I \to \mathbb{R}^3$ is

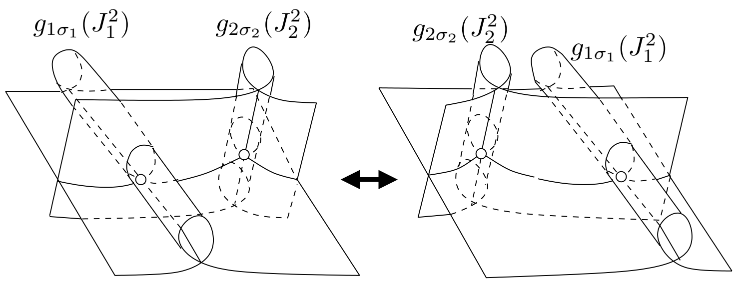

FIGURE 6. The pipe swapping.

defined as interchanging these pipes. We will call this modification a *pipe swapping*. The regular homotopy H induces a track: $\widehat{H} : I \times (J_1^2 \cup D_2^2) \to I \times \mathbb{R}^3$ defined by $\widehat{H}(t, x) = \big(t, H_t(x)\big)$ for $t \in I$ and $x \in (D_1^2 \cup D_1^2)$. Then the following holds [**16**].

Lemma 6.1. *The track $\widehat{H}$ induced from the pipe swapping deformation has an embedded lift into $\mathbb{R}^5$.*

7. Proof of Theorem 1.4

This section will be devoted to a proof of Theorem 1.4. First, we will discuss about a special immersed disc and its deformations. Then we will prove Theorem 1.4.

7.1. Pipe junctions.

In the bug construction with an embedded surface and an immersed circle on it, at a crossing point of the immersed circle, locally, a special immersed disc is created. We will discuss about this type of immersed disc and its deformation. Let B^3 be the closed unit 3–ball in $\boldsymbol{R}^3$ centred at the origin. Let D^2 be the closed unit disc in $\boldsymbol{R}^2 \times \{0\} \subset \boldsymbol{R}^3$. Let γ_0 be a straight line segment in D^2 from the point $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ to the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Assume that D^2 is oriented so that the vector $(0, 0, 1)$ is the normal vector for each point of D^2. Put a bug B_x along the x–axis so that the head b_1 and the tail b_2 of B_x are put at $(-1, 0)$ and $(1, 0)$ respectively, and the double segment of B_x lies along the x–axis. When we put the bug B_x on the disc, the line segment γ_0 becomes a curve, which is looped around B_x. We use the same notation γ_0 for the modified curve. We assume that γ_0 satisfies the following conditions.

(1) γ_0 does not have any self–intersection.

(2) γ_0 meets the double segment of B_x at two points.
Put another bug B_{γ_0} along γ_0 in the lower half of D^2; that is, $\gamma_0 \cap \{(x,y,0) : (x,y) \in D^2, y < 0\}$ such that B_{γ_0} does not meet B_x. We denote one of branch points of B_{γ_0} near B_x by b_3 and denote the other branch point by b_4. Extend the bug along γ_0 until b_3 reaches $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and b_4 reaches $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, so that the bug B_{γ_0} creates a pair of triple points p_0 and p_1 on the double segment of B_x. We denote the resulting singular disc with four branch points by Q'. ¿From this construction, obviously Q' is liftable into $\boldsymbol{R}^4$ [**16**] [**3**] [**18**]. Take disjoint small 3–ball neighbourhoods V_1, V_2, V_3 and V_4 of points $(-1,0)$, $(1,0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^3$ respectively, so that $V_i \cap (B_x \cup B_{\gamma_0}) \neq \varnothing$ $(i = 1, \ldots, 4)$. Remove $\bigcup_{i=1}^{4} (Q' \cap V_i)$ from Q' so that we obtain an immersed disc with two triple points. We call this immersed disc a *pipe junction* and we denote this by Q. The pre–image of Q and Q itself are depicted in Figure 7. Let B^3 be the closed unit 3–ball. Assume that Q is properly embedded in B^3. ¿From the construction, $B^3 \setminus Q$ is a union of open 3–balls. Thus the following holds.

Lemma 7.1. *Let Q be a pipe junction properly embedded in B^3. Then $B^3 \setminus Q$ is a union of open 3–balls.*

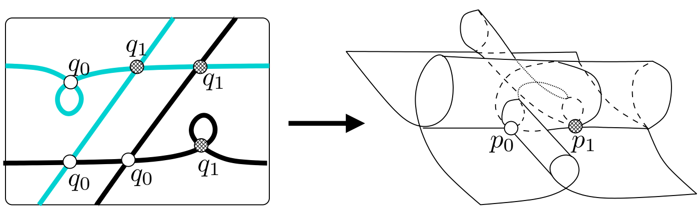

FIGURE 7

In the pre–image in Figure 7 we colour a pair of arcs, in the pre–images of the double curve of the pipe P_{γ_0}, with a and b. There are four types of colourings on the pre–image. It is not difficult to check that the pre–image satisfies the CS–conditions for each colouring. Thus we obtain an embedded lift of Q in $\mathbb{R}^4$ hence the following holds.

Lemma 7.2. *Let $g : F^2 \to \mathbb{R}^3$ be an embedding and let $\gamma : S^1 \to F^2$ be an immersion. Then we can construct a liftable immersion $g_\gamma : F^2 \to \mathbb{R}^3$ with its embedded lift $\widetilde{g}_\gamma$ into $\mathbb{R}^4$.*

7.2. Deformations of pipe junctions.

Let B^3 be the closed unit 3–ball and let Q be a properly embedded pipe junction in B^3. ¿From Lemma 7.1, the complement of Q in B^3 is a union of open 3–balls. We denote one pipe derived from the bug B_x by P_x and denote the pipe along γ_0 by P_{γ_0}. Note that P_{γ_0} is divided by P_x into three pipes. The pipe along γ_0 from the point $\gamma_0 \cap \partial B^3$ to p_0 is denoted by $P_{\gamma_0 1}$. The pipe between p_0 and p_1 is denoted by $P_{\gamma_0 2}$. The pipe along γ_0 from p_1 to the point $\gamma_0 \cap \partial B^3$ is denoted by $P_{\gamma_0 3}$. The portion bounded by ∂B^3, P_x and P_{γ_0} is denoted by V_x. The portion bounded by P_x and $P_{\gamma_0 2}$ is denoted by $V_{\gamma_0 2}$. The set of double curves in Q consists of two immersed curves α and β. α has two loops ℓ_0 and ℓ_1 based at triple points p_0 and p_1 respectively and β is a double curve containing p_0 and p_1 and no loops. Obviously, $\alpha \cap \beta = \{p_0, p_1\}$. Let $e : [0,1] \to \alpha$ be an embedded sub–arc of α such that there exists $0 < \varepsilon < 1/2$ with $e(\varepsilon) = p_0$ and $e(1-\varepsilon) = p_1$. We use the same notation e for the image. Take a tubular neighbourhood $N(e)$ of e in $\boldsymbol{R}^3$. The intersection $N(e) \cap Q$ consists of four discs D_0, D_1, D_2 and D_3 so that $D_0 \cap D_1 = e$, and D_2 and D_3 intersect to $D_0 \cup D_1$ transversely at p_0 and p_1 respectively, also $\ell_i \cap D_i \setminus p_i$ is a union of two double segments connecting to p_i, $(i = 0, 1)$. Take arcs e_{i0} in D_i, $(i = 0, 1)$ from a point in $q_i \in \ell_i \cap D_i$ to a point $r_i \in \beta \cap D_i$ such that e_{i0} is parallel to e and $\mathrm{Int}(e_{i0})$ does not meet the multiple points in Q. There also exists a simple arc e_{i1} on $\mathrm{Cl}(P_{\gamma_0} \cap V_x)$ from q_i to r_i such that $e_{i0} \cup e_{i1}$ is a simple closed curve on the boundary of $\mathrm{Cl}(V_x)$. Thus for each i, $e_{i0} \cup e_{i1}$ bounds a disc G_i inside $\mathrm{Cl}(V_x)$ The disc G_i is a descendent disc, which guides the type VI h–move so that the loop ℓ_0 and β will be connected (see Figure 8). Note that there are two disjoint discs

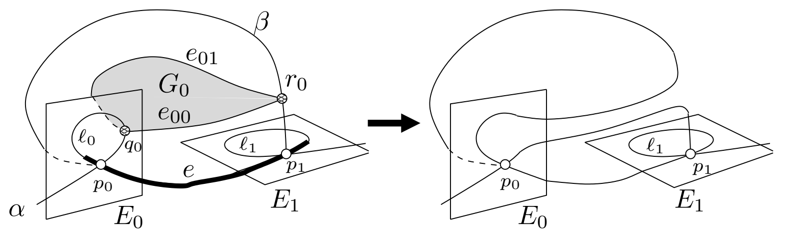

FIGURE 8

E_0 and E_1 in Q containing ℓ_0 and ℓ_1 respectively. E_0 and E_1 transversely intersect with P_{γ_0}. In order to see the above deformation, we look at a partial diagram of Q containing $P_{\gamma_0} \cup E_0 \cup E_1 \cup N(e_{i0}) \cup N(e_{i1})$. We deform the pipe P_{γ_0} by an isotopy so that E_0 and E_1 becomes parallel, $N(e_{i0})$ and $N(e_{i1})$ are also deformed and $e_{i0} \cup e_{i1}$ bound discs G_i $(i = 0, 1)$. We shall make a model of the deformed partial diagram of Q to see how we deform Q with the type VI h–move along G_0 or G_1. We will use the same notations of Q for corresponding part of the model. Consider an immersion $\ell : J \to J \times J$ with exactly one crossing point $\ell(0) = p$ such that $\ell(-1) = (-1, 0)$ and $\ell(1) = (0, 1)$, and also ℓ tangents to the x–axis at $\ell(-1)$ and it tangents to the y–axis at $\ell(1)$. We denote the immersed arc $\ell(J)$ by ℓ. Also we assume that ℓ lies in $([0,1] \times J) \cup (J \times [-1, 0])$ and $p \in ([0,1] \times J) \cap (J \times [-1, 0])$. We view $J \times \ell$ as a pipe disc in $J \times J \times J$. We denote this by P. Take a parallel discs $E_0 = \left\{\frac{1}{2}\right\} \times J \times J$ and $E_1 = \left\{-\frac{1}{2}\right\} \times J \times J$. These two discs intersect with the pipe P. Thus the intersection contains two triple points $\left\{\frac{1}{2}\right\} \times \{p\}$ and $\left\{-\frac{1}{2}\right\} \times \{p\}$. We denote them by p_0 and p_1 respectively, and denote the double loop based at p_i in $E_i \cap P$ by ℓ_i for $i = 0, 1$. We also denote the double curve between p_0 and p_1 on $J \times \ell$ by γ. Let L denote $E_0 \cup E_1 \cup P$. Add an orientation to L so that normal vectors to E_0 and E_1 are $(1, 0, 0)$, $(-1, 0, 0)$ respectively, and the normal vector to P at $(0, 0, 1)$ is $(0, -1, 0)$. Attach two 1–handles h^1_1 and h^1_2 to L such that the orientation of each handle is compatible with the orientation of L and the attaching edges of h^1_1 are: $\left\{\frac{1}{2}\right\} \times \left[\frac{1}{2}, 1\right] \times \{1\} \subset E_0$ and $\left[-\frac{1}{4}, \frac{1}{4}\right] \times \{0\} \times \{1\} \subset P$, and the attaching edges of h^1_2 are: $\left\{-\frac{1}{2}\right\} \times \{-1\} \times \left[-1, -\frac{1}{2}\right] \subset E_1$ and $\left[-\frac{1}{4}, \frac{1}{4}\right] \times \{-1\} \times \{0\} \subset P$. Also we assume that each handle is not twisted with respect to the orientation of L. Let us denote the set $L \cup h_1^1 \cup h_2^1$ by $\overline{L}$. Take a point q_i on ℓ_i and r_i on γ so that there is a pair of arcs e_{i0} form q_i to r_i through h_i^1 and e_{i1} from r_i to q_i on P. The arc $e_{i0} \cup e_{i1}$ bounds a disc denoted by G_i $(i = 0, 1)$. ¿From the construction of Q, it is liftable, and thus $\overline{L}$ is liftable. It is not difficult to see that we can assume that the height of E_0 is higher than the height of P and the height of E_1 is lower than that of P or vice versa. Here we assume that E_0 is higher than P and E_1 is lower than P. Now we consider the descendent disc G_0. After applying Type VI h–move, in the resulting immersed surface, the loop ℓ_0 disappear and a new pipe appears. The double segment of the new pipe between P_0 and P_1 goes around P. (see Figure 9). The resulting immersed disc is shown as the right picture of Figure 10. This shows

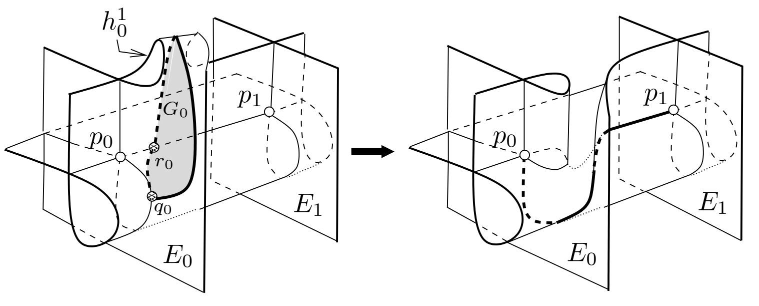

FIGURE 9

that there are two pipes intersecting each other with two triple points, p_0 and p_1. These pipes are deformed into disjoint pipes by a regular homotopy.

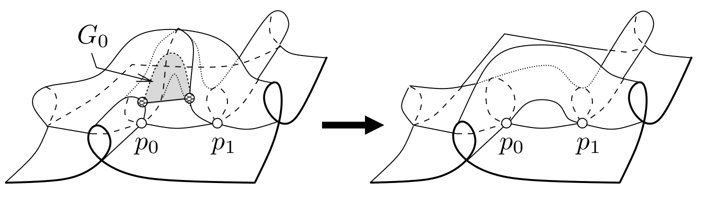

FIGURE 10

7.3. Pre–images of pipe junctions.

The deformation in the previous subsection deforms a pipe junction into a pair of disjoint pipes. We shall show that this deformation is liftable into $\boldsymbol{R}^5$. Pre–images of the above deformation on Q are depicted form Figure 11 to 13. In the left picture of Figure 11, dashed lines represent a pair of arcs, which are sent to the boundary of the descendent disc G_0 in $\boldsymbol{R}^3$. Crossing points q_0 and q_1 are sent to p_0 and p_1 in Q. If the dashed arcs joins different coloured curves, then the track does not have an embedded lift into $\mathbb{R}^5$ with fixing end maps. This can be deformed into the suitable diagram by the swapping pipe move. Thus we can assume that the pipe junction has an appropriate colourings. We apply the type VI h–move on the deformed pipe junction (see Figure 11). In Figure 12, the second VI h–move

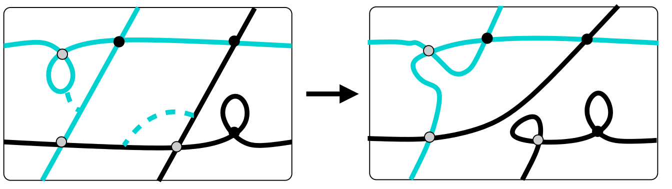

FIGURE 11

produces four proper arcs and a pair of circles (see the right picture of Figure 12). Obviously, this deformation is liftable.

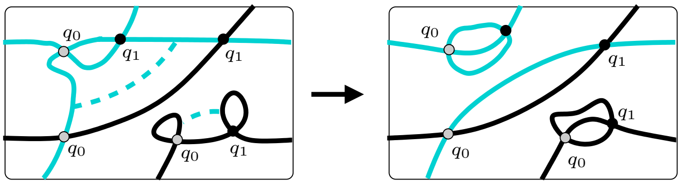

FIGURE 12

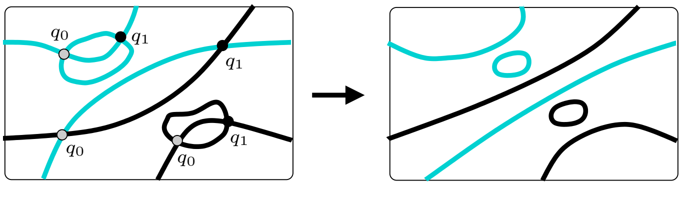

FIGURE 13

Note 7.3. It is shown that there exists a deformation from Q to the pair of disjoint pipes with sphere eversion tracks [**22**] so that it is not covered by an isotopy in $\mathbb{R}^4$.

7.3.1. *Smoothings.*

Let F^2 be a closed orientable surface and let $\gamma : S^1 \to F^2$ be an oriented immersion. For each crossing point p of $\gamma(S^1)$, we have a small disc neighbourhood $U(p)$ of p in F^2. Then $U(p)$ contains two proper arcs denoted by α and β. Let the boundary points of α and β be $\{a_0, a_1\}$ and $\{b_0, b_1\}$ respectively. Assume that α is an arc from a_0 to a_1 and β is an arc from b_0 to b_1. Then a *smoothing* is an operation that replace $\alpha \cup \beta$ in $U(p)$ with disjoint arcs α' from a_0 to b_1 and β' from b_0 to a_1 (see Figure 14). Let $g : F^2 \to \boldsymbol{R}^3$ be an embedding. We obtain an immersion g_γ with the bug construction along γ. Each crossing point of γ induces a pipe junction. Applying the above deformations to all pipe junctions of $g_\gamma(F^2)$ so that we obtain a new immersion g_δ where δ is a set of disjoint simple closed circles on F^2 which is obtained by applying smoothings on γ. The pre–image of a pipe junction of $g_\gamma(F^2)$ is depicted as the left picture of Figure 11. Thus crossing sets $C(g_\gamma)$ and $C(g_\delta)$ satisfy the (CS)–conditions. The colouring condition is consistent

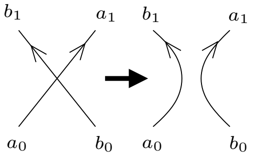

FIGURE 14

during the deformation. This implies that the regular homotopy track from g_γ to g_δ has an embedded lift into $\boldsymbol{R}^5$. Thus the lift gives an isotopy from $\widetilde{g}_\gamma$ to $\widetilde{g}_\delta$ in $\mathbb{R}^4$. Therefore, Theorem 1.4 is proved.

8. Constructing Immersed 3–Spheres in $\boldsymbol{R}^4$

A pipe junction contains six double curves consisting of two loops based at triple points, two arcs bounded by these triple points and four arcs joining the boundary of the immersed disc and triple points. We ignore double loops and one of double arcs bounded by triple points. Then we obtain a diagram as the left picture of Figure 15. The regular homotopy deformation described in Section 7 separates the diagram as in Figure 15. Let $g : S^2 \to \mathbb{R}^3$ be an embedding. Let $\bigcirc : S^1 \to S^2$ be an

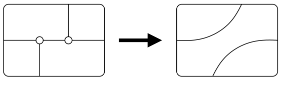

FIGURE 15

embedding and let $8 : S^1 \to S^2$ be the 8–immersion. Ignoring double loops and one double arc in the pipe junction as the above, then we have two types of diagrams representing constructed immersed sphere from g and the 8–immersion (see Figure 16). We denote those types by (a) and (b) respectively. In the diagram of type

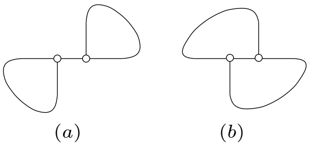

FIGURE 16

(a), double loops do not meet each other, while in the diagram of type (b), double loops meet at a double segment between the triple points. We denote the immersed sphere represented by the diagram (a) by $g_8^a(S^2)$ and denote the immersed sphere represented by the diagram (b) by $g_8^b(S^2)$. The deformation of the pipe junction deforms $g_8^a(S^2)$ into $g_\bigcirc(S^2)$ and it deforms $g_8^b(S^2)$ into $g_{\{\bigcirc \cup \bigcirc\}}(S^2)$, where $\bigcirc \cup \bigcirc$ is a disjoint union of two copies of the embedding $\bigcirc$. It is easy to see that the former track with the particular deformation on the pipe junction does not have an embedded lift into $\boldsymbol{R}^5$. Therefore, we can construct a non–liftable immersed 3–sphere in $\boldsymbol{R}^4$ with this track.

8.1. Proof of Theorem 1.6.

Proof. Using the deformation of a pipe junction, we can construct a non–liftable immersed 3–sphere in $\mathbb{R}^4$ without quadruple points into $\mathbb{R}^5$. We use the same notations in Example 1.5 and in the above arguments. Let $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection on xy–plane. Then we deform $i(S^2)$ to the immersed sphere $(\chi \circ i)_{\bigcirc}(S^2)$. This is done by the type I^+ h–move applying into $i(S^2)$. We can deform $(\chi \circ i)_{\bigcirc}(S^2)$ to $(\chi \circ i)^a_8(S^2)$ with the reverse deformation of the deformation on a pipe junction of type (a). Crossing sets of both maps are shown in Figure 17. Then we deform

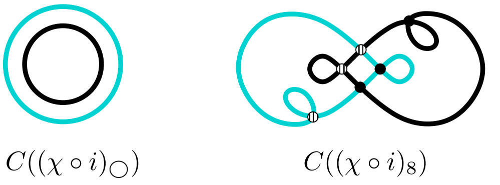

FIGURE 17

$(\chi \circ i)^a_8(S^2)$ to $(\chi \circ i)_{\bigcirc}(S^2)$ with the deformation of a pipe junction. Finally, we return to $i(S^2)$. Let $H : S^2 \times I \to \boldsymbol{R}^3$ be a regular homotopy from i to itself with above deformations passing through $(\chi \circ i)_{\bigcirc}(S^2)$, $(\chi \circ i)^a_8(S^2)$ and $(\chi \circ i)_{\bigcirc}(S^2)$. Then H induces a regular homotopy track $\widehat{H} : I \times S^2 \to I \times \boldsymbol{R}^3 \subset \boldsymbol{R}^4$. Obviously, $\widehat{H}$ is an immersion without quadruple points. We cap off both ends of this track with 3–balls so that we obtain an immersion $f : S^3 \to \boldsymbol{R}^4$. Hence, the immersed 3–sphere $f(S^3)$ does not contain quadruple points. The crossing set $C\big((\chi \circ i)^a_8\big)$ consists of two immersed circles (see Figure 17). As we have seen in Section 7.2, during the deformation from $(\chi \circ i)^a_8(S^2)$ to $(\chi \circ i)_{\bigcirc}(S^2)$, the type VI h–move joins those immersed components of $C(H_t)$ with distinct colours. This implies that $C(f)$ has one component, which is an immersed surface. This shows that f does not satisfy the condition (Z1) of Theorem 1.3 thus we have proved Theorem 1.6. □

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