A KHINTCHINE–GROSHEV TYPE THEOREM IN ABSOLUTE VALUE OVER COMPLEX NUMBERS

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Abstract. In this paper we prove a Khintchine–Groshev type theorem for simultaneously small linear forms, i.e. the linear forms which are close to the origin, over complex number system.

1. Introduction and Preliminaries

The fundamental result, undoubtedly, in Diophantine approximation is Dirichlet’s theorem (1842). A consequence of this result is that, any real number \( \alpha \) can be approximated by infinitely many rationals \( p/q \) with an error of approximation \( 1/q^2 \), see [12, Theorem 36 and Theorem 185]. By using continued fractions, Hurwitz (1895) [3, Theorem V] improved the error of approximation to best possible \( 1/\sqrt{5}q^2 \). To be precise, if the constant \( 1/\sqrt{5} \) is replaced with \( \frac{1}{\sqrt{5}+\epsilon} \) for any \( \epsilon > 0 \) then the numbers equivalent to the golden ratio \( \frac{1+\sqrt{5}}{2} \) can only be approximated by finitely many rationals \( p/q \). If the golden ratio is excluded from the set of real numbers then the error of approximation can further be improved to \( \frac{1}{2\sqrt{2}+\epsilon}q^2 \).

Disregarding real numbers such as those which are equivalent to the roots of the equations \( \alpha^2 + 2\alpha - 1 = 0 \) can improve this constant a bit more. Hermite and Hurwitz, in the 19th century, studied the approximation of complex numbers by Gaussian rationals, 

\[ \mathbb{Q}[i] := \left\{ \frac{p_1 + ip_2}{q_1 + iq_2} \in \mathbb{C} : p_1, p_2, q_1, q_2 \in \mathbb{Z} \right\}, \]

a natural analogue of approximation of real numbers by rationals. Diophantine approximation for complex numbers is more difficult than the Diophantine approximation for real numbers. An example can be seen when studying continued fractions, so simple and effective for real numbers yet are not so straightforward for complex numbers. For instance, as stated above, Hurwitz proof of the best possible error of approximation in Dirichlet’s theorem for real numbers follows easily from the continued fractions. It is still unknown whether the best possible error of approximation in Dirichlet’s theorem for complex numbers can be proved using continued fractions or not. We refer the reader to [4] and references therein for a recent account of the theory of continued fractions over complex numbers. The best possible analogue of Dirichlet’s theorem for complex numbers, however, does

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is satisfied for infinitely many vectors \( q \) where \( \psi \)

exist and it is due to Ford [11]. Ford used simple geometric methods to prove that for any complex irrational number \( z \), there are infinitely many Gaussian rationals \( \frac{p_1 + i p_2}{q_1 + i q_2} \) such that the best possible error of approximation is \( 1/\sqrt{3(q_1^2 + q_2^2)} \).

The main point of this discussion is that the weakening from requiring that all (real or complex) numbers enjoy a certain property to almost all numbers satisfy it is the central theme in metric theory of Diophantine approximation. In this theory, estimating the ‘size’ of such sets in terms of Lebesgue measure, Hausdorff measure and Hausdorff dimension is the main problem. This paper falls within this theory and is about estimating the Lebesgue measure of the set of complex points approximable infinitely often by ratio of Gaussian integers with an error of approximation.

Throughout, by an approximating function \( \psi \) we mean a decreasing function \( \psi : \mathbb{N} \rightarrow \mathbb{R}^+ \) such that \( \psi(n) \rightarrow 0 \) as \( n \rightarrow \infty \). An \( m \times n \) matrix

\[
Z := (z_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \in \mathbb{C}^{m \times n}
\]

is said to be \( \psi \)-approximable if the system of inequalities

\[
|q_1 z_{1j} + q_2 z_{2j} + \cdots + q_m z_{mj}| < \psi(|q|) \quad (1 \leq j \leq n)
\]

is satisfied for infinitely many vectors \( q \in \mathbb{Z}[i] \setminus \{0\} \). Here

\[
|q| = \max\{|[q_1]|, \cdots, |[q_m]|\},
\]

where

\[
|q_k|_2 = \sqrt{|q_{k1}|^2 + |q_{k2}|^2}, \quad q_k = q_{k1} + iq_{k2} \in \mathbb{Z}[i],
\]

and \([x]\) denotes the integer part of the real number \( x \). Throughout, the sum on the left hand side of (1), for 1 \( \leq j \leq n \), will be written more concisely as \( qZ \) or in other words \( qZ \) will denote the vector of the following linear forms

\[
(q_1 z_{11} + \cdots + q_m z_{m1}, \cdots, q_1 z_{1n} + \cdots + q_m z_{mn}).
\]

For convenience, we restrict to the \( mn \)-dimensional unit disc \( D := (\mathbb{C} \cap \Omega)^{mn} \), where \( \Omega = \{a + ib : 0 \leq a, b < 1\} \), instead of considering the full space \( \mathbb{C}^{mn} \). Let \( W_{m,n}(\psi) \) denote the set of \( \psi \)-approximable points in \( D \), i.e.

\[
W_{m,n}(\psi) := \{ Z \in D : |qZ| < \psi(|q|) \} \quad \text{for i.m.} \quad q \in \mathbb{Z}[i] \setminus \{0\}.
\]

Where ‘i. m.’ stands for ‘infinitely many’. It is worth relating the above formulation to the set of \( \psi \)-approximable matrices as is often studied in classical Diophantine approximation. The main focus of research, in the classical settings, has been on studying the metrical properties of the \( \limsup \) set

\[
\Lambda_{m,n}(\psi) := \{ Z \in D : |qZ - p| < \psi(|q|) \} \quad \text{for i.m.} \quad (p, q) \in \mathbb{Z}[i] \times \mathbb{Z}[i] \setminus \{0\},
\]

where \( |qZ - p| \) denotes the distance of \( qZ \) to the nearest Gaussian integer vector \( p \). The main result in this setting is the Khintchine-Groshev theorem which gives an elegant answer to the question of the ‘size’ of the set \( \Lambda_{m,n}(\psi) \). The result links the measure of the set to the convergence or otherwise of a series that depends only on the approximating function. It is clear then that the set \( W_{m,n}(\psi) \) is obtained by fixing \( p = 0 \) in (2). In the real settings, the set is well studied in [14,15] and in weighted settings in [10] and in a mixed settings in [5].

It is readily verified that \( W_{1,1}(\psi) = \{0\} \) as any \( Z = (z_1, z_2, \ldots, z_n) \in W_{1,1}(\psi) \) must satisfy the inequality \( |qz_j| < \psi(|q|) \) infinitely often. Since \( \psi(|q|)/|q| \rightarrow 0 \) as
$|q| \to \infty$, this is only possible if $|z_j| = 0$ for all $j = 1, 2, ..., n$. Thus when $m = 1$ the set $W_{1,n}(\psi)$ is a singleton. We will therefore assume throughout that $m \geq 2$.

**Notation.** To simplify notation in the proofs below the Vinogradov symbols $\ll$ and $\gg$ will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$ we write $a \asymp b$, and say that the quantities $a$ and $b$ are comparable. Throughout the paper, for any set $A$, $\lambda(A)$ denote the Lebesgue measure of the set $A$ and $\Psi(r) = \frac{\psi(r)}{r}$.

Now we are in a position to state the main result of this paper.

**Theorem 1.** Let $\psi$ be an approximating function and $m > n$. Then

$$\lambda(W_{m,n}(\psi)) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} r^{2m-1} \Psi(r)^{2n} < \infty \\ \text{Full} & \text{if } \sum_{r=1}^{\infty} r^{2m-1} \Psi(r)^{2n} = \infty. \end{cases}$$

Note that $\lambda(W_{m,n}(\psi)) = \text{Full}$ means that $\lambda(D \setminus W_{m,n}(\psi)) = 0$. The convergent half of the theorem is independent from the condition that $m > n$ and also free from monotonic assumption on the approximating function. The conditions come into play only for the divergent case.

**Remark 2.** The cases $m \leq n$ are less significant. Indeed, for $m \leq n$, the set $W_{m,n}(\psi)$ is contained in an algebraic variety of dimension strictly lower than the dimension of the ambient space $D$, hence $\lambda(W_{m,n}(\psi)) = 0$ irrespective of the approximating function $\psi$. We refer the reader to [15] for a detailed discussion about this situation. Similar analogue extends to the settings of this paper.

An analogue of Theorem 1 for the conventional set $\Lambda_{m,n}(\psi)$, for $m = n = 1$, was proved by LeVeque [16] in 1952. In 1982, Sullivan [19] used Bianchi groups to prove more general Khintchine theorems for real and complex numbers. For the approximation of complex numbers, the rational approximates were ratios $p/q$ of integers $p, q$ from the imaginary quadratic fields $\mathbb{R}(i \sqrt{d})$, where $d$ is a squarefree natural number. The case $d = 1$ corresponds to the Picard group and approximation by Gaussian rationals. The result was also derived by Beresnevich, Dickinson and Velani as a consequence of ubiquity framework in [1, Theorem 7]. The set $\Lambda_{1,1}(\psi)$ may be considered the half way house of metric Diophantine approximation over quaternions which has been recently studied by Dodson and Everitt [6]. However there are hardly any similarities in the ideas presented in this paper and in their paper.

The critical sum in the above theorem, $\sum_{r=1}^{\infty} r^{2m-1} \Psi(r)^{2n}$, should be compared with the analogous critical sum when considering such linear forms over real numbers. For comparison and completeness we state the corresponding real analogue result here. Let

$$W_0(m, n; \psi) := \{X \in I^mn : |qX| < \psi(|q|) \text{ for i.m. } q \in \mathbb{Z}^m \setminus \{0\} \},$$

where $I = [-1/2, 1/2]$.

It was proved in [15] that,
Theorem 3. Let \( m > n \) and \( \psi \) be an approximating function. Then
\[
\lambda (W_0 (m, n; \psi)) = \begin{cases} 
0 & \text{if } \sum_{r=1}^{\infty} \psi(r)^{m-n-1} < \infty, \\
1 & \text{if } \sum_{r=1}^{\infty} \psi(r)^{m-n-1} = \infty.
\end{cases}
\]

2. Proof of Theorem 1

We start the proof by first expanding upon the structure of the set \( W_{m,n}(\psi) \).

2.1. The structure of \( W_{m,n}(\psi) \).

We start with a family \( C \) of resonant sets \( C_q \) in \( D \) i.e. for non-zero \( q \in \mathbb{Z}^m[i] \), define the sets
\[
C_q := \{ Z \in D : |qZ| = 0 \}
\]
and
\[
C = \{ C_q : q \in \mathbb{Z}^m[i] \setminus \{0\} \}.
\]

To understand the structure of \( W_{m,n}(\psi) \), we explain the first non-trivial case, which is when \( m = 2, n = 1 \). In this case, the set \( C_q \) reduces to
\[
C_q = \{(x_1 + iy_1, x_2 + iy_2) \in D : (q_{11} + iq_{12}, q_{21} + iq_{22}) \cdot (x_1 + iy_1, x_2 + iy_2) = 0\}
\]
\[
= \left\{(x_1 + iy_1, x_2 + iy_2) \in D : q_{11}x_1 + q_{12}x_2 - q_{12}y_1 - q_{22}y_2 = 0 \text{ and } q_{12}x_1 + q_{22}x_2 + q_{11}y_1 + q_{21}y_2 = 0 \right\}.
\]

From this it should be clear that approximating complex numbers by the ratios of Gaussian integers splits up into real and imaginary parts. Hence, for \( m = 2, n = 1 \), the problem essentially reduces to four variables in two linear forms which in turn implies that the resonant sets \( C_q \) are the hyperplanes of dimension 2 and are contained in \( W_{2,1}(\psi) \) for all functions \( \psi \).

Let \( \Psi(|q|) := \psi(|q|)/|q| \). Given an approximating function \( \psi \) and a resonant set \( C_q \), define the \( \Psi \)-neighbourhood of \( C_q \) as
\[
\Delta(C_q, \Psi(|q|)) = \{ Z \in D : \text{dist}(Z, C_q) \leq \Psi(|q|) \}
\]
where \( \text{dist}(Z, C_q) := \inf \{|Z - Y|_2 : Y \in C_q\} \).

Let \( \omega \) be a positive real increasing function such that
- \( \omega(t) \to \infty \) as \( t \to \infty \), and
- there exists a constant \( C > 1 \) so that for \( t \) sufficiently large \( \omega(2t) < C\omega(t) \).

Let
\[
\Delta_{m,n}(\Psi) = \{ Z \in D : Z \in \Delta(C_q, \Psi(|q|)) \text{ for i.m. } q \in \mathbb{Z}^m[i] \setminus \{0\} \}
\]
and for any \( t \in \mathbb{N} \), define
\[
\Delta(\Psi, t) := \bigcup_{q \in J(t)} \Delta(C_q, \Psi(|q|))
\]
(3)

where
\[
J(t) = \left\{ q \in \mathbb{Z}^m[i] \setminus \{0\} : \frac{2^t}{\omega(t)} < |q| < 2^t \right\}.
\]

Then, \( \Delta_{m,n}(\Psi) \) can be written as a lim sup as
\[
\Delta_{m,n}(\Psi) = \bigcap_{N=1}^{\infty} \bigcup_{t=N}^{\infty} \Delta(\Psi, t).
\]
It can be readily verified that
\[ \Delta_{m,n}(\Psi/m) \subset W_{m,n}(\psi) \subset \Delta_{m,n}(\Psi). \]

Returning back to the function \( \omega \) which is chosen in such a way that the convergence and divergence properties of the sums
\[
\sum_{r=1}^{\infty} r^{2m-1} \Psi(r)^{2n} \omega(r)^{-2n} \quad (4)
\]
\[
\sum_{r=1}^{\infty} r^{2m-1} \Psi(r)^{2n} \quad (5)
\]
are the same. That is, the sum (4) converges (or diverges) if and only if the sum (5) converges (or diverges). Clearly if the sum in (4) diverges then so does the sum in (5). On the other hand, suppose that the sum in (5) diverges. Then there exists a strictly increasing sequence of positive integers \( \{ r_i \} \in \mathbb{N} \) such that
\[
\sum_{r_{i-1} \leq r \leq r_i} r^{2m-1} \Psi(r)^{2n} > 1
\]
and \( r_i > 2r_{i-1} \). Let \( \omega \) be the step function defined by \( \omega(r) = i \) for \( r_{i-1} \leq r \leq r_i \). Then \( \omega \) satisfies the required properties.

2.2. The convergent case.

We prove the convergence part of Theorem 1 for the case \( m = 2, n = 1 \). Extending the proof to the higher dimensions is similar with obvious modifications and therefore details are being omitted. The set \( W_{2,1}(\psi) \) can be written using the resonant sets as
\[
W_{2,1}(\psi) = \bigcap_{N=1}^{\infty} \bigcup_{r > N} \bigcup_{|q| = r} \Delta(C_q, \Psi(|q|)).
\]

It follows that
\[
W_{2,1}(\psi) \subset \bigcup_{r > N} \bigcup_{|q| = r} \Delta(C_q, \Psi(|q|)).
\]
In other words \( W_{2,1}(\psi) \) has a natural cover \( \{ \Delta(C_q, \Psi(|q|)) : |q| > N \} \) for each \( N \). It can further be covered by a collection of 4-dimensional closed hypercubes (it should be clear from §2.1 ) with disjoint interior and side length comparable with \( \Psi(|q|) \). The needed number of such hypercubes is clearly \( \ll (\Psi(|q|))^{-2} \). Thus
\[
\lambda(W_{2,1}(\psi)) \leq \sum_{r = N}^{\infty} \sum_{C_q |q| = r} \lambda(\Delta(C_q, \Psi(|q|)))
\]
\[
\ll \sum_{r = N}^{\infty} \sum_{r \leq |q| < r + 1} (\Psi(|q|))^{-2} (\Psi(|q|))^4
\]
\[
= \sum_{r = N}^{\infty} (\Psi(r))^2 \sum_{r \leq |q| < r + 1} 1. \quad (6)
\]
Now it remains to count \( \sum_{r \leq |q| < r + 1} 1 \). We follow an argument from [7, p. 328] or [12, Th. 386] to conclude that \( \sum_{r \leq |q| < r + 1} 1 \ll r^3 \). Thus (6) becomes
\[
\lambda(W_{2,1}(\psi)) \ll \sum_{r=N}^{\infty} r^2 \psi(r)^2
\]

Now since the sum \(\sum_{r=N}^{\infty} r^2 \psi(r)^2 < \infty\), it follows from the Borel–Cantelli lemma that \(\lambda(W_{2,1}(\psi)) = 0\).

2.3. The divergent case. The divergent case is much more difficult and the ideas involved, particularly ubiquity, require some further definitions and notations.

2.3.1. The ubiquitous systems. We begin by stating a simplified version of a more abstract framework developed in [1]. In our case the required measure and intersection conditions in [1] are trivially satisfied.

**Definition 2.1.** Let \(\rho : \mathbb{R}^+ \to \mathbb{R}^+\) be a function such that \(\rho(r) \to 0\) as \(r \to \infty\). Let \(B(Z,r)\) be an arbitrary ball with centre \(Z \in D\) and radius at most \(r_0\). Suppose there exists an absolute constant \(\kappa > 0\) such that

\[
\lambda(B(Z,r) \cap \Delta(\rho,t)) \geq \kappa \lambda(B(Z,r)) \quad \text{for} \quad t \geq t_0(B(Z,r)).
\]

Where \(\Delta(\rho,t)\) is defined in (3). Then \(\mathcal{C}\) is said to be a locally ubiquitous system relative to \(\rho\).

Roughly speaking the definition of local ubiquity says that the set \(\Delta(\rho,t)\) locally approximates the underlying space \(D\) in terms of Lebesgue measure. The actual value of \(\kappa\) in the above definition is irrelevant, only its existence is important. Clearly if \(\lambda(\Delta(\rho,t)) \to 1\) as \(t \to \infty\) then \(\mathcal{R}\) is locally ubiquitous.

The following theorem is a simplified version of Theorem 1 from [2].

**Theorem 4.** Assume that there exists \(0 < l < 1\) such that the function \(\rho\) satisfies \(\rho(2^{t+1}) < lp(2^t)\) for all \(t \in \mathbb{N}\). Suppose that \(\mathcal{C}\) is locally ubiquitous relative to \(\rho\) and \(\psi\) is an approximating function. Then

\[
\lambda(W_{m,n}(\psi)) = \lambda(D) \quad \text{if} \quad \sum_{i=1}^{\infty} \left( \frac{\psi(2^i)}{p(2^i)} \right)^{2n} = \infty.
\]

To establish ubiquity we need two technical lemmas, the first of which is the analogue of Dirichlet’s theorem. A short proof of the complex version of Dirichlet’s theorem is given below. The constant here is not best possible but the result is all that is needed. Here we use Minkowski’s linear forms theorem which we state below for completeness.

**Theorem 5** (Minkowski’s linear forms theorem [12]). Let \(\mathcal{C}\) be an \(n\)-dimensional lattice of determinant \(\det(\mathcal{C})\) and let \(a_{ij}\) \((1 \leq i, j \leq n)\) be real numbers. Suppose that \(c_j > 0\) for \(1 \leq j \leq n\), are numbers such that

\[
c_1 \cdots c_n \geq \det(a_{ij}) \det(\mathcal{C}).
\]

Then there is a non-zero integer point \(u = (u_1, u_2, \cdots, u_n) \in \mathcal{C}\) satisfying

\[
\left| \sum_{j=1}^{n} a_{ij} u_j \right| \leq c_1 \quad \text{and} \quad \left| \sum_{j=1}^{n} a_{ij} u_j \right| \leq c_i \quad (1 < i < n).
\]
Lemma 6. Given any $Z \in D$ and $N \in \mathbb{N}$, there exists a non-zero Gaussian integer $q \in \mathbb{Z}[i]$ with $0 < |q| \leq N$ such that

$$|qZ| < c N^{-m/n+1}. \quad (7)$$

where $c > 0$ is an appropriate constant.

**Proof.** For clarity we prove the theorem for $m = 2, n = 1$. For the higher dimensions, similar arguments will work with obvious modifications. Let $Z = (x_1 + iy_1, x_2 + iy_2)$ and $q = (q_{11} + iq_{12}, q_{21} + iq_{22})$. Then

$$|qZ| = |q_{11}x_1 + q_{21}x_2 - q_{12}y_1 - q_{22}y_2 + i(q_{12}x_1 + q_{22}x_2 + q_{11}y_1 + q_{21}y_2)| < c/N$$

which is the case if

$$\max\{|q_{11}x_1 + q_{21}x_2 - q_{12}y_1 - q_{22}y_2|, |(q_{12}x_1 + q_{22}x_2 + q_{11}y_1 + q_{21}y_2)|\} < c/\sqrt{2}N$$

Consider the convex body

$$B = \{(q_{11}, q_{12}, q_{21}, q_{22}) : \max\{q_{11}^2 + q_{12}^2, q_{21}^2 + q_{22}^2\} \leq N^2, \Delta \leq R^2\}$$

where

$$\Delta = (q_{11}x_1 + q_{21}x_2 - q_{12}y_1 - q_{22}y_2)^2 + (q_{12}x_1 + q_{22}x_2 + q_{11}y_1 + q_{21}y_2)^2.$$ 

Then

$$\lambda(B) = \int_{\max\{q_{11}^2 + q_{12}^2, q_{21}^2 + q_{22}^2\} \leq N^2, \Delta \leq R^2} dq_{11} dq_{12} dq_{21} dq_{22}$$

$$= \pi^2 R^2 N^2 \geq 2^4$$

if $R > \frac{2^2}{\sqrt{2}N}$. Hence, by Minkowski’s theorem, (7) has a non-zero integer solution with $0 < |q| \leq N$. \hfill \Box

Lemma 7. The family $\mathcal{C}$ of resonant sets is locally ubiquitous with respect to the ubiquity function

$$\rho(t) = c \left(2^t\right)^{-\frac{m}{\omega(t)}} \omega(t).$$

**Proof.** Throughout the proof of this lemma we consider those integer vectors which are shown to exist in Lemma 6. We first remove those resonant sets for which the denominators are small i.e. we consider the set $E(t)$ consisting of those points $Z \in D$ such that the denominators of the Gaussian rationals satisfy $1 \leq |q| \leq \frac{2^t}{\omega(t)}$, then

$$E(t) \subseteq \bigcup_{1 \leq r \leq \frac{2^t}{\omega(t)}} \bigcup_{|q|=r} \left\{Z \in D : |qZ| \leq c \left(2^t\right)^{-m/n+1}\right\}.$$
Hence,
\[
\lambda(E(t)) \leq \sum_{1 \leq r \leq \frac{2^i}{\omega(t)}} \sum_{|q| = r} \lambda \left( \left\{ Z \in D : |qZ| \leq c \left( \frac{2^t}{\omega(t)} \right)^{-m/n+1} \right\} \right)
\]
\[
\leq \sum_{1 \leq r \leq \frac{2^i}{\omega(t)}} \sum_{|q| = r} \Delta \left( C_q, \frac{c \left( \frac{2^t}{\omega(t)} \right)^{-m/n+1}}{|q|} \right)
\]
\[
\leq \sum_{1 \leq r \leq \frac{2^i}{\omega(t)}} \sum_{|q| = r} \left( 2^t \right)^{-2m+2n} \leq \omega(t) \left( \frac{2^t}{\omega(t)} \right)^{2m-2n+1}
\]
\[
= \omega(t)^{2n-2m}.
\]
Since, \( \omega \) is a positive real increasing function such that \( \omega(t) \to \infty \) as \( t \to \infty \) and \( m > n \), it follows that, \( \lambda(E(t)) = 0 \).

From now on we consider the large denominators i.e. \( \frac{2^t}{\omega(t)} \leq |q| \leq 2^t \). Let
\[
F(t) = \left\{ Z \in D : \text{dist}(Z, \partial D) \geq 1/2^t \right\} \setminus E(t)
\]
and
\[
\Delta(\rho, t) = \bigcup_{q \in J(t)} \Delta \left( C_q, c \left( \frac{2^t}{\omega(t)} \right)^{-m/n} \omega(t) \right).
\]

We now show that \( F(t) \subseteq \Delta(\rho, t) \). Let \( Z \in F(t) \) and \( q \in \mathbb{Z}^m[i] \setminus \{0\} \) be such that \( \frac{2^t}{\omega(t)} \leq |q| \leq 2^t \) and \( |qZ| < c \left( \frac{2^t}{\omega(t)} \right)^{-m/n+1} \) by Lemma 6. By definition, \( |q| = ||q_i||_2 \) for some \( 1 \leq i \leq m \). Let
\[
\delta_j = -\frac{q \cdot z^{(i)}}{||q_i||_2}, \quad j = 1, 2, ..., n
\]
so that
\[
q \cdot \left( z^{(i)} + \delta_j e^{(i)} \right) = 0,
\]
where \( e^{(i)} \) denotes the \( i \)th basis vector. Also, by Lemma 6, and since \( q \in J(t) \),
\[
|\delta_j| = \left| -\frac{q \cdot z^{(i)}}{||q_i||_2} \right| \leq c \left( \frac{2^t}{\omega(t)} \right)^{-m} \omega(t).
\]
Therefore,
\[
U = \left( z^{(i)} + \delta_j e^{(i)} \right) = \left( z^{(i)} + \delta_1 e^{(i)}, z^{(2)} + \delta_2 e^{(i)}, ..., z^{(n)} + \delta_n e^{(i)} \right)
\]
is a point in \( \mathcal{R}_q \) and
\[
|Z - U| \leq c \left( \frac{2^t}{\omega(t)} \right)^{-m} \omega(t) = \rho(t).
\]
Hence \( Z \in \Delta(\rho, t) \) which implies that
\[
\lambda(D \setminus \Delta(\rho, t)) \to 0 \quad \text{as} \quad t \to \infty.
\]
Hence, $C$ is locally ubiquitous.  

2.3.2. Completing the proof of Theorem 1.

We are now in a position to apply Theorem 4. It is readily verified that $\rho$ is 2-regular. Since, $\psi$ is decreasing, by Cauchy’s condensation argument, it is straightforward to see that

$$\sum_{t=1}^{\infty} (2^t)^{2m-2n} \psi(2^t)^{2n} \omega(t)^{-2n} \asymp \sum_{r=1}^{\infty} r^{2m-1} \psi(r)^{2n} \omega(r)^{-2n} \quad (8)$$

Hence

$$\sum_{t=1}^{\infty} \left( \frac{\psi(2^t)^{2n}}{\rho(2^t)^{2n}} \right)^{2n} \asymp \sum_{t=1}^{\infty} (2^t)^{2m-2n} \psi(2^t)^{2n} \omega(t)^{-2n} \asymp \sum_{r=1}^{\infty} r^{2m-1} \psi(r)^{2n} \omega(r)^{-2n} \asymp \sum_{r=1}^{\infty} r^{2m-1} \psi(r)^{2n} = \infty.$$

This completes the proof of the Theorem 1.

3. An Application

The absolute value theory, in its various particular forms, has many applications in other research disciplines, for example, in Kolmogorov–Arnold–Moser theory [9], biological problems [8] and in inhomogeneous wave equations [13].

Recently, the usage of metric Diophantine approximation theory has emerged in various signal processing applications. This can be seen in [17] where the author (with researchers from Waterloo) obtained Khintchine–Groshev type theorems for a particular form of absolute value setup and subsequently used them in Multiple-Input Multiple-Output (MIMO) X channels. In that paper, $K \times 2$ and $2 \times K$ MIMO X channels with constant channel coefficients available at all transmitters and receivers were considered. A new alignment scheme, named layered interference alignment, in which both vector and real interference alignment are exploited in conjunction with joint processing at receiver sides. Using Khintchine–Groshev type theorems for absolute value linear forms, it was observed that $K \times 2$ and $2 \times K$ X channels with $M$ antennas at all transmitters/receivers enjoy duality in Degrees of Freedom (DoF) $\frac{2K M}{K+1}$.

In another direction, but still within MIMO setup, the convergent case of the Khintchine–Groshev theorem (Theorem 3) was used by Erez–Ordentlich in [18] to prove that a precoded integer-forcing scheme universally achieves the MIMO capacity to within a constant gap. The authors considered an open-loop single-user MIMO communication scheme where a transmitter encodes the data into independent streams all taken from the same linear code. The coded streams are then linearly precoded using the encoding matrix of a perfect linear dispersion space-time code. At the receiver side integer-forcing equalisation is applied. It was shown that this communication architecture achieves the capacity of any Gaussian MIMO channel up to a gap that depends only on the number of transmitting antennas. A
central result [18, Lemma 3] is the estimation of how small the norm $|qX|^2$ can be made as a function of the largest component in the integer-valued vector $q$. The typical behaviour of this minimal norm is the subject of Theorem 3.

The analysis of [18] was over the real number system. One of the reasons for working over the reals was that at the time it was the only setting in which metrical results concerning the absolute value setup existed. With the help of Theorem 1 combined with an analogue of the Banaszczyk theorem [18, Theorem 1] for complex numbers, it is not difficult to establish a more powerful complex numbers version of [18].

The above applications only appeal to the case when there are more variables than linear forms i.e. $m > n$. It is no surprise as far as applications are concerned, for example, in the layered interference alignment scheme the total DoF of the MIMO $X$ channel are zero whenever $m \leq n$. It is also verified in [18, p.10] that the analysis only mattered when $m > n$, otherwise the system has zero DoF and hence contributes nothing.

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References


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