MULTIOBJECTIVE MIXED SYMMETRIC DUALITY WITH INVEXITY

IZHAR AHMAD

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Abstract. The usual duality results are established for mixed symmetric multiobjective dual programs without nonnegativity constraints using the notion of invexity/generalized invexity which has allowed weakening various types of convexity/generalized convexity assumptions. This mixed symmetric dual formulation unifies two existing symmetric dual formulations in the literature.

1. Introduction

A pair of dual problems is called symmetric if the dual of the dual is the primal problem. Symmetric duality in nonlinear programming was introduced by Dorn [5]. Dantzig et al. [4], Mond [8], Bazarra and Goode [2] and Mond and Weir [10] etc. further developed the concept of symmetric duality.


Xu [12] introduced the mixed type duals in multiobjective programming and proved duality theorems. In this paper, we study invexity/generalized invexity for mixed type symmetric dual in multiobjective programming problems ignoring nonnegativity constraints of Bector et al. [1] but adjoining an additional condition on invexity/generalized invexity. Self duality for our programs is also incorporated.

2. Notations and Prerequisites

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. The following conventions of vectors in $\mathbb{R}^n$ will be followed throughout this paper: $x \leq y \iff x_i \leq y_i$, $i = 1, 2, \ldots, n$; $x \leq y \iff x \leq y$ and $x \neq y$; $x < y \iff x_i < y_i$, $i = 1, 2, \ldots, n$.

For $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$ let $J_1 \subseteq N$, $K_1 \subseteq M$ and $J_2 = N \setminus J_1$ and $K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in the subset of $J_1$. The other symbols $|J_2|$, $|K_1|$ and $|K_2|$ are defined similarly. Let $x^1 \in \mathbb{R}^{|J_1|}$, $x^2 \in \mathbb{R}^{|J_2|}$, then any $x \in \mathbb{R}^n$ can be written as $(x^1, x^2)$. Similarly for $y^1 \in \mathbb{R}^{|K_1|}$ and $y^2 \in \mathbb{R}^{|K_2|}$, $y \in \mathbb{R}^m$ can be written as $(y^1, y^2)$. Let $f : \mathbb{R}^{|J_1|} \times \mathbb{R}^{|K_1|} \to \mathbb{R}^l$ and $g : \mathbb{R}^{|J_2|} \times \mathbb{R}^{|K_2|} \to \mathbb{R}^l$ be twice differentiable functions and $e = (1, 1, \ldots, 1) \in \mathbb{R}^l$. 1991 Mathematics Subject Classification 90C26, 90C29, 90C30, 90C46. Key words and phrases: Mixed symmetric duality, differentiable programming, invexity, pseudoinvexity, self duality.
It may be noted that if \( J_1 = \phi \), then \( |J_1| = 0 \), \( J_2 = N \) and \( |J_2| = n \). In this case \( R^{|J_1|} \times R^{|J_2|} \) and \( R^{|J_1|} \times R^{|K_1|} \) will be zero-dimensional, \( n \)-dimensional and \( |K_1| \)-dimensional Euclidean spaces respectively. The other situations are \( J_2 = \phi \), \( K_1 = \phi \) or \( K_2 = \phi \). These give particular cases of the problems considered in this paper and are discussed in Section 5.

Let \( \nabla_{x,y} f(x,y) \) denotes the \( |J_1| \times l \) matrix of first order partial derivatives. For the scalar function \( \lambda f \) with \( \lambda \in R^l \), \( \nabla_x (\lambda f) \) and \( \nabla_y (\lambda f) \) denote gradient vectors with respect to \( x^1 \) and \( y^1 \) respectively; \( \nabla_{y^1 y^2} (\lambda f) \) and \( \nabla_{x^1 x^2} (\lambda f) \) denote respectively the \( |K_1| \times |K_1| \) and \( |J_1| \times |J_1| \) matrices of second order partial derivatives. \( \nabla_{x^2} (\lambda g) \), \( \nabla_{y^2} (\lambda g) \) \( \nabla_{x^2 y^2} (\lambda g) \) are defined similarly.

No notational distinction is made between row and column vectors. It should be clear from the context.

**Definition 2.1.** A function \( \psi : R^m \times R^m \rightarrow R \) is invex in \( x \) for each \( y \in R^m \), if there exists a function \( \eta_1 : R^m \times R^m \rightarrow R^m \) such that

\[
\psi(x,y) - \psi(u,y) \geq \eta_1(x,u) \nabla_u \psi(u,y), \text{ for all } x,u \in R^m,
\]

and \( \psi \) is incave in \( y \) if for each \( x \in R^m \), if there exists a function \( \eta_2 : R^m \times R^m \rightarrow R^m \) such that

\[
\psi(x,v) - \psi(x,y) \leq \eta_2(v,y) \nabla_y \psi(x,y), \text{ for all } v,y \in R^m.
\]

**Definition 2.2.** A function \( \psi : R^m \times R^m \rightarrow R \) is pseudoinvex in \( x \) for each \( y \in R^m \), if there exists a function \( \eta_3 : R^m \times R^m \rightarrow R^m \) such that

\[
\eta_3(x,u) \nabla_u \psi(u,y) \geq 0 \Rightarrow \psi(x,y) \geq \psi(u,y) \text{ for all } x,u \in R^m,
\]

and \( \psi \) is pseudoincave in \( y \) if for each \( x \in R^m \), if there exists a function \( \eta_4 : R^m \times R^m \rightarrow R^m \) such that

\[
\eta_4(v,y) \nabla_y \psi(x,y) \leq 0 \Rightarrow \psi(x,v) \leq \psi(x,y) \text{ for all } v,y \in R^m.
\]

### 3. Mixed Symmetric Dual Programs

We introduce the following pair of multiobjective mixed symmetric dual programs and establish duality theorems:

**Primal(VP).**

Minimize \( H(x^1, x^2, y^1, y^2, \lambda) = f(x^1, y^1) + g(x^2, y^2) - [y^1 \nabla_{y^1} \lambda f(x^1, y^1)] e. \)

Subject to

\[
\nabla_{y^1} \lambda f(x^1, y^1) \leq 0, \quad (1)
\]
\[
\nabla_{y^2} \lambda g(x^2, y^2) \leq 0, \quad (2)
\]
\[
y^2 \nabla_{y^2} \lambda g(x^2, y^2) \geq 0, \quad (3)
\]
\[
\lambda > 0, \quad \lambda e = 1. \quad (4)
\]
Dual (VD).
Maximize \( G(u^1, u^2, v^1, v^2, \lambda) = f(u^1, v^1) + g(u^2, v^2) - [u^1 \nabla_u^1 \lambda f(u^1, v^1)] e. \)
Subject to
\[
\begin{align*}
\nabla_u^1 \lambda f(u^1, v^1) & \geq 0, \\
\nabla_u^2 \lambda g(u^2, v^2) & \geq 0, \\
\lambda f(u^2, v^2) & \leq 0,
\end{align*}
\]
\( \lambda > 0, \quad \lambda e = 1. \) \hfill (8)

These are the mixed symmetric dual programs formulated by Bector et al. [1],
with the omission of constraints \((x^1, x^2) \geq 0\) from (VP) and \((u^1, u^2, v^1, v^2, \lambda)\) be feasible for (VD).

**Theorem 3.1** (Weak Duality). Let \((x^1, x^2, y^1, y^2, \lambda)\) be feasible for (VP) and \((u^1, u^2, v^1, v^2, \lambda)\) be feasible for (VD). Let
(i) \( f(\cdot, v^1) \) be invex in \( x^1 \) with respect to \( \eta_1 \) and \( f(x^1, \cdot) \) be incave with respect to \( \eta_2 \), with \( \eta_1(x^1, u^1) + u^1 \geq 0 \) and \( \eta_2(v^1, y^1) + y^1 \geq 0 \), and
(ii) \( \lambda g(\cdot, v^2) \) be pseudoincave in \( x^2 \) with respect to \( \eta_3 \) and \( \lambda g(x^2, \cdot) \) be pseudoincave with respect to \( \eta_4 \), with \( \eta_3(x^2, u^2) + u^2 \geq 0 \) and \( \eta_4(v^2, y^2) + y^2 \geq 0 \).

Then
\[
H(x^1, x^2, y^1, y^2, \lambda) \nleq G(u^1, u^2, v^1, v^2, \lambda).
\]

**Proof.** By the invexity of \( f(\cdot, v^1) \) and incavity of \( f(x^1, \cdot) \), we have
\[
\lambda f(x^1, v^1) - \lambda f(u^1, v^1) \geq \eta_1(x^1, u^1) \nabla_u^1 \lambda f(u^1, v^1),
\]
and
\[
\lambda f(x^1, v^1) - \lambda f(x^1, y^1) \leq \eta_2(v^1, y^1) \nabla_y^1 \lambda f(x^1, y^1).
\]
Adding the above inequalities, we obtain
\[
\lambda f(x^1, y^1) - \lambda f(u^1, v^1) \geq \eta_1(x^1, u^1) \nabla_u^1 \lambda f(u^1, v^1) - \eta_2(v^1, y^1) \nabla_y^1 \lambda f(x^1, y^1) \tag{9}
\]
or
\[
\begin{align*}
[\lambda f(x^1, y^1) - y^1 \nabla_y^1 \lambda f(x^1, y^1)] - [\lambda f(u^1, v^1) - u^1 \nabla_u^1 \lambda f(u^1, v^1)] & \\
\geq (\eta_1(x^1, u^1) + u^1)(\nabla_u^1 \lambda f(u^1, v^1)) - (\eta_2(v^1, y^1) + y^1)(\nabla_y^1 \lambda f(x^1, y^1)) & \\
\geq 0 \text{ (using (1), (5), } \eta_1(x^1, u^1) + u^1 \geq 0 \text{ and } \eta_2(v^1, y^1) + y^1 \geq 0). & \tag{10}
\end{align*}
\]

From (6) and \( \eta_3(x^2, u^2) + u^2 \geq 0 \), we have
\[
\eta_3(x^2, u^2) \nabla_u^2 \lambda g(u^2, v^2) \geq -u^2 \nabla_u^2 \lambda g(u^2, v^2) & \\
\geq 0 \text{ (by (7)),}
\]
which, in view of the pseudoincavity of \( \lambda g(\cdot, v^2) \), gives
\[
\lambda g(x^2, v^2) \geq \lambda g(u^2, v^2). \tag{11}
\]
Similarly
\[ \lambda g(x^2, v^2) \leq \lambda g(x^2, y^2). \] (12)
The relations (11) and (12) yield
\[ \lambda g(x^2, y^2) \geq \lambda g(u^2, v^2). \] (13)
From (10) and (13), we obtain
\[ \lambda f(x^1, y^1) + \lambda g(x^2, y^2) - y^1 \nabla_{y^1} \lambda f(x^1, y^1) \geq \lambda f(u^1, v^1) + \lambda g(u^2, v^2) - u^1 \nabla_{u^1} \lambda f(u^1, v^1). \]
Since \( \lambda e = 1 \), the above inequality can be written as
\[ \lambda \left[ f(x^1, y^1) + g(x^2, y^2) - \{y^1 \nabla_{y^1} \lambda f(x^1, y^1)\} e \right] \geq \lambda \left[ f(u^1, v^1) + g(u^2, v^2) - \{u^1 \nabla_{u^1} \lambda f(u^1, v^1)\} e \right]. \]
Hence
\[ H(x^1, x^2, y^1, y^2, \lambda) \not< G(u^1, u^2, v^1, v^2, \lambda). \]
The following theorem also serves to correct the proof of Theorem 3.2 in Bector et al. [1] as while applying Fritz John conditions to (VP), the constraint \( \lambda e = 1 \) has not been considered.

**Theorem 3.2** (Strong Duality). Let the hypotheses of Theorem 3.1 be satisfied, and \((x^1, x^2, \gamma^1, \gamma^2, \lambda)\) be a properly efficient solution of (VP). Fix \( \lambda = \lambda \) in (VD). Assume that
(a) the Hessian matrices \( \nabla_{y^1} \lambda f(x^1, \gamma^1) \) and \( \nabla_{y^2} \lambda g(x^2, \gamma^2) \) are positive definite or negative definite, and
(b) the set \( \{\nabla_{y^1} g^1(x^1, \gamma^1), \nabla_{y^2} g^2(x^2, \gamma^2), \ldots, \nabla_{y^2} g^l(x^2, \gamma^2)\} \) is linearly independent.
Then \((x^1, x^2, \gamma^1, \gamma^2, \lambda)\) is a properly efficient solution of (VD).

**Proof.** Since \((x^1, x^2, \gamma^1, \gamma^2, \lambda)\) is a properly efficient solution of (VP), it is also weak efficient. Hence there exist Lagrange multipliers \( \alpha, \mu \in R^l, \beta \in R^{|K_1|}, \nu \in R^{|K_2|} \) and \( \xi, \eta \in R \) such that the following Fritz John conditions [11] are satisfied at \((x^1, x^2, \gamma^1, \gamma^2, \lambda)\):
\[ \nabla_{x^1} \alpha f(x^1, \gamma^1) + (\beta - \alpha \nabla \gamma^1) \nabla f(x^1, \gamma^1) = 0, \] (14)
\[ \nabla_{x^2} \alpha g(x^2, \gamma^2) + (\nu - \xi \nabla \gamma^2) \nabla g(x^2, \gamma^2) = 0, \] (15)
\[ \nabla_{y^1} (\alpha - \alpha \lambda) f(x^1, \gamma^1) + (\beta - \alpha \nabla \gamma^1) \nabla f(x^1, \gamma^1) = 0, \] (16)
\[ \nabla_{y^2} (\alpha - \xi \lambda) g(x^2, \gamma^2) + (\nu - \xi \nabla \gamma^2) \nabla g(x^2, \gamma^2) = 0, \] (17)
\[ (\beta - \alpha \nabla \gamma^1) \nabla_{y^i} f(x^1, \gamma^1) + (\nu - \xi \nabla \gamma^2) \nabla_{y^i} g(x^2, \gamma^2) - \mu^i - \eta = 0, \quad i \in \{1, 2, \ldots, l\} \] (18)
\[ \beta \nabla_{y^1} \bar{x} f(x^1, y^1) = 0, \]  
(19)

\[ \nu \nabla_{y^2} \bar{\lambda} g(x^2, y^2) = 0, \]  
(20)

\[ \xi \nabla_{y^2} \bar{\lambda} g(x^2, y^2) = 0, \]  
(21)

\[ \mu \bar{\lambda} = 0 \]  
(22)

\[ (\alpha, \beta, \nu, \xi, \mu) \geq 0, \]  
(23)

\[ (\alpha, \beta, \nu, \xi, \mu, \eta) \neq 0. \]  
(24)

Since \( \bar{\lambda} > 0 \) and \( \mu \geq 0 \), equation (22) yields \( \mu = 0 \). Multiplying (18) by \( \alpha_i - \alpha e \lambda_i \), \( i \in \{1, 2, \cdots, l\} \), summing the resulting expression for all \( i \) and then using \( \lambda e = 1 \), we obtain

\[ (\beta - \alpha e \bar{\gamma}^1) \sum_{i=1}^{l} \nabla_{y^1} f(x^1, y^1) \left[ \alpha - \alpha e \bar{\lambda} \right] + (\nu - \xi \bar{\gamma}^2) \sum_{i=1}^{l} \nabla_{y^2} g(x^2, y^2) \left[ \alpha - \alpha e \bar{\lambda} \right] = 0, \]  
(25)

Equation (25) along with (20) and (21) yield

\[ (\beta - \alpha e \bar{\gamma}^1) \nabla_{y^1} f(x^1, y^1) \left[ \alpha - \alpha e \bar{\lambda} \right] + (\nu - \xi \bar{\gamma}^2) \nabla_{y^2} g(x^2, y^2)\alpha = 0. \]  
(26)

Now multiplying (16) by \( (\beta - \alpha e \bar{\gamma}^1) \) and (17) by \( (\nu - \xi \bar{\gamma}^2) \) and then adding, we get

\[ (\beta - \alpha e \bar{\gamma}^1) \nabla_{y^1} (\alpha - \alpha e \bar{\lambda}) f(x^1, y^1) + (\nu - \xi \bar{\gamma}^2) \nabla_{y^2} (\alpha - \alpha e \bar{\lambda}) g(x^2, y^2) \]

\[ + (\beta - \alpha e \bar{\gamma}^1) \left( \nabla_{y^1} \bar{x} f(x^1, y^1) \right) (\beta - \alpha e \bar{\gamma}^1) \]

\[ + (\nu - \xi \bar{\gamma}^2) \left( \nabla_{y^2} \bar{\lambda} g(x^2, y^2) \right) (\nu - \xi \bar{\gamma}^2) = 0, \]  
(27)

Using (20), (21) and (26) in (27), we obtain

\[ (\beta - \alpha e \bar{\gamma}^1) \left( \nabla_{y^1} \bar{x} f(x^1, y^1) \right) (\beta - \alpha e \bar{\gamma}^1) \]

\[ + (\nu - \xi \bar{\gamma}^2) \left( \nabla_{y^2} \bar{\lambda} g(x^2, y^2) \right) (\nu - \xi \bar{\gamma}^2) = 0 \]  
(28)

which by the hypothesis (a) imply

\[ \beta - \alpha e \bar{\gamma}^1 = 0 \]  
(29)
\[ \nu - \xi \bar{y}^2 = 0. \]  

(30)

Therefore from (18) it follows that \( \eta = 0. \)

From (17) and (30),

\[ \nabla y^2(\alpha - \xi \bar{x})g(\bar{x}, \bar{y}) = 0, \]

which by assumption (b) gives

\[ \alpha - \xi \bar{x} = 0. \]  

(31)

If \( \xi = 0, \) then equations (29), (30) and (31) give \( \beta = 0, \nu = 0 \) and \( \alpha = 0. \) Thus \( (\alpha, \beta, \nu, \xi, \mu, \eta) = 0, \) a contradiction to (24). Hence

\[ \xi > 0. \]  

(32)

From equations (14), (29), (31) and (32),

\[ \nabla x^1 \bar{x}f(\bar{x}^1, \bar{y}^1) = 0. \]  

(33)

Moreover, equations (15), (30), (31) and (32) gives

\[ \nabla x^2 \bar{x}g(\bar{x}^2, \bar{y}^2) = 0. \]  

(34)

The equations (33) and (34) give feasibility of \( (\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{x}) \) for (VD). Also from (33),

\[ \bar{x}^1 \nabla x^1 \bar{x}f(\bar{x}^1, \bar{y}^1) = 0, \]  

(35)

and from (19), (29) and (31),

\[ \bar{y}^1 \nabla y^1 \bar{x}f(\bar{x}^1, \bar{y}^1) = 0. \]  

(36)

Therefore

\[ \bar{x}^1 \nabla x^1 \bar{x}f(\bar{x}^1, \bar{y}^1) = \bar{y}^1 \nabla y^1 \bar{x}f(\bar{x}^1, \bar{y}^1) = 0, \]

i.e, two objectives are equal.

The proof that \( (\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{x}) \) is properly efficient for (VD) follows on the lines of Gulati et al. [6]. \( \square \)

A converse duality theorem may be stated, as its proof would be analogous to that of Theorem 3.2.

**Theorem 3.3** (Converse duality). *Let the hypothesis of Theorem 3.1 be satisfied, and \((\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{x})\) be a properly efficient solution of (VD). Fix \( \lambda = \bar{x} \) in (VP). Assume that*

(a) the Hessian matrices \( \nabla_{u^1 u^1} \bar{x}f(\bar{x}^1, \bar{x}^1) \) and \( \nabla_{u^2 u^2} \bar{x}g(\bar{x}^2, \bar{x}^2) \) are positive definite or negative definite, and

(b) the set \( \{ \nabla_{u^2} g^1(\bar{x}^1, \bar{x}^2), \nabla_{u^2} g^2(\bar{x}^1, \bar{x}^2), \ldots, \nabla_{u^2} g^l(\bar{x}^1, \bar{x}^2) \} \) is linearly independent.

*Then \((\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{x})\) is a properly efficient solution of (VP).*
4. Self Duality

A program is said to be self dual if primal and dual formulations are equivalent. In general (VP) and (VD) are not self dual programs without an added restriction on $f$ and $g$.

The vector functions $f : R^{|J_1|} \times R^{|J_2|} \rightarrow R^l$ and $g : R^{|J_2|} \times R^{|J_2|} \rightarrow R^l$ are said to be skew symmetric if for all $x^i, y^2 \in R^{|J_2|}$ and $x^2, y^2 \in R^{|J_2|}$,
\[
f(y^1, x^1) = -f(x^1, y^1) \quad \text{and} \quad g(y^2, x^2) = -g(x^2, y^2),
\]
i.e.,
\[
f'(y^1, x^1) = -f'(x^1, y^1) \quad \text{and} \quad g'(y^2, x^2) = g'(x^2, y^2), \quad i \in \{1, 2, \cdots, l\}.
\]

The dual problem (VD) can be written as:
\[
(VD_0) \quad \text{Minimize} \quad -f(u^1, v^1) - g(u^2, v^2) + [u^1 \nabla u^1 \lambda f(u^1, v^1)] e.
\]
Subject to
\[
- \nabla u^1 \lambda f(u^1, v^1) \leq 0, \\
- \nabla u^2 \lambda g(u^2, v^2) \leq 0, \\
- u^2 \nabla u^2 \lambda g(u^2, v^2) \geq 0, \\
\lambda > 0, \quad \lambda e = 1.
\]

Since $\nabla u^1 f(u^1, v^1) = -\nabla v^1 f(v^1, u^1)$ and $\nabla u^2 g(u^2, v^2) = -\nabla v^2 g(v^2, u^2)$, the above problem becomes:
\[
(VD_0) \quad \text{Minimize} \quad f(v^1, u^1) + g(v^2, u^2) - [u^1 \nabla v^1 \lambda f(v^1, u^1)] e.
\]
Subject to
\[
\nabla v^1 \lambda f(v^1, u^1) \leq 0, \\
\nabla v^2 \lambda g(v^2, u^2) \leq 0, \\
u^2 \nabla v^2 \lambda g(v^2, u^2) \geq 0, \\
\lambda > 0, \quad \lambda e = 1.
\]

This shows that (VD0) is formally identical to (VP), that is, the objective and constraint functions are identical. Thus the problem (VP) becomes self dual in the spirit of Dorn [5].

It is obvious that the feasibility of $(x^1, x^2, y^1, y^2, \lambda)$ for (VP) implies the feasibility of $(y^1, y^2, x^1, x^2, \lambda)$ for (VD) and vice versa.

We now state the following self duality theorem. Its proof follows on the lines of Mond and Weir [10].

**Theorem 4.1** (Self Duality). If $f$ and $g$ are skew symmetric, then (VP) is a self dual. Also, if (VP) and (VD) are dual programs and $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ is a joint properly efficient solution, then so is $(\bar{y}^1, \bar{y}^2, \bar{x}^1, \bar{x}^2, \bar{\lambda})$ and
\[
H(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}) = 0 = G(\bar{y}^1, \bar{y}^2, \bar{x}^1, \bar{x}^2, \bar{\lambda}).
\]
5. Special Cases

(i) Let \( l = 1 \). Then (VP) and (VD) are reduced to the mixed symmetric dual problems recently studied by Chandra et al. [3], wherein primal and dual problems include the nonnegativity constraints \((x^1, x^2) \geq 0\) and \((v^1, v^2) \geq 0\) respectively.

(ii) If we set \( J_2 = \phi, K_2 = \phi \), then (VP) and (VD) reduces to the Wolfe type symmetric dual programs of Gulati et al. [6]. Similarly for \( J_1 = \phi and K_1 = \phi \), we get the Mond–Weir type symmetric dual programs discussed in [10].

When \( \eta(x, u) = x - u \), then invexity/generalized invexity conditions reduce to the convexity/generalized convexity conditions of Bector et al. [1]. So the duality results of Section 3 improve the work of Bector et al. [1].

It may be noted that the symmetric duality between (VP) and (VD) can be utilized to establish mixed symmetric duality in integer and other related programming problems.

References


Izhar Ahmad
Department of Mathematics
Aligarh Muslim University
Aligarh–202 002
INDIA
iahmad@postmark.net