TRIPLE POINT NUMBERS AND QUANDLE COCYCLE INVARIANTS OF KNOTTED SURFACES IN 4–SPACE

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Abstract. The triple point number of a knotted surface in 4–space is the minimal number of triple points for all generic projections into 3–space. We give lower bounds of triple point numbers by using cocycle invariants of knotted surfaces. As an application, we give an infinite family of surface–knots of triple point number six. We also study the triple point numbers restricted to generic projections without branch points.

1. Introduction

The classical knot table has an important role in classical knot theory, which is usually made according to the crossing numbers of classical knots. Making tables of “knotted” surfaces (surfaces in Euclidian 4–space $\mathbb{R}^4$ up to ambient isotopies) has been also attempted in 2–dimensional knot theory; Carter–Saito [4], Yasuda [19], and Yoshikawa [20], for example. In this paper, we investigate the triple point numbers of knotted surfaces, whose definition is analogous to that of the crossing numbers of classical knots. The triple point number of a knotted surface $K$, denoted by $t(K)$, is the minimal number of triple points among all possible generic images of $K$ by a fixed projection of $\mathbb{R}^4$ onto $\mathbb{R}^3$. Here a projection image of $K$ is generic if it has double points, isolated triple points, and isolated branch points as singularities.

A characterization of knotted surfaces of $t(K) = 0$ was studied by Yajima [18] for knotted 2–spheres (see also [9]) and by the second author [16] for knotted tori. On the other hand, every knotted surface satisfies $t(K) \neq 1$ (cf. [12]). In the case of $t(K) > 1$, there are few examples known; for example, the first author determined the triple point numbers of some linked surfaces with non–orientable components in [13]. For orientable knotted surfaces, we have the only result on $t(K) = 4$ as follows, where the $k$–twist-spun knot for a non–negative integer $k$ is a knotted 2–sphere introduced by Zeeman [21].

**Theorem 1.1** ([15]). For each non–negative integer $g$, there is an infinite family of knotted surfaces of genus $g$ whose triple point numbers are equal to four. In particular, the 2–twist-spun trefoil, which is a knotted 2–sphere, has the triple point number four.

The homology and cohomology theory for quandles were introduced by Carter, Jelsovsky, Kamada, Langford, and Saito in [3]. It is a modification of the rack theory due to Fenn, Rourke, and Sanderson [6]. For a finite quandle $Q$ and an abelian group $G$, Carter et al. proved that each 3–cocycle $\theta \in Z^3(Q; G)$ gives

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an invariant of an oriented knotted surface $K$. It is called the cocycle invariant associated with $\theta$, and denoted by $\Phi_\theta(K) \in \mathbb{Z}[G]$, where $\mathbb{Z}[G]$ is the group ring over $G$. In this paper, we will define a non-negative integer $\tau(\theta)$ for each 3-cocycle $\theta$ by solving simultaneous (in) equations, which is independent of $K$. Let $0_G$ denote the zero element of $G$. Then we have the following.

**Theorem 1.4.** If $\Phi_\theta(K) \not\in \mathbb{Z}[0_G] \subset \mathbb{Z}[G]$, then we have $t(K) \geq \tau(\theta)$.

By Theorem 1.2, the cocycle invariants of knotted surfaces can be used to detect their triple point numbers. As an application of Theorem 1.2, we give an infinite family of knotted surfaces of genus $g$ having no branch points simultaneously. As another application of Theorem 1.3, the 3-twist-spun trefoil has the triple point number six.

It is known that every orientable knotted surface $K$ has a generic projection into $\mathbb{R}^3$ without branch points [7]. Hence it is natural to ask whether every orientable knotted surface has a generic projection into $\mathbb{R}^3$ realizing its triple point number and having no branch points simultaneously. As another application of Theorem 1.2, we give a negative answer to this question as follows.

**Theorem 1.4.** For each non-negative integer $g$, there is an infinite family of knotted surfaces of genus $g$ whose triple point numbers are equal to six. In particular, the 3-twist-spun trefoil has the triple point number six.

This paper is organized as follows. In Section 2, we review the definition of cocycle invariants of knotted surfaces. In Section 3, we define $\tau(\theta)$ for each 3-cocycle $\theta$, and prove Theorem 1.2. Section 4 contains several applications (Theorems 1.3 and 1.4). We also give an alternative proof of Theorem 1.1 in this section.

2. Preliminaries

A quandle [8, 10] is a non-empty set $Q$ with a binary operation $*$ satisfying that

(i) $a * a = a$ for any $a \in Q$,

(ii) the map $*a : Q \to Q$ defined by $x \mapsto x * a$ is bijective for each $a \in Q$, and

(iii) $(a * b) * c = (a * c) * (b * c)$ for any $a, b, c \in Q$.

We denote the inverse map of *a in (ii) by $a^{-1}$ formally. The homology and cohomology theory for quandles are developed in [3], that are similar to those of groups. For an abelian group $G$, let $C^n(Q; G)$ be the free abelian group generated by the maps $\theta : Q^n \to G$ satisfying that $\theta(x_1, \ldots , x_n) = 0_G$ if $x_k = x_{k+1}$ for some $k = 1, \ldots , n - 1$. The coboundary map $\delta^n : C^n(Q; G) \to C^{n+1}(Q; G)$ is given by

$$(\delta^n \theta)(x_1, \ldots , x_{n+1}) = \sum_{k=2}^{n+1} (-1)^{k-1} \{ \theta(x_1, \ldots , x_{k-1}, x_{k+1}, \ldots , x_{n+1})$$

$$- \theta(x_1 * x_k, \ldots , x_{k-1} * x_k, x_{k+1}, \ldots , x_{n+1}) \}.$$  

The quandle cohomology group $H^*(Q; G)$ is defined by $\{ C^*(Q; G), \delta^* \}$ in the usual manner, and the cocycle group is denoted by $Z^*(Q; G)$.  


We call a knotted surface in $\mathbb{R}^4$ a surface-knot simply, which is a closed surface smoothly embedded in $\mathbb{R}^4$. Throughout this paper, we always assume that all surface-knots are oriented. To visualize a surface-knot, we use its image by a fixed projection $\pi: \mathbb{R}^4 \to \mathbb{R}^3$ as well as a description of a classical knot in the plane. A (surface) diagram $D$ of a surface-knot $K$ is the image $\pi(K)$ by the projection $\pi$ which has double point curves, isolated triple points, and isolated branch points as singularities. We indicate crossing information at the singularities as follows. Two sheets intersect along a double point curve locally, one of which is under the other relative to the projection direction of $\pi$. The under-sheet is broken in $D$, which consists of connected regions separated by over-sheets. We denote by $\Sigma(D)$ the set of such connected regions of $D$, whose element is called a (broken) sheet. Refer to [5] for more details.

A map $C: \Sigma(D) \to Q$ into a quandle $Q$ is a $Q$-coloring of $D$ if it satisfies the following condition along every double point curve: if $a = C(\alpha_1)$ and $c = C(\alpha_2)$ are the colors of under-sheets $\alpha_1$ and $\alpha_2 \in \Sigma(D)$ separated by the over-sheet $\beta$ colored by $b = C(\beta)$, where the orientation normal of $\beta$ points from $\alpha_1$ to $\alpha_2$, then $a * b = c$ holds. See the left of Figure 1. We denote the set of such $Q$-colorings by $\text{Col}_Q(D)$.

Each triple point $t$ of $D$ has a sign $\varepsilon(t) = \pm 1$ induced from the orientation; specifically, $\varepsilon(t) = +1$ if and only if the ordered triple of the orientation normals of the top, middle, and bottom sheets, respectively, agrees with the orientation of $\mathbb{R}^3$. Given a $Q$-coloring $C \in \text{Col}_Q(D)$, the colors of the sheets near a triple point $t$ are characterized by three colors $a = C(\alpha), b = C(\beta)$ and $c = C(\gamma)$, where $\gamma$ is the top sheet, $\beta$ is the middle sheet from which the orientation normal of $\gamma$ points, and $\alpha$ is the bottom sheet from which the orientation normals of $\beta$ and $\gamma$ point. The ordered triple $(a, b, c)$ is called the color of $t$, and denoted by $C(t) \in Q^3$. See the right of Figure 1, where the sheets $\alpha, \beta,$ and $\gamma$ are shaded.

Assume that $Q$ is a finite quandle. Given a 3-cocycle $\theta \in Z^3(Q; G)$, we define the (Boltzmann) weight at $t$ by $W_\theta(t; C) = \theta(a, b, c)^{\varepsilon(t)} \in G$ according to $C \in \text{Col}_Q(D)$, where $C(t) = (a, b, c)$ and $G$ is written multiplicatively. Then the cocycle invariant of $K$ associated with $\theta$ is the state-sum

$$
\Phi_\theta(K) = \sum_{C \in \text{Col}_Q(D)} \prod_{t \in T(D)} W_\theta(t; C) \in \mathbb{Z}[G]
$$
valued in the group–ring \(\mathbb{Z}[G]\), where \(T(D)\) denotes the set of triple points of \(D\). It is proved in [3] to be an invariant of \(K\) which does not depend on the choice of a diagram \(D\) of \(K\).

3. Lower Bounds for Triple Point Numbers

Let \(Q\) be a finite quandle, and \(\theta \in Z^3(Q;G)\) a 3–cocycle with a coefficient group \(G\) written additively. For variables \(X = (X_{pqr}^i)^{i=\pm 1}_{p\neq q\neq r\in Q}\), we consider the set of linear functions \(\{F_\theta(X), G_{ab}^j(X)\}^{j=\pm 1}_{a\neq b\in Q}\) given by

\[
\begin{align*}
F_\theta(X) &= \sum_{k,p\neq q\neq r} X_{pqr}^k \cdot \theta(p,q,r), \\
G_{ab}^j(X) &= \sum_{x \neq a} X_{xab}^j - \sum_{x \neq a,b} X_{axb}^j + \sum_{x \neq b} X_{abx}^j \\
&\quad - \sum_{x \neq a} X_{ab}^{-j} x - \sum_{x \neq a,b} X_{a x^{-1}, x,b}^{-j} - \sum_{x \neq b} X_{a x^{-1}, b x^{-1}, x}^{-j}.
\end{align*}
\]

**Definition 3.1.** If the simultaneous (in)equations

\[
\{F_\theta(X) \neq 0, G_{ab}^j(X) = 0\}^{j=\pm 1}_{a\neq b\in Q}
\]

have an integral solution \(A = (A_{pqr}^i)^{i=\pm 1}_{p\neq q\neq r\in Q}\), we define \(\tau(\theta)\) to be the minimal number of

\[
\sum_{i=\pm 1, p\neq q\neq r\in Q} |A_{pqr}^i|
\]

for all integral solutions \(A\). Otherwise, we put \(\tau(\theta) = 0\).

For a diagram \(D\) of a surface–knot, let \(\Lambda(D)\) denote the set of regions of \(\mathbb{R}^3\) separated by the underlying generic surface of \(D\). A map \(\lambda : \Lambda(D) \rightarrow \{\pm 1\}\) is a checkerboard coloring if it satisfies the following condition: if \(R_1\) and \(R_2\) \(\in \Lambda(D)\) are regions of \(\mathbb{R}^3\) separated by a sheet of \(D\), then \(\lambda(R_1) = -\lambda(R_2)\). For any diagram \(D\), there exist such checkerboard colorings in two ways, and we fix one of them such that the unbounded region \(R_0 \in \Lambda(D)\) satisfies \(\lambda(R_0) = 0\).

The double point curves of \(D\) are separated by triple points into several arcs and circles. Such an arc is called an edge of \(D\), and we denote the set of edges by \(E(D)\). Note that the boundary points of an edge are triple points or branch points. For each edge \(e \in E(D)\), define the orientation tangent \(\vec{v}(e)\) such that the ordered triple of \(\vec{v}(e)\) and the orientation normals of the over– and under–sheets agrees with the orientation of \(\mathbb{R}^3\). The label of \(e\), denoted by \(\lambda(e) \in \mathbb{Z}_2\), is defined to be \(\lambda(R)\), where \(R \in \Lambda\) is the one of four regions near \(e\) from which the orientation normals of the over– and under–sheets point.

Along an edge \(e\), let \(a\) and \(b\) be the under– and over–sheets respectively, where the orientation normal of \(\beta\) points from \(\alpha\). When \(D\) is colored by \(C \in \text{Col}_Q(D)\), the ordered pair \((C(\alpha), C(\beta))\) is called color of \(e\) and denoted by \(C(e) \in Q^2\). Note that if one of the boundary points of \(e\) is a branch point, then \(C(e) = (a,a)\) for some \(a \in Q\). For each \(j \in \{\pm 1\}\) and \(a,b \in Q\), we denote by \(E_{ab}^j(D)\) the subset of \(E(D)\) consisting of edges \(e\) with \(\lambda(e) = j\) and \(C(e) = (a,b)\).
For a triple point \( t \in T(D) \), we define the label \( \lambda(t) \in \mathbb{Z}_2 \) to be \( \lambda(R) \), where \( R \in \Lambda(D) \) is the one of eight regions near \( t \) from which the orientation normals of the top, middle, and bottom sheets point. Note that \( t \) has also a sign \( \varepsilon(t) \) and a color \( C(t) \in Q^3 \) (if \( D \) is colored by \( C \in \text{Col}_Q(D) \)) as in Section 2. For each \( i, s \in \{\pm 1\} \), and \( p, q, r \in Q \), we denote by \( T_{pqrs}^i(D) \) the subset of \( T(D) \) consisting of triple points \( t \) with \( \lambda(t) = i \), \( \varepsilon(t) = s \) and \( C(t) = (p, q, r) \), and by \( A_{pqrs}^i \) the number of triple points in \( T_{pqrs}^i(D) \). Moreover, put \( A_{pqrs}^i = A_{pqrs}^{i+1} - A_{pqrs}^{i-1} \) and \( A_C = (A_{pqrs}^i)_{p \neq q \neq r \in Q} \).

Then we have the following.

**Lemma 3.2.** \( F_0(A_C) = \sum_{t \in T(D)} W_0(t; C) \).

**Proof.** The lemma follows from the definition of Boltzmann weights. \( \square \)

**Lemma 3.3.** \( G_{ab}^j(A_C) = 0 \) for any \( j = \pm 1 \) and \( a \neq b \in Q \).

**Proof.** Since \( a \neq b \in Q \), the boundary points of each edge \( e \in E_{ab}^j(D) \) are triple points. Instead of counting the number of edges in \( E_{ab}^j(D) \), we consider the terminal triple points according to the orientations of the edges, so that the number is given by

\[
\sum_{x \in Q} A_{xab}^{j+1} + \sum_{x \in Q} A_{xab}^{j-1} + \sum_{x \in Q} A_{xab}^j + \sum_{x \in Q} A_{xab}^{-j}
\]

This is obtained as follows. Assume that an edge \( e \in E_{ab}^j(D) \) has the terminal triple point \( t \in T_{pqrs}^i(D) \) according to the orientation \( \bar{v}(e) \) of \( e \). If \( s = +1 = \varepsilon(t) \), then \( e \) must be coincident with \( e_1, e_3, \) or \( e_5 \) as shown in Figure 2. For example, if \( e = e_1 \), then \( i = j \) and \( (q, r) = (a, b) \). Hence, we have \( \lambda(t) = j \) and \( C(t) = (x, a, b) \) for some \( x \in Q \), and the number of such edges are given by the first sum as above. The numbers of \( e = e_1 \) and \( e_5 \) are similarly given by the third and fifth sums, respectively. On the other hand, if \( s = -1 = \varepsilon(t) \), then the labels and colors of six edges incident to \( t \) are the same as in Figure 2, and the orientations are opposite. Hence, \( e \) must be coincident with \( e_2, e_4, \) or \( e_6 \). If \( e = e_6 \), then \( i = -j \) and \( (p, q, r) = (a, b) \). Hence, we have \( \lambda(t) = -j \) and \( C(t) = (a \ast x^{-1}, b \ast x^{-1}, x) \) for some \( x \in Q \), and the number of such edges are given by the sixth sum. The numbers of \( e = e_2 \) and \( e_4 \) are similarly given by the second and fourth sums, respectively.

Next, by counting the initial triple points of the edges, the number of the edges in \( E_{ab}^j(D) \) is given by

\[
\sum_{x \in Q} A_{xab}^{-j-1} + \sum_{x \in Q} A_{xab}^{-j} + \sum_{x \in Q} A_{xab}^{-j+1} + \sum_{x \in Q} A_{xab}^{-j}
\]

Since these numbers are equal, we have \( G_{ab}^j(A_C) = 0 \), where we cancel two \( X_{aab}^j \)'s in the first and second sums, two \( X_{aab}^j \)'s in the second and third sums, and etc. \( \square \)
Proof of Theorem 1.2. We take a diagram $D$ of $K$ realizing the triple point number $t(K)$. By assumption, there is a $Q$–coloring $C \in \text{Col}_Q(D)$ satisfying that $\sum_{t \in \mathcal{T}(D)} W_{\theta}(t; D) \neq 0_G$. By Lemmas 3.2 and 3.3, $A_C$ is an integral solution of \begin{align*} \left\{ F_{\theta}(X) &\neq 0_G, \quad \frac{G_{a,b}(X)}{a \neq b \in Q} = 0 \right\}_{a \neq b \in Q}, \end{align*} and hence we have $t(K) = \sum_{i,s = \pm 1, p,q,r \in Q} |A_{pqr}^{i,s}| \geq \sum_{i = \pm 1, p \neq q \neq r \in Q} |A_{pqr}^i| \geq \tau(\theta)$. □

4. Applications

In [3], Carter at el. calculated the cohomology groups for various quandles. In particular, they proved that $H^3(S_4; \mathbb{Z}_2) \cong \mathbb{Z}_2$ for a certain quandle $S_4$ with four elements, and gave a 3–cocycle $\theta_1 \in Z^3(S_4; \mathbb{Z}_2)$ representing the generator of $H^3(S_4; \mathbb{Z}_2)$. They also proved that the 3–twist-spun trefoil $K$ has the cocycle invariant $\Phi_{\theta_1}(K) = 4(1 + 3t)$ valued in the group ring $\mathbb{Z}[t^\pm 1]/(t^2 - 1)$ under $\mathbb{Z}_2 \cong \langle t \mid t^2 = 1 \rangle$ (see [15] also).

Proof of Theorem 1.3. For non–negative integers $n$ and $g$, let $K_g^n$ be the surface–knot of genus $g$ which is obtained from the connected sum of the 3–twist–spun trefoil and $n$ 0–twist–spun trefoils by attaching $g$ trivial 1–handles. It is easy to see that $\Phi_{\theta_1}(K_g^n) = 4^n \Phi_{\theta_1}(K_0^n) = 4^n(1 + 3t) / (t \in \mathbb{Z})$. Hence, we have $K_g^n \neq K_g^m$ if $n \neq m$.

By means of computer calculation, we can check that if $\sum |A_{pqr}^i| < 6$, then $A = (A_{pqr}^i)$ is not a solution of the simultaneous (in)equations in Definition 3.1. This implies that $\tau(\theta_1) \geq 6$, and it follows by Theorem 1.2 that $t(K_g^n) \geq 6$. On the other hand, we have $t(K_g^n) \leq t(K_0^n) \leq 6$ (cf. [15]). Thus it holds that $t(K_g^n) = 6$ for any $n$ and $g$. □

It is known that every (oriented) surface–knot $K$ has a diagram without branch points (cf. [7]). We define a non–negative integer $t_0(K)$ to be the minimal number of triple points for all diagrams of $K$ without branch points. By definition, we have $t(K) \leq t_0(K)$, and it holds that $t(K) = t_0(K)$ if and only if $K$ has a diagram realizing its triple point number $t(K)$ and having no branch points simultaneously.
The second author [17] proves that for any sphere–knot $K$, we have $t_0(K) \neq 2$. Note that $t_0(K)$ is always even by Banchoff’s congruence [2].

Let $Q$ be a finite quandle, and $\theta \in Z^3(Q; G)$ a 3–cocycle. For variables $\bar{X} = (X_{pqr}^i)^{i=\pm 1}_{p,q,r \in Q}$ extending to $p = q$ or $q = r$ in $X$, consider the set of linear functions $\{F_\theta(\bar{X}), G_{ab}^j(\bar{X})\}^{i=\pm 1}_{a,b \in Q}$ extending to $a = b$ in $G_{ab}^j$, given in Section 3. If the simultaneous (in)equations

$$\{F_\theta(\bar{X}) \neq 0, G_{ab}^i(\bar{X}) = 0\}^{i=\pm 1}_{a,b \in Q}$$

have an integral solution $\bar{A} = (A_{pqr}^i)^{i=\pm 1}_{p,q,r \in Q}$, we define $\tau_0(\theta)$ to be the minimal number of $\sum_{p,q,r \in Q} |A_{pqr}^i|$ for all integral solutions $\bar{A}$, and otherwise we put $\tau_0(\theta) = 0$. Then we have the following.

**Proposition 4.1.** If $\Phi_\theta(K) \notin \mathbb{Z}[G] \subset \mathbb{Z}[G]$, then we have $t_0(K) \geq \tau_0(K)$.

**Proof.** Let $D$ be a diagram of $K$ without branch points realizing $t_0(K)$. Put $A_C = (A_{pqr}^i)^{i=\pm 1}_{p,q,r \in Q}$ (the definition of $A_{pqr}^i$ is given in Section 3). The proof is very similar to those of Lemmas 3.2, 3.3, and Theorem 1.2. The only difference is that since $D$ has no branch points, the boundary points of any edge in $E_{ab}(D)$ are triple points. Hence, by counting the number of edges in $E_{ab}(D)$, we obtain $G_{ab}^i(A_C) = 0$ for any $a \in Q$. □

It is also proved in [3] that $H^3(R_3; \mathbb{Z}_3) \cong \mathbb{Z}_3$ for a certain quandle $R_3$ with three elements (cf. [11]). By using a 3–cocycle $\theta_2 \in Z^3(R_3; \mathbb{Z}_3)$ presenting a generator of $H^3(R_3; \mathbb{Z}_3)$, we see that the 2–twist–spun trefoil $L$ has the cocycle invariant $\Phi_{\theta_2}(L) = 3(1 + 2t)$ valued in $\mathbb{Z}[\mathbb{Z}_3] \cong \mathbb{Z}[t^\pm 1]/(t^3 - 1)$ under $\mathbb{Z}_3 \cong \langle t^3 \rangle = 1$ (see also [1, 14, 15]).

**Proofs of Theorems 1.1 and 1.4.** For non-negative integers $n$ and $g$, let $L_n^g$ be the surface–knot of genus $g$ which is obtained from the connected sum of the 2–twist–spun trefoil and $n$ 0–twist–spun trefoils by attaching $g$ trivial 1–handles. It is easy to see that $\Phi_{\theta_2}(L_n^g) = 3n + 1(1 + 2t) \notin \mathbb{Z}$. Hence, we have $L_n^g \neq L_n^m$ if $n \neq m$.

By means of computer calculation, we can check that if $\sum |A_{pqr}^i| < 4$, then $A = (A_{pqr}^i)$ is not a solution of the simultaneous (in)equations in Definition 3.1. This implies that $\tau(\theta_2) \geq 4$, and it follows by Theorem 1.2 that $t(L_n^g) \geq 4$. On the other hand, we have $t(L_n^g) \leq 4$ (cf. [15]). Thus it holds that $t(L_n^4) = 4$.

Similarly, we have $\tau_0(\theta_2) \geq 5$ by computer calculation, and $t_0(L_n^g) \geq \tau_0(\theta_2) \geq 5$ by Proposition 4.1. Hence, it holds that $t(L_n^g) < t_0(L_n^g)$ for any $n$ and $g$. □

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