ON MODELS OF $q$–REPRESENTATIONS OF $\text{sl}(2, \mathbb{C})$ AND $q$–CONFLUENT HYPERGEOMETRIC FUNCTIONS

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Abstract. We construct new two variable models of irreducible $q$–representations of the Lie algebra $\text{sl}(2, \mathbb{C})$. The $q$–Mellin transformation is defined and used to transform the models in the form of $q$–difference dilation models, involving basic hypergeometric function $2\phi_1$ as basis functions. All the models culminate in many special function identities and recurrence relations.

1. Introduction

Manocha [6] introduced the idea of irreducible $q$–representations of the special complex Lie algebra $\text{sl}(2, \mathbb{C})$ and developed theory centering around it. A $q$–analogue of fractional calculus technique [1, 7] was used to construct models of irreducible $q$–representations of $\text{sl}(2, \mathbb{C})$ followed by the Weisner’s method [18] to obtain identities involving $q$–special functions of one and several variables. These models were in terms of $q$–differintegral operators acting on $q$–hypergeometric functions $2\phi_1$. Further, in [15], $q$–differintegral operator models of $\text{sl}(2, \mathbb{C})$ with basis functions involving $k^{2+1}\phi_{2k}$ functions were constructed and a good number of identities were derived from them.

The theory of $q$–representation of Lie algebras together with the theory of $q$–integral transformations has also been a rich source of results in $q$–special function theory. In Sahai [12, 13, 14], the $q$–Euler integral transformation, which is motivated by $q$–integral representation of beta function, is utilized to obtain models in terms of $q$–difference dilation operators of $\text{sl}(2, \mathbb{C})$ along with identities involving $q$–special functions. Both these models, viz. $q$–differintegral operator models and $q$–difference dilation operator models, are of $q$–representations $D_q(u, \alpha)$ and $\uparrow_q(u)$ of $\text{sl}(2, \mathbb{C})$ in the particular case $u = 0$.

In this paper, we construct new models of $q$–representations $D_q(u, \alpha)$ and $\uparrow_q(u)$ of $\text{sl}(2, \mathbb{C})$ corresponding to $u \neq 0$ acting on $q$–confluent hypergeometric functions $1\phi_1$. We then define $q$–Mellin transformation, which is motivated by $q$–integral representation of gamma function. The $q$–Mellin transformation is the $q$–analogue of Mellin integral transformation used in [5, 10]. We utilize this transformation to transform these models of $\text{sl}(2, \mathbb{C})$ to new models involving $q$–difference dilation operators with the $q$–hypergeometric functions $2\phi_1$ as basis functions. The discussion culminates in interesting identities and recurrence relations involving $1\phi_1$ and $2\phi_1$ functions.

Section–wise treatment is as follows.

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In Section 2, we give definitions involving hypergeometric and basic hypergeometric series, needed for our discussion. In Section 3, Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is presented along with its $q$–deformed version. In Section 4, we construct model of $D_q(-\gamma/2, \alpha)$ and $\Phi_q(-\gamma/2)$ in two variables acting on $\phi_1$ and obtain identities and recurrence relations based on it. In Section 5, we introduce $q$–Mellin transformation and then utilize it to transform the models given in Section 4 to $q$–difference dilation models acting on basis functions $\phi_1$. The transformed models are further used in obtaining identities and recurrence relations involving $\phi_1$. Finally, in Section 6, we give an application of these techniques to $q$–Laguerre polynomials.

2. Preliminaries

The generalized basic or $q$–hypergeometric series $r \phi_s$ is defined as \[ 1 \]

$$r \phi_s \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n (q; q)_n} \left[ (-1)^n q^{\frac{1}{q}} \right]^{1+q^{-n}} x^n,$$

where $q$–shifted factorial $(a; q)_n$ is defined by

$$ (a; q)_n = \begin{cases} (1 - a) (1 - aq) \cdots (1 - aq^{n-1}), & n = 1, 2, \ldots \\ 1; & n = 0 \end{cases} $$

The series $r \phi_s$ converges absolutely for all $x$ if $n \leq s$; and for $|x| < 1$ if $r = s + 1$. If $|q| > 1$ and $|x| < |\frac{b_1 - b_s}{a_1 - a_r}|$, then also $r \phi_s$ converges absolutely. It diverges for $x \neq 0$ if $0 < |q| < 1$ and $r > s + 1$, and if $|q| > 1$ and $|x| > |\frac{b_1 - b_s}{a_1 - a_r}|$, unless it terminates.

The generalized hypergeometric series $F_s$ is defined by \[ 2 \]

$$F_s \left( \frac{\alpha_1, \ldots, \alpha_s}{\beta_1, \ldots, \beta_s}; z \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n n!} z^n,$$

where $(\alpha)_n$ is Pochhammer’s symbol defined by

$$ (\alpha)_0 = 1; \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}; \quad n = 1, 2, \ldots $$

The $q$–analogue of the binomial function is

$$ 1 \Phi_0 \left( \frac{a}{x}; q, x \right) = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |q| < 1, \ |x| < 1. $$

The $q$–analogue of the exponential functions are

$$ e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_\infty}, \quad |q| < 1 $$
We shall also make use of the function
\[ q^{-\alpha} (1 - q)^{1-\alpha} \]
defined for \( \alpha \neq 0, -1, -2, \ldots \). This is a \( q \)-analogue of the gamma function and satisfies the functional equation
\[ \Gamma_q (\alpha + 1) = \frac{1 - q^\alpha}{1 - q} \Gamma_q (\alpha). \]

We need the \( q \)-analogue of Kampé de Fériet function [16]
\[ F_{C,E;E'}^{A,B,B'} \left( (\alpha); (\beta); (\gamma); (\delta); (\delta') : x, y \right) = \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{A} (\alpha_j)_m \prod_{j=1}^{B} (\beta_j)_m \prod_{j=1}^{B'} (\beta'_j)_n \prod_{j=1}^{C} (\gamma_j)_m \prod_{j=1}^{E} (\delta_j)_n \prod_{j=1}^{E'} (\delta'_j)_n x^m y^n}{\prod_{j=1}^{C} (\gamma_j)_m \prod_{j=1}^{E} (\delta_j)_n \prod_{j=1}^{E'} (\delta'_j)_n m! n!} \]
in the form [6]
\[ \phi_{C,E;E'}^{A,B,B'} \left( (a); (b); (b') \ (c); (c'); (e'); q, q' \right) \]
\[ = \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{A} (a_j; q)_m \prod_{j=1}^{B} (b_j; q)_m \prod_{j=1}^{B'} (b'_j; q)_n \prod_{j=1}^{C} (c_j; q)_m \prod_{j=1}^{E} (e_j; q)_n \prod_{j=1}^{E'} (e'_j; q)_n (q; q)_m (q'; q')_n x^m y^n}{\prod_{j=1}^{C} (c_j; q)_m \prod_{j=1}^{E} (e_j; q)_n \prod_{j=1}^{E'} (e'_j; q)_n (q; q)_m (q'; q')_n}. \]

For \( r = s = 0 \), we denote the l.h.s. of (2.12) simply by
\[ \phi_{C,E;E'}^{A,B,B'} \left( (a); (b); (b') \ (c); (c'); (e'); x, y \right). \]

The \( q \)-derivative operator is defined by
\[ \Delta_x (f(x)) = \frac{f(x) - f(qx)}{(1 - q)x} = \frac{1 - T_x}{(1 - q)x} f(x), \]
where the \( q \)-dilation operator \( T_x \) is given by
\[ T_x (f(x)) = f(qx). \]

The \( q \)-integral [3] is defined by
\[ \int_0^\infty f(t) \, d_q t = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^n. \]
3. Irreducible \(q\)-representations of \(\mathfrak{sl}(2, \mathbb{C})\)

The complex Lie algebra \(\mathfrak{sl}(2, \mathbb{C}) = L\{SL(2, \mathbb{C})\}\), the Lie algebra of complex Lie group

\[
SL(2, \mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}
\]

(3.1)

consists of all \(2 \times 2\) matrices with trace zero, that is,

\[
\mathfrak{sl}(2, \mathbb{C}) = \left\{ \begin{bmatrix} e & f \\ g & -e \end{bmatrix} : e, f, g \in \mathbb{C} \right\}
\]

(3.2)

It has a basis

\[
J^+ = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad J^- = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad J^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

(3.3)

satisfying the commutation relations

\[
[J^0, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = 2J^0.
\]

(3.4)

Let \(V_q\) be a complex vector space consisting of \(q\)-special functions with a basis \(\{ \phi_\lambda | \lambda \in S \}\) such that the functions \(\{ f_\lambda = \lim_{q \to 1} \phi_\lambda | \lambda \in S \}\) form a basis of a vector space, say \(V\). Let \(A(V_q)\) be the associative algebra of all linear operators on \(V_q\) over the complex field.

A \(q\)-representation of \(\mathfrak{sl}(2, \mathbb{C})\) on \(V_q\) is a mapping \(\rho_q : \mathfrak{sl}(2, \mathbb{C}) \to A(V_q)\) satisfying the following conditions:

(i) \(\rho_q(ax + by) = a\rho_q(x) + b\rho_q(y)\)

(ii) There exists a Lie algebra representation \(\rho\) of \(\mathfrak{sl}(2)\) on \(V\) such that \(\lim_{q \to 1} \rho_q(x) \phi_\lambda = \rho(x) f_\lambda\), for all \(x, y \in \mathfrak{sl}(2, \mathbb{C})\) and \(a, b \in \mathbb{C}\).

The representation \(\rho_q\) of \(\mathfrak{sl}(2, \mathbb{C})\) is said to be irreducible if there is no proper subspace \(W_q\) of \(V_q\) which is invariant under \(\rho_q\).

Define

\[
J^+_q = \rho_q(J^+), \quad J^-_q = \rho_q(J^-), \quad J^0_q = \rho_q(J^0),
\]

(3.5)

where \(J^+_q, J^-_q, J^0_q \in A(V_q)\).

Manocha [6] has defined the following commutator rules for \(J_q\)-operators:

\[
J^0_q J^+_q - qJ^+_q J^0_q = J^+_q,
qJ^0_q J^-_q - J^-_q J^0_q = -J^-_q,
qJ^+_q J^-_q - J^-_q J^+_q = 2q^{2u} J^0_q - (1 - q)q^{2u} J^0_q J^0_q, \quad u \in \mathbb{C}.
\]

(3.6)

These commutator rules were later generalized by Sahai [11] for the 4-dimensional complex Lie algebra \(\mathfrak{g}(a, b)\), \(a, b \in \mathbb{C}\). Further, the commutation relations (3.6) are also equivalent to those introduced by Jimbo [4]. For details, see [6, 14, 15].

If we define an operator \(C_q\) on \(V_q\) by

\[
C_q = qJ^+_q J^-_q + q^{2u} J^0_q J^0_q - q^{2u} J^0_q
\]

(3.7)
then it is easy to check that
\[
q J_q^+ C_q = C_q J_q^+,
\]
\[
J_q^- C_q = q C_q J_q^-,
\]
\[
J_q^0 C_q = C_q J_q^0.
\]
(3.8)
As \(q \to 1\), the operators \(J_q^+, J_q^-, J_q^0\) reduce to \(J^+, J^-, J^0\) which satisfies the commutation relations obeyed by \(\mathfrak{sl}(2, \mathbb{C})\) and the operator \(C_q\) reduces to the Casimir operator \(C\). Hence, as \(q \to 1\), \(\rho_q\) reduces to a Lie algebra representation \(\rho\) of \(\mathfrak{sl}(2, \mathbb{C})\). For more details, see [6, 8].

4. Models of Irreducible \(q\)-representations

We give below new two variable models for each of the \(q\)-representations \(D_q(u, \alpha)\) and \(\uparrow_q(u)\) corresponding to \(u = -\gamma/2\), in which case they will be denoted by \(D_q(-\gamma/2, \alpha)\) and \(\uparrow_q(-\gamma/2)\), respectively.

4.1. Representation \(D_q(-\gamma/2, \alpha)\).
Consider an irreducible representation \(D_q(-\gamma/2, \alpha)\) of \(\mathfrak{sl}(2, \mathbb{C})\), \(\alpha, \gamma \in \mathbb{C}\) and \(\gamma \neq 0, -1, -2, \ldots\) on the representation space \(V_q\) with basis \(\{f_\lambda | \lambda = \alpha + n, n \in \mathbb{Z}\}\), such that action of \(\mathfrak{sl}(2, \mathbb{C})\) on \(V_q\) is given by
\[
J_q^+ f_\lambda = \frac{q^{-\gamma} - q^{\lambda-\gamma}}{1-q} f_{\lambda+1},
\]
\[
J_q^- f_\lambda = \frac{1 - q^{\lambda-\gamma}}{1-q} f_{\lambda-1},
\]
\[
J_q^0 f_\lambda = \frac{1 - q^{\lambda-\gamma/2}}{1-q} f_\lambda,
\]
\[
(q J_q^+ J_q^- + q^{-\gamma} J_q^0 J_q^0 - q^{-\gamma} J_q^0) f_\lambda = \frac{(1-q^{-\gamma/2})(1-q^{-\gamma/2+1})}{(1-q)^2} q^{\lambda-\gamma} f_\lambda. \tag{4.1}
\]
To find a realization of (4.1) in terms of \(q\)-dilation operators, we choose
\[
J_q^+ = \frac{t}{1-q} (q^{-\gamma} - q^{-\gamma}T_2 T_1),
\]
\[
J_q^- = \frac{t^{-1}}{1-q} T_z^{-1} (q^{-\gamma} T_1 - T_z - q^{-\gamma} z T_2 T_1),
\]
\[
J_q^0 = \frac{1 - q^{-\gamma/2} T_z}{1-q},
\]
\[
C_q = q J_q^+ J_q^- + q^{-\gamma} J_q^0 J_q^0 - q^{-\gamma} J_q^0,
\]  
(4.2)
with basis functions
\[
f_\lambda = \phi_1 \left( q^\lambda, q^{-\lambda}; q, -z \right) t^\lambda, \quad \lambda = \alpha, \alpha \pm 1, \alpha \pm 2, \ldots .
\]
Model (4.2) obeys (4.1) and the following

\[
\begin{align*}
J^0_q J^+_q - qJ^+_q J^0_q &= J^+_q, \\
quJ^0_q J^-_q - J^-_q J^0_q &= -J^-_q, \\
quJ^+_q J^-_q - J^-_q J^+_q &= 2q^{-\gamma}J^0_q - (1-q)q^{-\gamma}J^0_q, \\
&\quad \gamma \in \mathbb{C}
\end{align*}
\]

as well as

\[
\begin{align*}
qJ^+_q C_q &= C_q J^+_q, \\
J^-_q C_q &= qC_q J^-_q, \\
J^0_q C_q &= C_q J^0_q.
\end{align*}
\]

4.1.1. Identities based on model (4.2).

As

\[
f_\lambda (z, t) = \phi_1 \left( \frac{q^\lambda}{q^\gamma} : q, -z \right) t^\lambda
\]
satisfies

\[
C_q f_\lambda (z, t) = \frac{(1-q^{-\gamma/2}) (1-q^{-\gamma/2+1})}{(1-q)^2} q^{\gamma-\gamma} f_\lambda (z, t).
\]

It immediately follows that

\[
u (z, t) = \phi_1^{1:0:2} \left( \frac{a: -; b', b''}{-; q^\gamma; c', q, 1} \right) t^\alpha
\]

\[
= \sum_{n=0}^{\infty} \frac{(a; q)_n (b'; q)_n (b''; q)_n}{(c'; q)_n (q; q)_n} \phi_1 \left( \frac{q^\lambda}{q^\gamma} : q, -z \right) t^\lambda,
\]

where \( q^\alpha = a \), also satisfies

\[
C_q u (z, t) = \frac{(1-q^{-\gamma/2}) (1-q^{-\gamma/2+1})}{(1-q)^2} q^{\gamma-\gamma} u (z, t).
\]

Using the fact that

\[
e_q (qsJ^+_q) C_q = C_q e_q (sJ^+_q),
\]

which follows from (4.4), we have

\[
C_q \left[ e_q (sJ^+_q) u \right] (z, t) = \left( \frac{q^{-\ast t} (1-q^{-1})}{q^{-\ast t}} ; q \right)_\infty \frac{(1-q^{-\gamma/2}) (1-q^{-\gamma/2+1})}{(1-q)^2} q^{-\gamma}
\]

\[
\times \phi_1^{1:0:2} \left( \frac{a: -; b', b''}{-; q^\gamma; c', q, 1} \right) (tq)^\alpha.
\]

where

\[
\left[ e_q (sJ^+_q) u \right] (z, t) = \left( \frac{q^{-\ast t} (1-q^{-1})}{q^{-\ast t}} ; q \right)_\infty \phi_1^{1:0:2} \left( \frac{a: -; b', b''}{-; q^\gamma; c', q, 1} ; z, t \right) t^\alpha.
\]
Using the expansion
\[ [e_q (s J_q^+ u)] (z, t) = \sum_{n=0}^{\infty} i_n f_{\alpha+n} (z, t), \] (4.10)
where \( i_n \) are obtained by putting \( z = 0 \), we get the following identity, after suitable rescaling:
\[ \left( \frac{(at; q)_{\infty}}{(q; q)_{\infty}} \right)_{1+0,2} \phi_{1; 1; 1} \left( \frac{a t}{q}; b', b''; \frac{z}{q}, t/w \right) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (q^{-\gamma} t)^n 3\phi_1 \left( q^{-n}, b', b''; \frac{q^{-2\gamma+n}}{w}; q, -z \right) \phi_1 \left( aq^n; q^{-\gamma}; q, -z \right), \] (4.11)
where \( w = -\frac{s}{1-q} \).

4.1.2. Recurrence relations from model (4.2).

The recurrence relations originating from the action of \( J_q^+ \), \( J_q^- \) on \( f_\lambda \) are given by
\[ (1 - q^\lambda) \phi_1 (\lambda + 1; z) + q^\lambda \phi_1 (\lambda; qz) = \phi_1 (\lambda; z), \] (4.12)
and
\[ zq^{\lambda-\gamma} \phi_1 (\lambda; qz) + \phi_1 (\lambda; qz) - q^{\lambda-\gamma} \phi_1 (\lambda; z) = (1 - q^{\lambda-\gamma}) \phi_1 (\lambda - 1; qz) \] (4.13)
respectively, where \( \phi_1 (\lambda; z) \) stands for \( \phi_1 \left( q^\lambda; q^{-\gamma}; q, -z \right) \).

4.2. Representation \( \uparrow_q (-\gamma/2) \).

We look for an irreducible representation \( \uparrow_q (-\gamma/2) \) of \( \text{sl}(2, \mathbb{C}) \), \( \gamma \in \mathbb{C} - \{0, -1, -2, \ldots\} \), on the representation space \( V_q \) with basis functions \( \{ f_\lambda \mid \lambda = 0, 1, 2, \ldots \} \) such that the action of \( \text{sl}(2, \mathbb{C}) \) on \( V_q \) is given by
\[ J_q^+ f_\lambda = \frac{q^{-\gamma} - q^\lambda}{1 - q} f_{\lambda+1}, \]
\[ J_q^- f_\lambda = -\frac{1 - q^\lambda}{1 - q} f_{\lambda-1}, \]
\[ J_q^0 f_\lambda = \frac{1 - q^{\lambda+\gamma/2}}{1 - q} f_{\lambda}, \]
\[ (q J_q^+ J_q^- + q^{-\gamma} J_q^0 J_q^0 - q^{-\gamma} J_q^0) f_\lambda = \frac{(1 - q^{-\gamma/2}) (1 - q^{-\gamma/2+1})}{(1-q)^2} q^\lambda f_\lambda. \] (4.14)
To meet this requirement, we choose

\[
J_q^+ = \frac{t}{1-q} \left( q^{-\gamma} - T_z T_t - z T_z T_t \right),
\]

\[
J_q^- = \frac{t^{-1}}{1-q} T_z^{-1} \left( T_t - T_z \right),
\]

\[
J_q^0 = \frac{1-q^{\gamma/2} T_t}{1-q},
\]

\[
C_q = q J_q^+ J_q^- + q^{-\gamma} J_q^0 J_q^0 - q^{-\gamma} J_q^0,
\]

(4.15)

with basis functions

\[
f_\lambda = \phi_1 \left( \frac{q^{-\lambda}}{q^\gamma} : q, -q^{\lambda+\gamma z} \right) t^\lambda, \quad \lambda = 0, 1, 2, \ldots .
\]

Model (4.15) satisfies (4.3) as well as (4.4).

4.2.1. Identities based on model (4.15).

As

\[
f_\lambda = \phi_1 \left( \frac{q^{-\lambda}}{q^\gamma} : q, -q^{\lambda+\gamma z} \right) t^\lambda,
\]

it immediately follows that

\[
u(z, t) = \phi_1^{1:0;0} \left( \frac{a: \gamma; -z; -q^\gamma z t, t}{c': q^\gamma; \gamma; -q^{\gamma+\gamma z} z t, q^\gamma} \right)
\]

\[
= \sum_{n=0}^{\infty} (a; q)_n (c'; q)_{n} (q; q)_n \phi_1 \left( \frac{q^{-n}}{q^\gamma} : q, -q^{n+\gamma z} \right) t^n,
\]

(4.17)

satisfies

\[
C_q u(z, t) = \left( 1 - q^{-\gamma/2} \right) \left( 1 - q^{-\gamma/2+1} \right) q^n u(z, t).
\]

(4.18)

Using the fact that

\[
E_q \left( \frac{1}{q} s J_q^- \right) C_q = C_q E_q \left( s J_q^- \right),
\]

(4.19)

which follows from (4.4), we have

\[
C_q \left[ E_q \left( s J_q^- \right) u \right](z, t) = \left( 1 - q^{-\gamma/2} \right) \left( 1 - q^{-\gamma/2+1} \right)
\]

\[
\times \phi_1^{1:0;1} \left( \frac{a: \gamma; -z; -q^\gamma z t, t}{c': q^\gamma; \gamma; -q^{\gamma+1} z t, q^\gamma} \right),
\]

(4.20)
where
\[ [E_q (sJ^-_q) u] (z, t) = \phi^{1,0:1}_{1,1;0} \left( \frac{a}{e^t}; -\frac{t}{q^\gamma z}, -q^\gamma z, t \right). \] (4.21)

The expansion
\[ [E_q (sJ^-_q) u] (z, t) = \sum_{n=0}^{\infty} A_n f_n (z, t), \] (4.22)
leads to the following identity
\[
\phi^{1,0:1}_{1,1;0} \left( \frac{a}{e^t}; -\frac{t}{q^\gamma z}, -q^\gamma z, t \right) = \sum_{n=0}^{\infty} \left( \frac{a}{c^\gamma q} \right)^n \left( \frac{q^{-n}}{q^\gamma}; -q^\gamma z \right) t^n. \] (4.23)

4.2.2. Recurrence relations.

The recurrence relations arising from model (4.15) are given by
\[
\phi_1 (\lambda; z) - q^{\lambda+\gamma} \phi_1 (\lambda; qz) - q^{\lambda+\gamma} z \phi_1 (\lambda; qz) = (1 - q^{\lambda+\gamma}) \phi_1 (\lambda + 1; z), \] (4.24)
\[
\phi_1 (\lambda; qz) - q^{\lambda} \phi_1 (\lambda; z) = (1 - q^{\lambda}) \phi_1 (\lambda - 1; qz), \] (4.25)
where \( \phi_1 (\lambda; z) \) stands for \( \phi_1 \left( \frac{q^{-\lambda}}{q^\gamma}; q, -q^{\lambda+\gamma} z \right). \)

5. Transformed Models of \( sl(2, \mathbb{C}) \)

We introduce the \( q \)-Mellin transformation \( I_q \).

Define
\[
h (\beta, x) = I_q [f (ux)] = \frac{q^{\beta(\beta-1)} x}{\Gamma_q (\beta)} \int_0^\infty e_q (-u) u^{\beta-1} f (ux) d_q u, \] (5.1)
which is motivated from the \( q \)-integral representation of gamma function, as in [17],
\[ \Gamma_q (\gamma) = q^{\frac{\gamma(\gamma+1)}{2}} \int_0^\infty e_q (-u) u^{\gamma-1} d_q u. \] (5.2)

Using the substitution \( z = ux \), which gives \( z \Delta_x u \Delta_u = x \Delta_x \) and \( T_z f (z) = f (qz) = T_u f (ux) \), we have the following transforms of certain operator expressions which are needed for the discussion.
\[
I_q [uf (ux)] = \frac{q^{-\beta} - 1}{1 - q} E_{\beta} h (\beta, x),
\]
\[
I_q [u \Delta_u f (ux)] = x \Delta_x h (\beta, x), \] (5.3)
where \( E_{\beta} h (\beta, x) = h (\beta + 1, x). \)
To obtain transformed models of $D_q(-\gamma/2, \alpha)$ and $\uparrow_q (-\gamma/2)$, we put $z = ux$ in models (4.2) and (4.15). Note that if $\rho_q$ is an irreducible representation of $\text{sl}(2, \mathbb{C})$ on the representation space $V_q$ having the basis vectors $\{f_\lambda \mid \lambda \in S\}$ in terms of $\{J^+_q, J^-_q, J^0_q\}$, then the transformation $I_q$ induces another irreducible representation $\sigma_q$ of $\text{sl}(2, \mathbb{C})$ on the representation space $W_q = I_q V_q$ in terms of $\{K^+_q = I_q J^+_q I_q^{-1}, K^-_q = I_q J^-_q I_q^{-1}, K^0_q = I_q J^0_q I_q^{-1}\}$ with basis vectors $\{h_\lambda = I_q f_\lambda \mid \lambda \in S\}$.

### 5.1. Transformed Model of $D_q (-\gamma/2, \alpha)$

\begin{align}
K^+_q &= \frac{tq^{-\gamma}}{1-q} (1 - T_x T_t), \\
K^-_q &= \frac{t^{-1}}{1-q} T_x^{-1} \left( q^{-\gamma} T_t - T_t - \frac{q^{-\beta} - 1}{1-q} E_\beta q^{-\gamma} x T_x T_t \right), \\
K^0_q &= \frac{1 - q^{-\gamma}/2 T_t}{1-q},
\end{align}

(5.4)

with basis functions as

\[ h_\lambda(x, t) = 2\phi_1 \left( q^\lambda, q^\beta, \frac{x}{(1-q) q^q} \right) t^\lambda. \]

Indeed,

\begin{align}
qK^0_q K^0_q - q K^+_q K^-_q &= K^+_q, \\
qK^0_q K^-_q - K^-_q K^0_q &= -K^-_q, \\
qK^+_q K^-_q - K^-_q K^+_q &= 2q^{-\gamma} K^0_q - (1-q) q^{-\gamma} K^0_q K^0_q \quad (5.5)
\end{align}

and

\begin{align}
K^+_q h_\lambda &= q^{-\gamma} \frac{1 - q^\lambda}{1-q} h_{\lambda+1}, \\
K^-_q h_\lambda &= -\frac{1 - q^{\lambda-\gamma}}{1-q} h_{\lambda-1}, \\
K^0_q h_\lambda &= \frac{1 - q^{\lambda-\gamma}/2}{1-q} h_{\lambda}, \\
C'_q h_\lambda &= \left( qK^+_q K^-_q + q^{-\gamma} K^0_q K^0_q - q^{-\gamma} K^0_q \right) h_{\lambda} \\
&= \frac{(1-q^{-\gamma/2}) (1-q^{-\gamma/2+1})}{(1-q)^2} q^{\lambda-\gamma} h_{\lambda}. \quad (5.6)
\end{align}

Moreover,

\begin{align}
qK^+_q C'_q &= C'_q K^+_q, \\
K^-_q C'_q &= q C'_q K^-_q, \\
K^0_q C'_q &= C'_q K^0_q. \quad (5.7)
\end{align}
5.1.1. Identities based on transformed model $D_q (-\gamma/2, \alpha)$.

As

$$h_\lambda (x, t) = 2\phi_1 \left( q^\lambda, q^\gamma : q, \frac{x}{(1-q)q^\alpha} \right) t^\lambda$$

is a solution of

$$C_q' h_\lambda (x, t) = \frac{(1-q^{-\gamma/2}) (1-q^{-\gamma/2+1})}{(1-q)^2} q^{\lambda-\gamma} h_\lambda (x, t), \quad (5.8)$$

the function

$$u (x, t) = \phi^{4:1;2}_{01:11} \left( a : q^\beta, b', b'': q^\epsilon q^{\gamma} - : q^\gamma q^{\epsilon} \right) t^\alpha$$

$$= \sum_{n=0}^{\infty} \frac{(q; q)_n (b'; q)_n (b''; q)_n}{(c'; q)_n (q; q)_n} 2\phi_1 \left( q^\lambda, q^\gamma : q, \frac{x}{(1-q)q^\alpha} \right) t^\lambda \ (5.9)$$

is also a solution of

$$C_q' u (x, t) = \frac{(1-q^{-\gamma/2}) (1-q^{-\gamma/2+1})}{(1-q)^2} q^{\lambda-\gamma} u (x, t). \quad (5.10)$$

Using $qK_q^+ C_q^* = C_q^* K_q^+$, we have

$$C_q' [e_q (sK_q^+) u] (x, t) = \left( \frac{q^{\alpha t} (1-q)}{q^{\alpha t} (1-q)} : q \right) \infty q^{-\gamma} \frac{(1-q^{-\gamma/2}) (1-q^{-\gamma/2+1})}{(1-q)^2}$$

$$\times \phi^{4:1;2}_{11:11} \left( a : q^\beta, b', b'' : q^\epsilon q^{\gamma} \right) \infty (q t)^\alpha. \quad (5.11)$$

where

$$[e_q (sK_q^+) u] (x, t) = \left( \frac{q^{\alpha t} (1-q)}{q^{\alpha t} (1-q)} : q \right) \infty \phi^{4:1;2}_{11:11} \left( a : q^\beta, b', b'' : q^\epsilon q^{\gamma} \right) \infty (q t)^\alpha. \quad (5.12)$$

From the expansion

$$[e_q (sK_q^+) u] (x, t) = \sum_{n=0}^{\infty} A_n h_n (x, t),$$

where $A_n$ is found by putting $x = 0$, we arrive at the following identity after suitable rescaling:

$$\frac{(a; q)_n}{(q; q)_n} \phi^{4:1;2}_{11:11} \left( a : q^\beta, b', b'' : q^\epsilon q^{\gamma} \right) \frac{t}{q^{\alpha t}}$$

$$= \sum_{n=0}^{\infty} \frac{(a; q)_n (q^{-\gamma} t)^n}{(q; q)_n} 2\phi_1 \left( q^{-n}, b', b'' : q, \frac{x}{(1-q)q^\alpha} \right) \infty \phi_1 \left( a q^n, q^\beta : q, \frac{x}{(1-q)q^\alpha} \right). \quad (5.13)$$
5.1.2. Recurrence relations from model (5.4).

The recurrence relations originating from the action of $K_q^+$ and $K_q^-$ on $h_\lambda$ are given by

\begin{equation}
2\phi_1 (\lambda; \beta; x) - q^\lambda 2\phi_1 (\lambda; \beta; qx) = (1 - q^\lambda) 2\phi_1 (\lambda + 1; \beta; x), \tag{5.14}
\end{equation}

\begin{equation}
x q^{\lambda-\gamma-\beta} \frac{1-q^\beta}{1-q} 2\phi_1 (\lambda; \beta + 1; qx) + 2\phi_1 (\lambda; \beta; qx) - q^{\lambda-\gamma} 2\phi_1 (\lambda; \beta; x)
\end{equation}

\begin{equation}
= (1 - q^{\lambda-\gamma}) 2\phi_1 (\lambda - 1; \beta; qx), \tag{5.15}
\end{equation}

where $2\phi_1 (\lambda; \beta; x)$ stands for $2\phi_1 \left( \frac{q^\lambda, q^\beta, q^{x/(1-q^\gamma)}; q}{(1-q^\gamma)} \right)$.

5.2. Transformed Model of $t_q (-\gamma/2)$.

\begin{equation}
K_q^+ = \frac{t}{1-q} \left( q^{-\gamma} - T_x T_t - \frac{q^{-\beta} - 1}{1-q} E_\beta x T_x T_t \right), \tag{5.16}
\end{equation}

\begin{equation}
K_q^- = \frac{t^{-1}}{1-q} T_x^{-1} (x \Delta_x - t \Delta_t),
\end{equation}

\begin{equation}
K_q^0 = \frac{1 - q^{\gamma/2} T_t}{1-q},
\end{equation}

with basis functions as

\begin{equation}
h_\lambda (x, t) = 2\phi_1 \left( \frac{q^{-\lambda}, q^\beta, q^{\lambda+\gamma-\beta}; q}{1-q} x \right) t^{\lambda}. \tag{5.17}
\end{equation}

Model (5.16) satisfies (5.5) and (5.7) and

\begin{equation}
K_q^+ h_\lambda = \frac{q^{-\gamma} - q^\lambda}{1-q} h_{\lambda+1},
\end{equation}

\begin{equation}
K_q^- h_\lambda = -\frac{1 - q^\lambda}{1-q} h_{\lambda-1},
\end{equation}

\begin{equation}
K_q^0 h_\lambda = \frac{1 - q^{\lambda+\gamma/2}}{1-q} h_\lambda,
\end{equation}

\begin{equation}
C_q h_\lambda = (q K_q^+ K_q^- + q^{-\gamma} K_q^0 - q^{-\gamma} K_q^0) h_\lambda
\end{equation}

\begin{equation}
= (1 - q^{-\gamma/2}) \frac{1 - q^{-\gamma/2+1}}{(1-q)^2} q^\lambda h_\lambda. \tag{5.17}
\end{equation}
5.2.1. Identities based on transformed model \( \Upsilon_q (-\gamma/2) \).

As

\[
h_\lambda(x,t) = 2\phi_1 \left( q^{-\lambda}, q^2 ; q, q^{\lambda+\gamma-\beta} \frac{x}{1-q} \right) t^\lambda, \quad \lambda = 0, 1, 2, \ldots
\]
satisfies

\[
C_q^t h_\lambda(x,t) = \frac{(1-q^{-\gamma/2})(1-q^{-\gamma/2+1})}{(1-q)^2} q^\lambda h_\lambda(x,t).
\] (5.18)

This, in turn, gives

\[
u(x,t) = \phi_1^{1:1:0} \left( a : q^3 ; -\frac{q^{\frac{3-\gamma}{2}}x}{1-q}, t \right)
\]

\[
\sum_{n=0}^{\infty} \frac{(a;q)_n}{(c';q)_n} \frac{(q;q)_n}{(q;q)_n} 2\phi_1 \left( q^{-n}, q^3 ; q, q^{\frac{n+\gamma-\beta}{1-q}} x \right) t^n,
\] (5.19)
satisfies

\[
C_q^t u(x,t) = \frac{(1-q^{-\gamma/2})(1-q^{-\gamma/2+1})}{(1-q)^2} q^\nu u(x,t).
\] (5.20)

Using \( K_q C_q = q C_q K_q^- \), we have

\[
C_q^t \left[ E_q \left( sK_q^- \right) u \right](x,t) = \frac{(1-q^{-\gamma/2})(1-q^{-\gamma/2+1})}{(1-q)^2}
\]

\[
\times \phi_1^{1:1:1} \left( a : q^3 ; -\frac{q^{\frac{3-\gamma}{2}}x}{1-q}, t \right)
\] (5.21)

where

\[
\left[ E_q \left( sK_q^- \right) u \right](x,t) = \phi_1^{1:1:1} \left( a : q^3 ; -\frac{q^{\frac{3-\gamma}{2}}x}{1-q}, t \right)
\] (5.22)

The expansion

\[
\left[ E_q \left( sK_q^- \right) u \right](x,t) = \sum_{n=0}^{\infty} C_n h_n(x,t),
\] (5.23)
gives rise to the following identity:

\[
\phi_1^{1:1:1} \left( a : q^3 ; -\frac{q^{\frac{3-\gamma}{2}}x}{1-q}, t \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \frac{(q;q)_n}{(q;q)_n} \phi_1 \left( aq^n ; q, -w \right) 2\phi_1 \left( q^{-n}, q^3 ; q, q^{\frac{n+\gamma-\beta}{1-q}} x \right) t^n.
\] (5.24)
5.2.2. Recurrence relations.

The recurrence relations that come from (5.17) are given by

\[ 2\phi_1(\lambda; \beta; x) - q^{\lambda+\gamma} 2\phi_1(\lambda; \beta; qx) - xq^{\lambda+\gamma-\beta} \frac{1-q^\beta}{1-q} 2\phi_1(\lambda; \beta+1; qx) = (1-q^\lambda) 2\phi_1(\lambda+1; \beta; x), \] (5.25)

\[ 2\phi_1(\lambda; \beta; qx) - q^\lambda 2\phi_1(\lambda; \beta; x) = (1-q^\lambda) 2\phi_1(\lambda-1; \beta; qx), \] (5.26)

where \( 2\phi_1(\lambda; \beta; x) \) stands for \( 2\phi_1\left(\frac{q^{-\lambda}, q^\beta}{q^\gamma : q, \frac{q^\lambda q_q^{\lambda+\gamma-\beta}}{1-q} x}\right) \).

6. Conclusion

In this section, we give an application of our models to \( q \)-Laguerre polynomials. Let \( \alpha > -1 \) and \( 0 < q < 1 \). Then \( q \)-Laguerre polynomials are defined by

\[ L_n^\alpha(z; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \phi_1\left(\frac{q^{-n}}{q^{\alpha+1} : q, -2q^{n+\alpha+1}}\right). \] (6.1)

If we take \( z = (1-q)x \) then it can be easily seen that \( \lim_{q \to 1^-} L_n^\alpha(z; q) = L_n^\alpha(x) \), where \( L_n^\alpha(x) \) are classical Laguerre polynomials. Consider the model (4.15) in the particular case \( \gamma = 1 + \alpha \) (that is, a model of representation \( \uparrow q^{-\frac{1+\alpha}{2}} \)) with basis functions involving \( q \)-Laguerre polynomials, given as

\[ f_n(z, t) = L_n^\alpha(z; q) t^n. \] (6.2)

The action of these \( J_q \)-operators on \( f_n(z, t) \) will take the following form:

\[ J_q^+ f_n = q^{-(1+\alpha)} \frac{1-q^{n+1}}{1-q} f_{n+1}, \]

\[ J_q^- f_n = -\frac{1-q^{\alpha+n+1}}{1-q} f_{n-1}, \]

\[ J_q^0 f_n = \frac{1-q^{\frac{1+\alpha+n}{2}}}{1-q} f_n. \] (6.3)

We can obtain identities involving \( q \)-Laguerre polynomials using the method explained above. For example, taking

\[ u(z, t) = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q^{\alpha+1}; q)_n} \phi_1\left(\frac{q^{-n}}{q^{\alpha+1} : q, -q^{n+\alpha+1}}\right) t^n \] (6.4)
and proceeding as in Section 4.2.1, we arrive at the following identity:

\[
\phi_{1;0;1}^{1;0;1} \left( q^{\alpha+1} : \frac{w}{c} : q^{1+\alpha} ; - \frac{w}{c} : q^{2} \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{t^{n}}{(c; q)_{n}} L_{n}^{\alpha} (z; q) \psi_{1} \left( q^{n+1} : \frac{w}{c} c' q^{n} : q^{1+\alpha} ; - q^{\alpha+1} \right),
\]

which is a special case of (4.23).

The recurrence relations involving \(q\)-Laguerre polynomials can also be obtained by using first two equations of (6.3). These will shape as

\[
L_{n}^{\alpha} (z; q) - q^{1+\alpha+n} L_{n}^{\alpha} (qz; q) - z q^{n+1} L_{n}^{\alpha} (qz; q) = (1 - q^{n+1}) L_{n-1}^{\alpha} (z; q),
\]

and

\[
q^{n} L_{n}^{\alpha} (z; q) - L_{n}^{\alpha} (qz; q) = (1 - q^{\alpha+n}) L_{n-1}^{\alpha} (qz; q),
\]

respectively. Eliminating \(L_{n}^{\alpha} (z; q)\) from (6.6) and (6.7) gives rise to a three term recurrence relation as follows

\[
(1 - q^{\alpha+n}) L_{n-1}^{\alpha} (qz; q) + (z q^{1+\alpha+2n} - (1 - q^{1+\alpha+2n})) L_{n}^{\alpha} (qz; q)
\]

\[
+ q^{n} (1 - q^{n+1}) L_{n+1}^{\alpha} (z; q) = 0,
\]

\(n = 1, 2, \ldots\).

Note that (6.8) is a \(q\)-analogue of the three term recurrence relation for classical Laguerre polynomials [2],

\[
(\alpha + n) L_{n-1}^{\alpha} (x) + (x - 1 - \alpha - 2n) L_{n}^{\alpha} (x) + (n + 1) L_{n+1}^{\alpha} (x) = 0.
\]

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