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ON MODELS OF q–REPRESENTATIONS OF sl(2, $\mathbb{C}$) AND q–CONFLUENT HYPERGEOMETRIC FUNCTIONS

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Abstract. We construct new two variable models of irreducible q–representations of the Lie algebra sl$(2,\mathbb{C})$. The q–Mellin transformaton is defined and used to transform the models in the form of q–difference dilation models, involving basic hypergeometric function ${}_2\phi_1$ as basis functions. All the models culminate in many special function identities and recurrence relations.

1. Introduction

Manocha [**6**] introduced the idea of irreducible q–representations of the special complex Lie algebra $\mathrm{sl}\,(2,\mathbb{C})$ and developed theory centering around it. A q–analogue of fractional calculus technique [**1, 7**] was used to construct models of irreducible q–representations of $\mathrm{sl}\,(2,\mathbb{C})$ followed by the Weisner's method [**18**] to obtain identities involving q–special functions of one and several variables. These models were in terms of q–differintegral operators acting on q–hypergeometric functions ${}_2\phi_1$. Further, in [**15**], q–differintegral operator models of $\mathrm{sl}\,(2,\mathbb{C})$ with basis functions involving ${}_{k^2+1}\phi_{k^2}$ functions were constructed and a good number of identities were derived from them.

The theory of q–representation of Lie algebras together with the theory of q–integral transformations has also been a rich source of results in q–special function theory. In Sahai [**12, 13, 14**], the q–Euler integral transformation, which is motivated by q–integral representation of beta function, is utilized to obtain models in terms of q–difference dilation operators of $\mathrm{sl}\,(2,\mathbb{C})$ along with identities involving q–special functions. Both these models, *viz.* q–differintegral operator models and q–difference dilation operator models, are of q–representations $D_q\,(u,\alpha)$ and $\uparrow_q\,(u)$ of $\mathrm{sl}\,(2,\mathbb{C})$ in the particular case $u=0$.

In this paper, we construct new models of q–representations $D_q\,(u,\alpha)$ and $\uparrow_q\,(u)$ of $\mathrm{sl}\,(2,\mathbb{C})$ corresponding to $u\neq 0$ acting on q–confluent hypergeometric functions ${}_1\phi_1$. We then define q–Mellin transformation, which is motivated by q–integral representation of gamma function. The q–Mellin transformation is the q–analogue of Mellin integral transformation used in [**5, 10**]. We utilize this transformation to transform these models of $\mathrm{sl}\,(2,\mathbb{C})$ to new models involving q–difference dilation operators with the q–hypergeometric functions ${}_2\phi_1$ as basis functions. The discussion culminates in interesting identities and recurrence relations involving ${}_1\phi_1$ and ${}_2\phi_1$ functions.

Section–wise treatment is as follows.

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In Section 2, we give definitions involving hypergeometric and basic hypergeometric series, needed for our discussion. In Section 3, Lie algebra $\mathrm{sl}(2,\mathbb{C})$ is presented along with its q–deformed version. In Section 4, we construct model of $D_q(-\gamma/2,\alpha)$ and $\uparrow_q(-\gamma/2)$ in two variables acting on ${}_1\phi_1$ and obtain identities and recurrence relations based on it. In Section 5, we introduce q–Mellin transformation and then utilize it to transform the models given in Section 4 to q–difference dilation models acting on basis functions ${}_2\phi_1$. The transformed models are further used in obtaining identities and recurrence relations involving ${}_2\phi_1$. Finally, in Section 6, we give an application of these techniques to q–Laguerre polynomials.

2. Preliminaries

The generalized basic or q–hypergeometric series ${}_r\phi_s$ is defined as [**3**]

$$ {}_r\phi_s\left(\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_s \end{array};q,x\right) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n}{(b_1;q)_n\cdots(b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}x^n, \tag{2.1} $$

where q–shifted factorial $(a;q)_n$ is defined by

$$ (a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty},\,(a;q)_\infty=\prod_{r=0}^{\infty}(1-q^r a)\,. \tag{2.2} $$

In other words,

$$ (a;q)_n = \begin{cases} (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right), & n=1,2,\dots \\ 1; & n=0 \\ \left[\left(1-aq^{-1}\right)\left(1-aq^{-2}\right)\cdots\left(1-aq^{-n}\right)\right]^{-1}; & n=-1,-2,\dots. \end{cases} \tag{2.3} $$

The series ${}_r\phi_s$ terminates if one of the numerator parameter is of the form q^{-m}, $m=0,1,2,\dots$, and $q\neq 0$. When $0<|q|<1$, the series ${}_r\phi_s$ converges absolutely for all x if $r\le s$; and for $|x|<1$ if $r=s+1$. If $|q|>1$ and $|x|<\frac{|b_1\cdots b_s|}{|a_1\cdots a_r|}$, then also ${}_r\phi_s$ converges absolutely. It diverges for $x\neq 0$ if $0<|q|<1$ and $r>s+1$, and if $|q|>1$ and $|x|>\frac{|b_1\cdots b_s|}{|a_1\cdots a_r|}$, unless it terminates.

The generalized hypergeometric series ${}_rF_s$ is defined by [**9**]

$$ {}_rF_s\left(\begin{array}{c} \alpha_1,\dots,\alpha_s \\ \beta_1,\dots,\beta_s \end{array};z\right) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n\,n!}z^n, \tag{2.4} $$

where $(\alpha)_n$ is Pochhammer's symbol defined by

$$ (\alpha)_n = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1)=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}; & n=1,2,\dots \\ 1; & n=0\,. \end{cases} \tag{2.5} $$

The q–analogue of the binomial function is

$$ {}_1\phi_0\left(\begin{array}{c} a \\ - \end{array};x\right) = \frac{(ax;q)_\infty}{(x;q)_\infty}, \qquad |q|<1,\ |x|<1. \tag{2.6} $$

The q–analogues of the exponential functions are

$$ e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}=\frac{1}{(x;q)_\infty}, \qquad |q|<1 \tag{2.7} $$

and

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} = (-x;q)_\infty, \qquad |q| < 1. \tag{2.8}$$

We shall also make use of the function

$$\Gamma_q(\alpha) = \frac{e_q(q^\alpha)}{e_q(q)} (1-q)^{1-\alpha} \tag{2.9}$$

defined for $\alpha \neq 0, -1, -2, \ldots$. This is a q–analogue of the gamma function and satisfies the functional equation

$$\Gamma_q(\alpha+1) = \frac{1-q^\alpha}{1-q} \Gamma_q(\alpha). \tag{2.10}$$

We need the q–analogue of Kampé de Fériet function [**16**]

$$F^{A:B;B'}_{C:E;E'}\left(\begin{smallmatrix}(\alpha):(\beta);(\beta')\\(\gamma):(\delta);(\delta')\end{smallmatrix}; x, y\right) = \sum_{m,n=0}^{\infty} \frac{\Pi_{j=1}^{A}(\alpha_j)_{m+n} \Pi_{j=1}^{B}(\beta_j)_m \Pi_{j=1}^{B'}(\beta'_j)_n \, x^m y^n}{\Pi_{j=1}^{C}(\gamma_j)_{m+n} \Pi_{j=1}^{E}(\delta_j)_m \Pi_{j=1}^{E'}(\delta'_j)_n \, m!n!} \tag{2.11}$$

in the form [**6**]

$$\phi^{A:B;B'}_{C:E;E'}\left(\begin{matrix}(a):(b);(b')\\(c):(e);(e')\end{matrix}; \begin{matrix}x, y\\q^r, q^s\end{matrix}\right)$$

$$= \sum_{m,n=0}^{\infty} \frac{\Pi_{j=1}^{A}(a_j;q)_{m+n} \Pi_{j=1}^{B}(b_j;q)_m \Pi_{j=1}^{B'}(b'_j;q)_n \, q^{r\binom{m}{2}} q^{s\binom{n}{2}} x^m y^n}{\Pi_{j=1}^{C}(c_j;q)_{m+n} \Pi_{j=1}^{E}(e_j;q)_m \Pi_{j=1}^{E'}(e'_j;q)_n \, (q;q)_m (q;q)_n}. \tag{2.12}$$

For $r = s = 0$, we denote the l.h.s. of (2.12) simply by

$$\phi^{A:B;B'}_{C:E;E'}\left(\begin{matrix}(a):(b);(b')\\(c):(e);(e')\end{matrix}; x, y\right).$$

The q–derivative operator is defined by

$$\Delta_x(f(x)) = \frac{f(x) - f(qx)}{(1-q)x} = \frac{(1-T_x)}{(1-q)x} f(x), \tag{2.13}$$

where the q–dilation operator T_x is given by

$$T_x(f(x)) = f(qx). \tag{2.14}$$

The q–integral [**3**] is defined by

$$\int_0^\infty f(t)\, d_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n. \tag{2.15}$$

3. Irreducible q–representations of sl(2, $\mathbb{C}$)

The complex Lie algebra $\mathrm{sl}\,(2,\mathbb{C}) = L\,\{SL\,(2,\mathbb{C})\}$, the Lie algebra of complex Lie group

$$SL\,(2,\mathbb{C}) \;=\; \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{C}, ad-bc = 1 \right\}, \tag{3.1}$$

consists of all 2×2 matrices with trace zero, that is,

$$\mathrm{sl}\,(2,\mathbb{C}) \;=\; \left\{ \begin{bmatrix} e & f \\ g & -e \end{bmatrix} : e,f,g \in \mathbb{C} \right\}. \tag{3.2}$$

It has a basis

$$\mathcal{J}^{+} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{J}^{-} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{J}^{0} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \tag{3.3}$$

satisfying the commutation relations

$$\left[\mathcal{J}^{0}, \mathcal{J}^{\pm}\right] = \pm \mathcal{J}^{\pm}, \qquad \left[\mathcal{J}^{+}, \mathcal{J}^{-}\right] = 2\mathcal{J}^{0}. \tag{3.4}$$

Let V_q be a complex vector space consisting of q–special functions with a basis $\{\phi_\lambda \mid \lambda \in S\}$ such that the functions $\{f_\lambda = \lim_{q\to 1} \phi_\lambda \mid \lambda \in S\}$ form a basis of a vector space, say V. Let $A(V_q)$ be the associative algebra of all linear operators on V_q over the complex field.

A q–representation of $\mathrm{sl}\,(2,\mathbb{C})$ on V_q is a mapping $\rho_q : \mathrm{sl}\,(2,\mathbb{C}) \to A\,(V_q)$ satisfying the following conditions:

(i) $\rho_q\,(ax+by) = a\rho_q\,(x) + b\rho_q\,(y)$

(ii) There exists a Lie algebra representation ρ of sl(2) on V such that

$$\lim_{q\to 1} \rho_q\,(x)\,\phi_\lambda \;=\; \rho\,(x)\,f_\lambda,$$

for all $x,y \in \mathrm{sl}(2,\mathbb{C})$ and $a,b \in \mathbb{C}$.

The representation ρ_q of $\mathrm{sl}(2,\mathbb{C})$ is said to be irreducible if there is no proper subspace W_q of V_q which is invariant under ρ_q.

Define

$$J_q^{+} = \rho_q(\mathcal{J}^{+}), \quad J_q^{-} = \rho_q(\mathcal{J}^{-}), \quad J_q^{0} = \rho_q(\mathcal{J}^{0}), \tag{3.5}$$

where $J_q^{+},\ J_q^{-},\ J_q^{0} \in A\,(V_q)$.

Manocha [**6**] has defined the following commutator rules for J_q–operators:

$$\begin{aligned} J_q^0 J_q^+ - qJ_q^+ J_q^0 &= J_q^+, \\ qJ_q^0 J_q^- - J_q^- J_q^0 &= -J_q^-, \\ qJ_q^+ J_q^- - J_q^- J_q^+ &= 2q^{2u} J_q^0 - (1-q)q^{2u} J_q^0 J_q^0, \qquad u \in \mathbb{C}. \end{aligned} \tag{3.6}$$

These commutator rules were later generalized by Sahai [**11**] for the 4–dimensional complex Lie algebra $\mathcal{G}\,(a,b)$, $a,b \in \mathbb{C}$. Further, the commutation relations (3.6) are also equivalent to those introduced by Jimbo [**4**]. For details, see [**6, 14, 15**].

If we define an operator C_q on V_q by

$$C_q \;=\; qJ_q^+ J_q^- + q^{2u} J_q^0 J_q^0 - q^{2u} J_q^0 \tag{3.7}$$

then it is easy to check that

$$\begin{aligned} qJ_q^+ C_q &= C_q J_q^+, \\ J_q^- C_q &= qC_q J_q^-, \\ J_q^0 C_q &= C_q J_q^0 \end{aligned} \tag{3.8}$$

As $q \to 1$, the operators J_q^+, J_q^-, J_q^0 reduce to J^+, J^-, J^0 which satisfies the commutation relations obeyed by $\mathrm{sl}\,(2,\mathbb{C})$ and the operator C_q reduces to the Casimir operator C. Hence, as $q \to 1$, ρ_q reduces to a Lie algebra representation ρ of $\mathrm{sl}\,(2,\mathbb{C})$. For more details, see [**6, 8**].

4. Models of Irreducible q–representations

We give below new two variable models for each of the q–representations $D_q\,(u,\alpha)$ and $\uparrow_q\,(u)$ corresponding to $u = -\gamma/2$, in which case they will be denoted by $D_q\,(-\gamma/2,\alpha)$ and $\uparrow_q\,(-\gamma/2)$, respectively.

4.1. Representation $D_q\,(-\gamma/2,\alpha)$.

Consider an irreducible representation $D_q\,(-\gamma/2,\alpha)$ of $\mathrm{sl}\,(2,\mathbb{C})$, $\alpha,\gamma \in \mathbb{C}$ and $\gamma \neq 0,-1,-2,\ldots$ on the representation space V_q with basis $\{f_\lambda \mid \lambda = \alpha + n,\ n \in \mathbb{Z}\}$, such that action of $\mathrm{sl}\,(2,\mathbb{C})$ on V_q is given by

$$J_q^+ f_\lambda = \frac{q^{-\gamma} - q^{\lambda-\gamma}}{1-q} f_{\lambda+1},$$

$$J_q^- f_\lambda = -\frac{1 - q^{\lambda-\gamma}}{1-q}\, f_{\lambda-1},$$

$$J_q^0 f_\lambda = \frac{1 - q^{\lambda-\gamma/2}}{1-q}\, f_\lambda,$$

$$\left(qJ_q^+J_q^- + q^{-\gamma}J_q^0J_q^0 - q^{-\gamma}J_q^0\right) f_\lambda = \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\, q^{\lambda-\gamma} f_\lambda. \tag{4.1}$$

To find a realization of (4.1) in terms of q–dilation operators, we choose

$$\begin{aligned} J_q^+ &= \frac{t}{1-q}\left(q^{-\gamma} - q^{-\gamma}T_zT_t\right), \\ J_q^- &= \frac{t^{-1}}{1-q}\, T_z^{-1}\left(q^{-\gamma}T_t - T_z - q^{-\gamma}\, zT_zT_t\right), \\ J_q^0 &= \frac{1 - q^{-\gamma/2}T_t}{1-q}, \\ C_q &= qJ_q^+J_q^- + q^{-\gamma}J_q^0J_q^0 - q^{-\gamma}J_q^0, \end{aligned} \tag{4.2}$$

with basis functions

$$f_\lambda \;=\; {}_1\phi_1\left(\begin{matrix} q^\lambda \\ q^\gamma \end{matrix}\,;q,-z\right) t^\lambda, \qquad \lambda = \alpha,\ \alpha\pm 1,\ \alpha\pm 2,\ \ldots\,.$$

Model (4.2) obeys (4.1) and the following

$$\begin{aligned} J_q^0 J_q^+ - q J_q^+ J_q^0 &= J_q^+, \\ q J_q^0 J_q^- - J_q^- J_q^0 &= -J_q^-, \\ q J_q^+ J_q^- - J_q^- J_q^+ &= 2q^{-\gamma} J_q^0 - (1-q) q^{-\gamma} J_q^0 J_q^0, \qquad \gamma \in \mathbb{C} \end{aligned} \tag{4.3}$$

as well as

$$\begin{aligned} q J_q^+ C_q &= C_q J_q^+, \\ J_q^- C_q &= q C_q J_q^-, \\ J_q^0 C_q &= C_q J_q^0. \end{aligned} \tag{4.4}$$

4.1.1. *Identities based on model (4.2).*
As

$$f_\lambda(z,t) = {}_1\phi_1\left(\begin{array}{c} q^\lambda \\ q^\gamma \end{array}; q, -z\right) t^\lambda$$

satisfies

$$C_q f_\lambda(z,t) = \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\, q^{\lambda-\gamma} f_\lambda(z,t). \tag{4.5}$$

It immediately follows that

$$\begin{aligned} u(z,t) &= \phi_{0:1;1}^{1:0;2}\left(\begin{array}{c} a : -; b', b'' \\ - : q^\gamma; c' \end{array}, \begin{array}{c} z, t \\ q, 1 \end{array}\right) t^\alpha \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n (b';q)_n (b'';q)_n}{(c';q)_n (q;q)_n}\, {}_1\phi_1\left(\begin{array}{c} q^\lambda \\ q^\gamma \end{array}; q, -z\right) t^\lambda, \end{aligned} \tag{4.6}$$

where $q^\alpha = a$, also satisfies

$$C_q u(z,t) = \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\, q^{\lambda-\gamma}\, u(z,t). \tag{4.7}$$

Using the fact that

$$e_q\left(qsJ_q^+\right) C_q = C_q e_q\left(sJ_q^+\right), \tag{4.8}$$

which follows from (4.4), we have

$$C_q\left[e_q\left(sJ_q^+\right)u\right](z,t) = \frac{\left(\frac{ast}{q^{\gamma-1}(1-q)};q\right)_\infty}{\left(\frac{st}{q^{\gamma-1}(1-q)};q\right)_\infty} \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\, q^{-\gamma}$$

$$\times\ \phi_{1:1;1}^{1:0;2}\left(\begin{array}{c} a : -; b', b'' \\ \frac{ast}{q^{\gamma-1}(1-q)} : q^\gamma; c' \end{array}; \begin{array}{c} z, qt \\ q, 1 \end{array}\right)(tq)^\alpha$$

where

$$\left[e_q\left(sJ_q^+\right)u\right](z,t) = \frac{\left(\frac{ast}{q^{\gamma}(1-q)};q\right)_\infty}{\left(\frac{st}{q^{\gamma}(1-q)};q\right)_\infty}\, \phi_{1:1;1}^{1:0;2}\left(\begin{array}{c} a : -; b', b'' \\ \frac{ast}{q^{\gamma}(1-q)} : q^\gamma; c' \end{array}; \begin{array}{c} z, t \\ q, 1 \end{array}\right) t^\alpha. \tag{4.9}$$

Using the expansion

$$\left[e_q\left(sJ_q^+\right)u\right](z,t) \;=\; \sum_{n=0}^{\infty} i_n\, f_{\alpha+n}\,(z,t)\,, \tag{4.10}$$

where i_n are obtained by putting $z=0$, we get the following identity, after suitable rescaling:

$$\frac{\left(\frac{at}{q^\gamma};q\right)_\infty}{\left(\frac{t}{q^\gamma};q\right)_\infty}\,\phi^{1:0;2}_{1:1;1}\left(\begin{array}{c} a:-;b',b'' \\ \frac{at}{q^\gamma}:q^\gamma;c' \end{array};\begin{array}{r} z,-t/w \\ q,1 \end{array}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\left(q^{-\gamma}t\right)^n\;{}_3\phi_1\left(\begin{array}{r} q^{-n},b',b'' \\ c' \end{array};q,-\frac{q^{n+\gamma}}{w}\right){}_1\phi_1\left(\begin{array}{c} aq^n \\ q^\gamma \end{array};q,-z\right), \tag{4.11}$$

where $w=-\frac{s}{1-q}$.

4.1.2. *Recurrence relations from model (4.2)*.

The recurrence relations originating from the action of J_q^+, J_q^- on f_λ are given by

$$\left(1-q^\lambda\right){}_1\phi_1\,(\lambda+1;z)+q^\lambda{}_1\phi_1\,(\lambda;qz) \;=\; {}_1\phi_1\,(\lambda;z)\,, \tag{4.12}$$

and

$$zq^{\lambda-\gamma}{}_1\phi_1\,(\lambda;qz)+{}_1\phi_1\,(\lambda;qz)-q^{\lambda-\gamma}{}_1\phi_1\,(\lambda;z) \;=\; \left(1-q^{\lambda-\gamma}\right){}_1\phi_1\,(\lambda-1;qz) \tag{4.13}$$

respectively, where ${}_1\phi_1\,(\lambda;z)$ stands for ${}_1\phi_1\left(\begin{array}{c} q^\lambda \\ q^\gamma \end{array};q,-z\right)$.

4.2. Representation $\uparrow_q\,(-\gamma/2)$.

We look for an irreducible representation $\uparrow_q\;(-\gamma/2)$ of $\mathrm{sl}\,(2,\mathbb{C})$, $\gamma\in\mathbb{C}-\{0,-1,-2,\dots\}$, on the representation space V_q with basis functions $\{f_\lambda\mid\lambda=0,1,2,\dots\}$ such that the action of $\mathrm{sl}\,(2,\mathbb{C})$ on V_q is given by

$$J_q^+f_\lambda=\frac{q^{-\gamma}-q^\lambda}{1-q}\,f_{\lambda+1},$$

$$J_q^-f_\lambda=-\frac{1-q^\lambda}{1-q}\,f_{\lambda-1},$$

$$J_q^0f_\lambda=\frac{1-q^{\lambda+\gamma/2}}{1-q}\,f_\lambda,$$

$$\left(qJ_q^+J_q^-+q^{-\gamma}J_q^0J_q^0-q^{-\gamma}J_q^0\right)f_\lambda=\frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\,q^\lambda f_\lambda. \tag{4.14}$$

To meet this requirement, we choose

$$\begin{aligned} J_q^+ &= \frac{t}{1-q}\left(q^{-\gamma} - T_z T_t - z T_z T_t\right), \\ J_q^- &= \frac{t^{-1}}{1-q}\, T_z^{-1}\left(T_t - T_z\right), \\ J_q^0 &= \frac{1-q^{\gamma/2}T_t}{1-q}, \\ C_q &= q J_q^+ J_q^- + q^{-\gamma} J_q^0 J_q^0 - q^{-\gamma} J_q^0, \end{aligned} \tag{4.15}$$

with basis functions

$$f_\lambda = {}_1\phi_1\left(\begin{matrix} q^{-\lambda} \\ q^{\gamma} \end{matrix} ; q, -q^{\lambda+\gamma} z \right) t^\lambda, \qquad \lambda = 0, 1, 2, \dots .$$

Model (4.15) satisfies (4.3) as well as (4.4).

4.2.1. ***Identities based on model (4.15).***

As

$$f_\lambda = {}_1\phi_1\left(\begin{matrix} q^{-\lambda} \\ q^{\gamma} \end{matrix} ; q, -q^{\lambda+\gamma} z \right) t^\lambda,$$

satisfies

$$C_q f_\lambda(z,t) = \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\, q^\lambda f_\lambda(z,t), \tag{4.16}$$

it immediately follows that

$$\begin{aligned} u(z,t) &= \phi^{1:0:0}_{1:1;0}\left(\begin{matrix} a : -; - \\ c' : q^\gamma; - \end{matrix} ; \begin{matrix} -q^\gamma z t, t \\ q^2, 1 \end{matrix} \right) \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(c';q)_n (q;q)_n}\, {}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\gamma} \end{matrix} ; q, -q^{n+\gamma} z \right) t^n, \end{aligned} \tag{4.17}$$

satisfies

$$C_q u(z,t) = \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\, q^n u(z,t). \tag{4.18}$$

Using the fact that

$$E_q\left(\frac{1}{q} s J_q^-\right) C_q = C_q E_q\left(s J_q^-\right), \tag{4.19}$$

which follows from (4.4), we have

$$C_q\left[E_q\left(sJ_q^-\right) u\right](z,t) = \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2} \times \phi^{1:0;1}_{1:1;0}\left(\begin{matrix} a : -; \frac{s}{(1-q)t} \\ c' : q^\gamma; - \end{matrix} ; \begin{matrix} -q^{\gamma+1} z t, qt \\ q^2, 1 \end{matrix} \right), \tag{4.20}$$

where

$$\left[E_q\left(sJ_q^-\right)u\right](z,t) \;=\; \phi_{1:1;0}^{1:0;1}\left(\begin{array}{c} a:-;\frac{s}{(1-q)t} \\ c':q^{\gamma};- \end{array};\begin{array}{c} -q^{\gamma}zt,t \\ q^2,1 \end{array}\right). \qquad (4.21)$$

The expansion

$$\left[E_q\left(sJ_q^-\right)u\right](z,t) \;=\; \sum_{n=0}^{\infty} A_n\, f_n\left(z,t\right), \qquad (4.22)$$

leads to the following identity

$$\phi_{1:1;0}^{1:0;1}\left(\begin{array}{c} a:-;-w/t \\ c':q^{\gamma};- \end{array};\begin{array}{c} -q^{\gamma}zt,t \\ q^2,1 \end{array}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n\,(c';q)_n}\,{}_1\phi_1\left(\begin{array}{c} aq^n \\ c'q^n \end{array};q,-w\right){}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\gamma} \end{array};q,-q^{n+\gamma}z\right)t^n. \qquad (4.23)$$

4.2.2. *Recurrence relations*.

The recurrence relations arising from model (4.15) are given by

$${}_1\phi_1\left(\lambda;z\right) - q^{\lambda+\gamma}{}_1\phi_1\left(\lambda;qz\right) - q^{\lambda+\gamma}\,z\,{}_1\phi_1\left(\lambda;qz\right) \;=\; \left(1-q^{\lambda+\gamma}\right){}_1\phi_1\left(\lambda+1;z\right), \qquad (4.24)$$

$${}_1\phi_1\left(\lambda;qz\right) - q^{\lambda}{}_1\phi_1\left(\lambda;z\right) \;=\; \left(1-q^{\lambda}\right){}_1\phi_1\left(\lambda-1;qz\right), \qquad (4.25)$$

where ${}_1\phi_1\left(\lambda;z\right)$ stands for ${}_1\phi_1\left(\begin{array}{c} q^{-\lambda} \\ q^{\gamma} \end{array};q,-q^{\lambda+\gamma}z\right)$.

5. Transformed Models of $\mathbf{sl}\left(2,\mathbb{C}\right)$

We introduce the q–Mellin transformation I_q.
Define

$$h\left(\beta,x\right) = I_q\left[f\left(ux\right)\right] = \frac{q^{\frac{\beta(\beta-1)}{2}}}{\Gamma_q\left(\beta\right)}\int_0^{\infty} e_q\left(-u\right)\,u^{\beta-1}f\left(ux\right)\,d_qu, \qquad (5.1)$$

which is motivated from the q–integral representation of gamma function, as in [**17**],

$$\Gamma_q\left(\gamma\right) \;=\; q^{\frac{\gamma(\gamma-1)}{2}}\int_0^{\infty} e_q\left(-u\right)\,u^{\gamma-1}\,d_qu. \qquad (5.2)$$

Using the substitution $z = ux$, which gives $z\Delta_z = u\Delta_u = x\Delta_x$ and $T_zf\left(z\right) = f\left(qz\right) = T_uf\left(ux\right)$, we have the following transforms of certain operator expressions which are needed for the discussion.

$$\begin{aligned} I_q\left[uf\left(ux\right)\right] \;&=\; \frac{q^{-\beta}-1}{1-q}\,E_{\beta}h\left(\beta,x\right), \\ I_q\left[u\Delta_uf\left(ux\right)\right] \;&=\; x\Delta_xh\left(\beta,x\right), \end{aligned} \qquad (5.3)$$

where $E_{\beta}h\left(\beta,x\right) = h\left(\beta+1,x\right)$.

To obtain transformed models of $D_q(-\gamma/2,\alpha)$ and $\uparrow_q(-\gamma/2)$, we put $z = ux$ in models (4.2) and (4.15). Note that if ρ_q is an irreducible representation of $\mathrm{sl}(2,\mathbb{C})$ on the representation space V_q having the basis vectors $\{f_\lambda \mid \lambda \in S\}$ in terms of $\left\{J_q^+, J_q^-, J_q^0\right\}$, then the transformation I_q induces another irreducible representation σ_q of $\mathrm{sl}(2,\mathbb{C})$ on the representation space $W_q = I_q V_q$ in terms of $\left\{K_q^+ = I_q J_q^+ I_q^{-1}, K_q^- = I_q J_q^- I_q^{-1}, K_q^0 = I_q J_q^0 I_q^{-1}\right\}$ with basis vectors $\{h_\lambda = I_q f_\lambda \mid \lambda \in S\}$.

5.1. Transformed Model of $D_q(-\gamma/2,\alpha)$.

$$\begin{aligned} K_q^+ &= \frac{tq^{-\gamma}}{1-q}\left(1 - T_x T_t\right), \\ K_q^- &= \frac{t^{-1}}{1-q}\, T_x^{-1}\left(q^{-\gamma} T_t - T_x - \frac{q^{-\beta}-1}{1-q}\, E_\beta q^{-\gamma} x T_x T_t\right), \\ K_q^0 &= \frac{1-q^{-\gamma/2} T_t}{1-q}, \end{aligned} \tag{5.4}$$

with basis functions as

$$h_\lambda(x,t) = {}_2\phi_1\left(\begin{matrix} q^\lambda, q^\beta \\ q^\gamma \end{matrix}; q, \frac{x}{(1-q)\,q^\beta}\right) t^\lambda.$$

Indeed,

$$\begin{aligned} K_q^0 K_q^+ - q K_q^+ K_q^0 &= K_q^+, \\ q K_q^0 K_q^- - K_q^- K_q^0 &= -K_q^-, \\ q K_q^+ K_q^- - K_q^- K_q^+ &= 2q^{-\gamma} K_q^0 - (1-q)\, q^{-\gamma} K_q^0 K_q^0 \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} K_q^+ h_\lambda &= q^{-\gamma}\frac{1-q^\lambda}{1-q}\, h_{\lambda+1}, \\ K_q^- h_\lambda &= -\frac{1-q^{\lambda-\gamma}}{1-q}\, h_{\lambda-1}, \\ K_q^0 h_\lambda &= \frac{1-q^{\lambda-\gamma/2}}{1-q}\, h_\lambda, \\ C_q' h_\lambda &= \left(q K_q^+ K_q^- + q^{-\gamma} K_q^0 K_q^0 - q^{-\gamma} K_q^0\right) h_\lambda \\ &= \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\, q^{\lambda-\gamma} h_\lambda. \end{aligned} \tag{5.6}$$

Moreover,

$$\begin{aligned} q K_q^+ C_q' &= C_q' K_q^+, \\ K_q^- C_q' &= q C_q' K_q^-, \\ K_q^0 C_q' &= C_q' K_q^0. \end{aligned} \tag{5.7}$$

5.1.1. *Identities based on transformed model $D_q(-\gamma/2,\alpha)$.*

As

$$h_\lambda(x,t) = {}_2\phi_1\left(\begin{matrix} q^\lambda, q^\beta \\ q^\gamma \end{matrix};q,\frac{x}{(1-q)q^\beta}\right)t^\lambda$$

is a solution of

$$C_q' h_\lambda(x,t) = \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\, q^{\lambda-\gamma} h_\lambda(x,t), \tag{5.8}$$

the function

$$\begin{aligned} u(x,t) &= \phi^{1:1;2}_{0:1;1}\left(\begin{matrix} a: q^\beta; b', b'' \\ -: q^\gamma; c' \end{matrix}, \frac{q^{-\beta}x}{1-q}, t\right)t^\alpha \\ &= \sum_{n=0}^{\infty}\frac{(a;q)_n (b';q)_n (b'';q)_n}{(c';q)_n (q;q)_n}\, {}_2\phi_1\left(\begin{matrix} q^\lambda, q^\beta \\ q^\gamma \end{matrix};q,\frac{x}{(1-q)q^\beta}\right)t^\lambda \end{aligned} \tag{5.9}$$

is also a solution of

$$C_q' u(x,t) = \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\, q^{\lambda-\gamma}\, u(x,t). \tag{5.10}$$

Using $qK_q^+ C_q' = C_q' K_q^+$, we have

$$\begin{aligned} C_q'\left[e_q\left(sK_q^+\right)u\right](x,t) &= \frac{\left(\frac{ast}{q^{\gamma-1}(1-q)};q\right)_\infty}{\left(\frac{st}{q^{\gamma-1}(1-q)};q\right)_\infty}\, \frac{q^{-\gamma}\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2} \\ &\quad\times\ \phi^{1:1;2}_{1:1;1}\left(\begin{matrix} a: q^\beta; b', b'' \\ \frac{ast}{q^{\gamma-1}(1-q)}: q^\gamma; c' \end{matrix}; \frac{q^{-\beta}x}{1-q}, qt\right)(qt)^\alpha, \end{aligned} \tag{5.11}$$

where

$$\left[e_q\left(sK_q^+\right)u\right](x,t) = \frac{\left(\frac{ast}{q^\gamma(1-q)};q\right)_\infty}{\left(\frac{st}{q^\gamma(1-q)};q\right)_\infty}\,\phi^{1:1;2}_{1:1;1}\left(\begin{matrix} a: q^\beta; b', b'' \\ \frac{ast}{q^\gamma}: q^\gamma; c' \end{matrix}; \frac{q^{-\beta}x}{1-q}, t\right)t^\alpha. \tag{5.12}$$

From the expansion

$$\left[e_q\left(sK_q^+\right)u\right](x,t) = \sum_{n=0}^{\infty} A_n h_{\alpha+n}(x,t),$$

where A_n is found by putting $x=0$, we arrive at the following identity after suitable rescaling:

$$\begin{aligned} &\frac{\left(\frac{at}{q^\gamma};q\right)_\infty}{\left(\frac{t}{q^\gamma};q\right)_\infty}\,\phi^{1:1;2}_{1:1;1}\left(\begin{matrix} a: q^\beta; b', b'' \\ \frac{at}{q^\gamma}: q^\gamma; c' \end{matrix}; \frac{q^{-\beta}x}{1-q}, -t/w\right) \\ &= \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\left(q^{-\gamma}t\right)^n {}_3\phi_1\left(\begin{matrix} q^{-n}, b', b'' \\ c' \end{matrix};q,-\frac{q^{n+\gamma}}{w}\right){}_2\phi_1\left(\begin{matrix} aq^n, q^\beta \\ q^\gamma \end{matrix};q,\frac{x}{(1-q)q^\beta}\right). \end{aligned} \tag{5.13}$$

5.1.2. *Recurrence relations from model (5.4)*.

The recurrence relations originating from the action of K_q^+ and K_q^- on h_λ are given by

$$ {}_2\phi_1(\lambda;\beta;x) - q^\lambda {}_2\phi_1(\lambda;\beta;qx) = \left(1-q^\lambda\right) {}_2\phi_1(\lambda+1;\beta;x), \tag{5.14} $$

$$ xq^{\lambda-\gamma-\beta}\frac{1-q^\beta}{1-q}\, {}_2\phi_1(\lambda;\beta+1;qx) + {}_2\phi_1(\lambda;\beta;qx) - q^{\lambda-\gamma} {}_2\phi_1(\lambda;\beta;x) = \left(1-q^{\lambda-\gamma}\right) {}_2\phi_1(\lambda-1;\beta;qx), \tag{5.15} $$

where ${}_2\phi_1(\lambda;\beta;x)$ stands for ${}_2\phi_1\left(\begin{matrix} q^\lambda, q^\beta \\ q^\gamma \end{matrix}; q, \frac{x}{(1-q)q^\beta}\right)$.

5.2. Transformed Model of $\uparrow_q(-\gamma/2)$.

$$ K_q^+ = \frac{t}{1-q}\left(q^{-\gamma} - T_xT_t - \frac{q^{-\beta}-1}{1-q}\, E_\beta\, xT_xT_t\right), \tag{5.16} $$

$$ K_q^- = \frac{t^{-1}}{1-q} T_x^{-1}\left(x\Delta_x - t\Delta_t\right), $$

$$ K_q^0 = \frac{1-q^{\gamma/2}T_t}{1-q}, $$

with basis functions as

$$ h_\lambda(x,t) = {}_2\phi_1\left(\begin{matrix} q^{-\lambda}, q^\beta \\ q^\gamma \end{matrix}; q, \frac{q^{\lambda+\gamma-\beta}}{1-q}\, x\right) t^\lambda. $$

Model (5.16) satisfies (5.5) and (5.7) and

$$ \begin{aligned} K_q^+ h_\lambda &= \frac{q^{-\gamma}-q^\lambda}{1-q} h_{\lambda+1}, \\ K_q^- h_\lambda &= -\frac{1-q^\lambda}{1-q} h_{\lambda-1}, \\ K_q^0 h_\lambda &= \frac{1-q^{\lambda+\gamma/2}}{1-q}\, h_\lambda, \\ C_q' h_\lambda &= \left(qK_q^+K_q^- + q^{-\gamma}K_q^0K_q^0 - q^{-\gamma}K_q^0\right) h_\lambda \\ &= \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2}\, q^\lambda h_\lambda. \end{aligned} \tag{5.17} $$

5.2.1. *Identities based on transformed model $\uparrow_q(-\gamma/2)$*.

As

$$h_\lambda(x,t) = {}_2\phi_1\left(\begin{matrix} q^{-\lambda}, q^\beta \\ q^\gamma \end{matrix}; q, \frac{q^{\lambda+\gamma-\beta}}{1-q}x\right)t^\lambda, \qquad \lambda = 0,1,2,\dots$$

satisfies

$$C'_q h_\lambda(x,t) = \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2} q^\lambda h_\lambda(x,t). \tag{5.18}$$

This, in turn, gives

$$\begin{aligned} u(x,t) &= \phi^{1:1;0}_{1:1;0}\left(\begin{matrix} a:q^\beta;- \\ c':q^\gamma;- \end{matrix}; \begin{matrix} -\frac{q^{\gamma-\beta}xt}{1-q}, t \\ q, 1 \end{matrix}\right) \\ &= \sum_{n=0}^\infty \frac{(a;q)_n}{(c';q)_n(q;q)_n} {}_2\phi_1\left(\begin{matrix} q^{-n}, q^\beta \\ q^\gamma \end{matrix}; q, \frac{q^{n+\gamma-\beta}}{1-q}x\right)t^n, \end{aligned} \tag{5.19}$$

satisfies

$$C'_q u(x,t) = \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2} q^n u(x,t). \tag{5.20}$$

Using $K_q^- C'_q = qC'_q K_q^-$, we have

$$\begin{aligned} C'_q\left[E_q\left(sK_q^-\right)u\right](x,t) = {}& \frac{\left(1-q^{-\gamma/2}\right)\left(1-q^{-\gamma/2+1}\right)}{(1-q)^2} \\ & \times \phi^{1:1;1}_{1:1;0}\left(\begin{matrix} a:q^\beta;\frac{s}{q(1-q)t} \\ c':q^\gamma;- \end{matrix}; \begin{matrix} -\frac{q^{\gamma-\beta+1}xt}{1-q}, qt \\ q, 1 \end{matrix}\right), \end{aligned} \tag{5.21}$$

where

$$\left[E_q\left(sK_q^-\right)u\right](x,t) = \phi^{1:1;1}_{1:1;0}\left(\begin{matrix} a:q^\beta;\frac{s}{(1-q)t} \\ c':q^\gamma;- \end{matrix}; \begin{matrix} -\frac{q^{\gamma-\beta}xt}{1-q}, t \\ q, 1 \end{matrix}\right). \tag{5.22}$$

The expansion

$$\left[E_q\left(sK_q^-\right)u\right](x,t) = \sum_{n=0}^\infty C_n h_n(x,t), \tag{5.23}$$

gives rise to the following identity:

$$\begin{aligned} &\phi^{1:1;1}_{1:1;0}\left(\begin{matrix} a:q^\beta;-\frac{w}{t} \\ c':q^\gamma;- \end{matrix}; \begin{matrix} -\frac{q^{\gamma-\beta}xt}{1-q}, t \\ q, 1 \end{matrix}\right) \\ &= \sum_{n=0}^\infty \frac{(a;q)_n}{(q;q)_n(c';q)_n} {}_1\phi_1\left(\begin{matrix} aq^n \\ c'q^n \end{matrix}; q, -w\right) {}_2\phi_1\left(\begin{matrix} q^{-n}, q^\beta \\ q^\gamma \end{matrix}; q, \frac{q^{n+\gamma-\beta}}{1-q}x\right)t^n. \end{aligned} \tag{5.24}$$

5.2.2. ***Recurrence relations.***

The recurrence relations that come from (5.17) are given by

$$
\begin{aligned}
{}_2\phi_1(\lambda;\beta;x) - q^{\lambda+\gamma}{}_2\phi_1(\lambda;\beta;qx) - xq^{\lambda+\gamma-\beta}\frac{1-q^{\beta}}{1-q}\,{}_2\phi_1(\lambda;\beta+1;qx) \\
= \left(1-q^{\lambda}\right){}_2\phi_1(\lambda+1;\beta;x), \quad (5.25)
\end{aligned}
$$

$$
{}_2\phi_1(\lambda;\beta;qx) - q^{\lambda}{}_2\phi_1(\lambda;\beta;x) = \left(1-q^{\lambda}\right){}_2\phi_1(\lambda-1;\beta;qx), \quad (5.26)
$$

where ${}_2\phi_1(\lambda;\beta;x)$ stands for ${}_2\phi_1\left(\begin{matrix} q^{-\lambda}, q^{\beta} \\ q^{\gamma} \end{matrix}; q, \frac{q^{\lambda+\gamma-\beta}}{1-q}x\right)$.

6. Conclusion

In this section, we give an application of our models to q–Laguerre polynomials. Let $\alpha > -1$ and $0 < q < 1$. Then q–Laguerre polynomials are defined by [**3**]

$$
L_n^{\alpha}(z;q) = \frac{\left(q^{\alpha+1};q\right)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix}; q, -zq^{n+\alpha+1}\right). \quad (6.1)
$$

If we take $z = (1-q)\,x$ then it can be easily seen that $\lim_{q\to 1^-} L_n^{\alpha}(z;q) = L_n^{\alpha}(x)$, where $L_n^{\alpha}(x)$ are classical Laguerre polynomials. Consider the model (4.15) in the particular case $\gamma = 1+\alpha$ (that is, a model of representation $\uparrow_q(-(1+\alpha)/2)$) with basis functions involving q–Laguerre polynomials, given as

$$
\begin{aligned}
J_q^{+} &= \frac{t}{1-q}\left(q^{-(1+\alpha)} - T_zT_t - zT_zT_t\right), \\
J_q^{-} &= \frac{t^{-1}}{1-q}\,T_z^{-1}\left(T_t - T_z\right), \\
J_q^{0} &= \frac{1-q^{(1+\alpha)/2}T_t}{1-q}, \\
f_n(z,t) &= L_n^{\alpha}(z;q)\,t^n. \quad (6.2)
\end{aligned}
$$

The action of these J_q–operators on $f_n(z,t)$ will take the following form:

$$
\begin{aligned}
J_q^{+}f_n &= q^{-(1+\alpha)}\frac{1-q^{n+1}}{1-q}f_{n+1}, \\
J_q^{-}f_n &= -\frac{1-q^{\alpha+n}}{1-q}\,f_{n-1}, \\
J_q^{0}f_n &= \frac{1-q^{\frac{1+\alpha}{2}+n}}{1-q}\,f_n. \quad (6.3)
\end{aligned}
$$

We can obtain identities involving q–Laguerre polynomials using the method explained above. For example, taking

$$
u(z,t) = \sum_{n=0}^{\infty}\frac{\left(q^{\alpha+1};q\right)_n}{(c';q)_n\,(q;q)_n}\,{}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix}; q, -zq^{n+\alpha+1}\right)t^n \quad (6.4)
$$

and proceeding as in Section 4.2.1, we arrive at the following identity:

$$\phi^{1:0;1}_{1:1;0}\left(\begin{array}{c} q^{\alpha+1} : -; -\frac{w}{t} \\ c' : q^{1+\alpha}; - \end{array} ; \begin{array}{r} -q^{\alpha+1}zt, t \\ q^2, 1 \end{array}\right)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(c';q)_n} L_n^{\alpha}(z;q)\, {}_1\phi_1\left(\begin{array}{c} q^{\alpha+1+n}, \\ c'q^n \end{array} ; q, -w\right), \quad (6.5)$$

which is a special case of (4.23).

The recurrence relations involving q–Laguerre polynomials can also be obtained by using first two equations of (6.3). These will shape as

$$L_n^{\alpha}(z;q) - q^{1+\alpha+n} L_n^{\alpha}(qz;q) - zq^{n+\alpha+1} L_n^{\alpha}(qz;q) \;=\; \left(1-q^{n+1}\right) L_{n+1}^{\alpha}(z;q) \quad (6.6)$$

and

$$q^n L_n^{\alpha}(z;q) - L_n^{\alpha}(qz;q) \;=\; -\left(1-q^{\alpha+n}\right) L_{n-1}^{\alpha}(qz;q), \quad (6.7)$$

respectively. Eliminating $L_n^{\alpha}(z;q)$ from (6.6) and (6.7) gives rise to a three term recurrence relation as follows

$$\left(1-q^{\alpha+n}\right) L_{n-1}^{\alpha}(qz;q) + \left(zq^{1+\alpha+2n} - \left(1-q^{1+\alpha+2n}\right)\right) L_n^{\alpha}(qz;q)$$

$$+\; q^n \left(1-q^{n+1}\right) L_{n+1}^{\alpha}(z;q) \;=\; 0, \quad (6.8)$$

$n = 1, 2, \ldots$.

Note that (6.8) is a q–analogue of the three term recurrence relation for classical Laguerre polynomials [**2**],

$$(\alpha+n) L_{n-1}^{\alpha}(x) + (x-1-\alpha-2n) L_n^{\alpha}(x) + (n+1) L_{n+1}^{\alpha}(x) \;=\; 0. \quad (6.9)$$

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