MOCK THETA FUNCTIONS AND THETA FUNCTIONS

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1. Introduction

In his last letter to Hardy, Ramanujan gave a list of seventeen functions $F(q)$, where $q$ is a complex number and $|q| < 1$, and called them “mock theta functions”. He called them mock theta function as they were not theta functions. He further stated that as $q$ radially approaches any point $e^{2\pi i r}$ ($r$ rational) there is a theta function $\theta_r(q)$, such that $F(q) - \theta_r(q) = O(1)$. Moreover there is no single theta function which works for all $r$ i.e. for every theta function $\theta(q)$, there is some root of unity $r$ for which $F(q) - \theta(q)$ is unbounded as $q \to e^{2\pi i r}$ radially.

Mock theta functions are mysterious functions and not much is known about these functions. Watson [16,17], Andrews [2,6], Andrews and Garvan [7], Andrews and Hickerson [8] have done much work on these mysterious mock theta functions. Ramanujan [14, p. 9] gave eight identities involving four mock theta functions and Choi [9] called them of order ten.

Ramanujan listed seventeen mock theta functions and divided them into “third order”, “fifth order” and “seventh order”, but did not explain what he meant by the “order” of the mock theta functions, he did not give any rigorous definition of the “order”. Recently Gordon and McIntosh [12] defined the “order” of a mock theta functions by its behaviour under the action of the modular group. By this definition the tenth order mock theta functions of Ramanujan, as Choi called them, are actually of order five, but these tenth order mock theta functions are not connected with the fifth order mock theta functions listed in Ramanujan’s last letter, because in their transformation formulae the Mordell integrals involved are not related [12].

Recently Gordon and McIntosh [11] by performing half-shift transformations on the ordinary theta series obtained Ramanujan’s mock theta functions. Using this method they constructed eight functions and called them of order eight. Gordon and McIntosh [12] pointed out that four of these functions are of eighth order and the other four are of lower order (considering their behaviour under the action of modular group).

Nobody knows what Ramanujan had in his mind when he mentioned the “order” of the mock theta functions. We find that by applying half-shift transformations on the theta series, the order of the obtained mock theta functions is the same as order of the theta series, defining order as the level of a modular form. The four eighth order mock theta functions of Gordon and McIntosh are obtained by applying the half-shift transformation on eighth order theta series. In this paper we obtain Ramanujan’s tenth order mock theta functions (which are actually of order

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by performing half-shift transformation on the fifth order theta series. This further shows that these tenth order mock theta functions are actually of order five. It is interesting that if we give a left half-shift transformation to certain \(q\)-series, instead of getting mock theta functions \(X_R(q)\) and \(\chi_R(q)\) we get 
\[ -iq^{-\frac{1}{2}} X_R(q) \] 
and 
\[ -iq^{-\frac{9}{4}} \chi_R(q). \]

Later, generalizing a theta series to two variable and on specializing we get theta functions and mock theta functions. We also give a first order non homogeneous \(q\)-difference equation for these generalized functions and express them as polynomials. These polynomials have combinatorial interpretation given by Andrews [4].

2. Notation

We shall use the following usual basic hypergeometric notations: For a complex number \(q\) with \(|q^k| < 1\),

\[
(a; q^k)_n = (1 - a)(1 - aq^k)...(1 - aq^{(n-1)k}), n \geq 1
\]
\[
(a; q^k)_0 = 1,
\]
\[
(a; q^k)_\infty = \prod_{j=0}^{\infty} (1 - aq^{jk}),
\]
\[
(a_1, a_2, \cdots, a_m; q^k)_n = (a_1; q^k)_n(a_2; q^k)_n\cdots(a_m; q^k)_n,
\]
\[
(a; q^k)_n = (a)_n,
\]

For non-negative integers \(n\), we have

\[
(a; q^k)_n = \frac{(a; q^k)_\infty}{(aq^{nk}; q^k)_\infty},
\]

and we will take this as the definition of \((a; q^k)_n\) for real \(n\) also.

3. Tenth Order Mock Theta Functions

The four tenth order mock theta functions, as defined by Ramanujan, are

\[
\Phi_R(q) := \sum_{n=0}^{\infty} \frac{q^n(n+1)/2}{(q; q^2)_{n+1}},
\]
\[
\Psi_R(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q; q^2)_{n+1}},
\]
\[
X_R(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(-q; q)_{2n}},
\]
and
\[
\chi_R(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q; q)_{2n+1}}.
\]
4. Mock Theta Functions And Theta Functions

In this section we apply left half-shift transformation on theta series to obtain Ramanujan’s mock theta functions.

(i) We shall first obtain the mock theta function $\Phi_R(q^2)$. Consider the identity of Slater [15, eq. (20)] viz;

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4; q^4)_n} = \frac{1}{(-q^2; q^2)_\infty (q; q^5)_\infty (q^4; q^5)_\infty} = \frac{1}{(-q^2; q^2)_\infty} G(q),
\]

where the last equality is the first Rogers-Ramanujan identity. Now

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4; q^4)_n} = \frac{1}{(q^2; q^4)_\infty} \sum_{n=0}^{\infty} q^{n^2} (q^{4n+4}; q^4)_\infty.
\]

So

\[
(q^4; q^4)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4; q^4)_n} = \sum_{n=0}^{\infty} q^{n^2} (q^{4n+4}; q^4)_\infty = \sum_{n=0}^{\infty} a_n, \quad \text{(say),}
\]

where $a_n$ is defined for all real $n$. Making a left half-shift transformation and summing $a_n$ over the positive half-integers $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ instead of the non-negative integers. Define $b_n = a_n - \frac{1}{4}$.

Then

\[
\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n - \frac{1}{4} = \sum_{n=0}^{\infty} q^{n^2-n+\frac{1}{4}} (q^{4n+2}; q^4)_\infty = q^{\frac{1}{4}} (q^2; q^4)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q^2; q^4)_n} = q^{\frac{1}{4}} (q^2; q^4)_\infty (\Phi_R(q^2) + 1),
\]

since

\[
\Phi_R(q^2) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2; q^4)_{n+1}} = \sum_{n=1}^{\infty} \frac{q^{n^2-n}}{(q^2; q^4)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2-n}}{(q^2; q^4)_n} - 1, \quad \text{by (1)}
\]

Thus by (6) and (7) we obtain $q^{\frac{1}{4}} (q^2; q^4)_\infty (\Phi_R(q^2) + 1)$, by applying a left half-shift on $(q^4; q^4)_\infty \left[\frac{1}{(-q^2; q^2)_\infty} G(q)\right]$, where $\Phi_R(q)$ is Ramanujan’s tenth order mock theta function.

(ii) We now obtain the mock theta function $\Psi_R(q^2)$.

Consider the identity of Slater [15, eq. 6], viz.,
\[
\sum_{n=0}^{\infty} q^{n^2+2n} (q^4; q^4)_n = \frac{1}{(-q^2; q^2)_{\infty}(q^4; q^4)_{\infty}(q^5; q^5)_{\infty}} \\
= \frac{1}{(-q^2; q^2)_{\infty}} H(q),
\]

where the last equality is the second Rogers-Ramanujan identity.

Similarly we obtain
\[
q^{-\frac{3}{4}} (q^2; q^4)_{\infty} \left(\Psi_R(q^2) + 1\right),
\]
by applying the left half-shift transformation on \((q^4; q^4)_{\infty} \left[\frac{1}{(-q^2; q^2)_{\infty}} H(q)\right],\) where \(\Psi_R(q)\) is the Ramanujan’s tenth order mock theta function.

As pointed earlier, both \(\Phi_R(q)\) and \(\Psi_R(q)\) have been obtained by applying left half-shift transformation on ‘genuine’ fifth order theta series \(G(q)\) and \(H(q)\).

(iii) Now we obtain the mock theta function \(X_R(q)\).

Consider the function, Rogers [4, p. 92, (9.30)],
\[
\sum_{n=0}^{\infty} (-1)^n q^{n^2+n} (q; q)_{2n+1} = \sum_{n=0}^{\infty} q^{n(n+1)} (1 - q^{2n+1}).
\]

Let
\[
\sum_{n=0}^{\infty} (-1)^n q^{n^2+n} (q; q)_{2n+1} = \sum_{n=0}^{\infty} a_n, \text{ (say)}.
\]

The term on the right-hand side is defined for all real \(n\). We make a left half-shift transformation and sum over the positive half-integers \(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\).

Let \(b_n = a_{n-\frac{1}{2}}\), then
\[
\sum_{n=0}^{\infty} b_n = -i \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2-\frac{1}{4}}}{(-q; q)_{2n}}
\]
\[
= -iq^{-\frac{1}{4}} X_R(q).
\]

Hence \(-iq^{-\frac{1}{4}} X_R(q)\) can be obtained by applying a left half-shift transformation on \(\sum_{n=0}^{\infty} (-1)^n q^{n^2+n} (q; q)_{2n+1}\).

(iv) Now we obtain the mock theta function \(X_R(q)\).

Applying a left half-shift transformation on the series
\[
\sum_{n=0}^{\infty} (-1)^n q^{n^2+3n} (q; q)_{2n+2}
\]
we obtain \(-iq^{-\frac{1}{4}} X_R(q)\).

5. Asymptotic Expansion

By McIntosh [13, pp 126-127], since the sequences \((a)_n\) and \((b)_n\) are unimodal for a fixed \(q \in (0, 1)\), we have
\[
\sum_{n=0}^{\infty} b_n = \left[ \sum_{n=0}^{\infty} a_n \right] \left[ 1 + O(e^{-\frac{1}{4}}) \right], \quad (11)
\]
where $\delta$ is a positive constant. By (5), (6) and (7), we have
\[-\frac{t}{4} + \log(q^2; q^4) + \log(1 + \Phi_R(q^2)) = \log(q^2; q^2) + \log G(q) + O(e^{-\frac{t}{2}}). \tag{12}\]

By Watson \[16, pp 57-58\], we have
\[(q)_{\infty} = \sqrt{\frac{2\pi}{t}} \exp \left[-\frac{\pi^2}{6t} + \frac{t}{24}\right] + o(1).\]
Hence
\[(q^2; q^2)_{\infty} = \sqrt{\frac{\pi}{t}} \exp \left[-\frac{\pi^2}{12t} + \frac{t}{12}\right] + o(1)\]
and
\[(q^2; q^4)_{\infty} = (q^2; q^2)_{\infty} (q^4; q^4)_{\infty} = \sqrt{2} \exp \left[-\frac{\pi^2}{24t} - \frac{t}{12}\right] + o(1).\]
Putting these in (12), we have
\[
\log(\Phi_R(q^2) + 1) = -\frac{1}{2} \log t + \frac{2t}{5} + \frac{\pi^2}{40t} + \frac{1}{2} \log \frac{\pi}{5 - \sqrt{5}} + O(t^p), \tag{13}
\]
where $p$ is a positive integer.
This shows that $\log(\Phi_R(q^2) + 1)$ has an asymptotic expansion of the form
\[
\log(\Phi_R(q^2) + 1) = \frac{A}{t} + B \log t + \sum_{k=0}^{N} C_k t^k + O(t^p), \tag{14}
\]
where $p$ is a positive integer $> N$.
Similarly $\log(\Psi_R(q^2) + 1)$ has an asymptotic expansion of the form
\[
\log(\Psi_R(q^2) + 1) = -\frac{1}{2} \log t - \frac{3t}{5} + \frac{\pi^2}{40t} + \frac{1}{2} \log \frac{\pi}{5 + \sqrt{5}} + O(t^p).
\]

6. Two Variable Generalized Functions

We define the following two variable generalized functions:

(a) $f_\phi(q, t) = \sum_{n=0}^{\infty} \frac{t^{2n} q^{n^2}}{(t^4; q^4)_{n+1}}$

(b) $f_\psi(q, t) = \sum_{n=0}^{\infty} \frac{t^{2n} q^{n^2+2n}}{(t^4; q^4)_{n+1}}$

(c) $f_X(q, t) = \sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(t; q)_{2n+2}}$

(d) $f_X(q, t) = \sum_{n=0}^{\infty} \frac{t^n q^{n^2+3n}}{(t; q)_{2n+3}}$

We shall now show that these functions have the following properties:
(i) They can be written as $\sum_{n=0}^{\infty} D_n(q) t^n$, where $D_n(q)$ are polynomials.
(ii) We can determine $\lim_{t \to \infty} D_n(q)$.
(iii) They satisfy a first-order non homogeneous $q$-difference equation.
We shall start with the (iii) property.
\[ f_\phi(q, t) = \sum_{n=0}^{\infty} \frac{t^{2n}q^{n^2}}{(t^4; q^4)_n + 1} \]

\[ = \frac{1}{1 - t^4} + \sum_{n=1}^{\infty} \frac{t^{2n}q^{n^2}}{(t^4; q^4)_n + 1} \]

\[ = \frac{1}{1 - t^4} + \frac{1}{1 - t^4} \sum_{n=0}^{\infty} \frac{q^{2n+2}q^{n^2+2n+1}}{(t^4q^4; q^4)_{n+1}} \]

\[ = \frac{1}{1 - t^4} + \frac{qt^2}{1 - t^4} f_\phi(q, tq), \]

or

\[ (1 - t^4) f_\phi(q, t) = 1 + qt^2 f_\phi(q, tq), \]

which proves (iii).

**Proof of (i).** Using a relation of Andrews [1, Theorem 3.3, p. 36]

\[ f_\phi(q, t) = \sum_{n=0}^{\infty} t^{2n}q^{n^2} \sum_{m=0}^{\infty} \frac{q^{n^2[N-m]_4}}{(q^4)_m} \]

\[ = \sum_{N=0}^{\infty} \frac{t^{2N}}{1 - t^4} D_N(q), \]

where

\[ D_N(q) = \sum_{0 \leq 2m \leq N} q^{n^2[N-m]_4}, \]

which proves (i).

**Proof of (ii).** By Abel’s lemma [18, p. 57]

\[ \lim_{n \to \infty} D_n(q) = \lim_{t \to 1^{-}} (1 - t) f_\phi(q, t) \]

\[ = \frac{1}{4} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4; q^4)_n} \]

\[ = \frac{1}{4(-q^2; q^2)_{4\infty}(q; q^2)_{4\infty}(q^4; q^4)_{4\infty}}, \] by (5)

which proves (ii).

Since all these generalized series are so similar, it can be easily seen that they have the properties (i)-(iii).
For $t = \sqrt{q}$, we have

$$f_\phi(q, \sqrt{q}) = \Phi_R(q^2).$$

For $t = \sqrt{q}$, we have

$$f_\psi(q, \sqrt{q}) = q^{-2}\Psi_R(q^2).$$

For $t = -q^{-1}$, we have

$$f_X(q, -q^{-1}) = \frac{q}{2(1+q)}X_R(q).$$

For $t = -q^{-1}$, we have

$$f_\chi(q, -q^{-1}) = \frac{1}{2(1+q)}\chi_R(q).$$

**Combinatorial Significance**

The combinatorial significance of the polynomial $D_n(q)$ are given in detail in Andrews [4].

**References**


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