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INJECTIVE AND PROJECTIVE MODULES RELATIVE TO A TORSION THEORY

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Abstract. Injective and projective modules are studied relative to a hereditary torsion theory τ on Mod–R. It is shown that every τ–projective module is projective if and only if the torsion ideal of R is a semisimple direct summand of R. Hereditary torsion theories are determined for which every τ–injective module is injective and several conditions are given that characterize right hereditary rings. We also show that the right global τ–projective dimension of R is equal to the right global projective dimension of R modulo its torsion ideal.

The purpose of this paper is to investigate injective and projective modules relative to a hereditary torsion theory τ on Mod–R. In Section 1 we study conditions under which τ–projective modules are projective while Section 2 is devoted to studying similar conditions for τ–injective modules. In Section 3 we look at right hereditary rings in a torsion theoretical setting and in Section 4 the τ–projective dimension of a module and the right global τ–projective dimension of a ring are considered.

Throughout, R will denote an associative ring with identity 1 and R–module or module will mean unitary right R–module. If $\tau = (\mathsf{T},\mathsf{F})$ is a hereditary torsion theory on Mod–R, then $\mathcal{F}_\tau = \{I \mid I$ is a right ideal of R and $R/I \in \mathsf{T}\}$ will denote the (Gabriel) filter [**6**] of right ideals of R uniquely determined by τ. Modules in T will be called *τ–torsion* and those in F are said to be *τ–torsion free*. An element m of a module M is a *τ–torsion element* of M if there is an $I \in \mathcal{F}_\tau$ such that $mI = 0$. The set $t_\tau(M) = \{m \in M \mid mR$ is τ–torsion$\}$ is a submodule of M and $t_\tau(R)$ is an ideal of R called the *τ–torsion submodule* and the *τ–torsion ideal*, respectively. Moreover, $M \in \mathsf{T}$ if and only if $t_\tau(M) = M$ and $M \in \mathsf{F}$ if and only if $t_\tau(M) = 0$. A submodule N of M is said to be *τ–dense* in M if M/N is τ–torsion and *τ–pure* in M if M/N is τ–torsion free. A class $\mathcal{C}$ of modules is *closed under extensions* if whenever $0 \to L \to M \to N \to 0$ is exact and $L, N \in \mathcal{C}$, then $M \in \mathcal{C}$. If $\tau = (\mathsf{T},\mathsf{F})$ is a hereditary torsion theory on Mod–R, then T is closed under submodules, homomorphic images, arbitrary direct sums and extensions while F is closed under submodules, arbitrary direct products, extensions and injective hulls.

In the following, τ will denote an arbitrary hereditary torsion theory on Mod–R, the category of unitary right R–modules. Standard results on torsion theory can be found in [**3**], [**7**] and [**15**] while general information on rings and modules can be found in [**1**] and [**12**].

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1. Relative Projective Modules

An R–module M is *τ–projective* if for each exact sequence $0 \to L \to X \to N \to 0$ of R–modules, where L is τ–torsion free, $\mathrm{Hom}_R(M, X) \to \mathrm{Hom}_R(M, N) \to 0$ is exact. This definition of a τ–projective module agrees with that given in [**3**], but such a module is called codivisible in [**2**]. Note that if M is a τ–projective R–module and N is a τ–torsion submodule of M, then M/N is τ–projective. A proof of this fact can be found in [**2**].

The following result from [**13**] will be used several times in what is to follow.

Theorem 1.1. *An R–module M is τ–projective if and only if $M/Mt_\tau(R)$ is a projective $R/t_\tau(R)$–module.*

The class of projective modules is closed under extensions. The following theorem shows that this is also true for the class of τ–projective modules.

Theorem 1.2. *If N is a submodule of M and N and M/N are τ–projective, then M is τ–projective.*

Proof. The short exact sequence

$$0 \to \big(N + Mt_\tau(R)\big)/Mt_\tau(R) \to M/Mt_\tau(R) \to (M/N)/\big((M/N)t_\tau(R)\big) \to 0$$

splits since $(M/N)/\big((M/N)t_\tau(R)\big)$ is, by Theorem 1.1, a projective $R/t_\tau(R)$–module. Now $R/t_\tau(R)$ is a τ–torsion free R–module, so any projective $R/t_\tau(R)$–module is a τ–torsion free R–module. Using Theorem 1.1 again, we see that $N/Nt_\tau(R)$ is a projective $R/t_\tau(R)$–module, so $N/Nt_\tau(R)$ is τ–torsion free. But $\big(N \cap Mt_\tau(R)\big)/Nt_\tau(R)$ is a τ–torsion submodule of $N/Nt_\tau(R)$, so $N \cap Mt_\tau(R) = Nt_\tau(R)$. Hence $\big(N + Mt_\tau(R)\big)/Mt_\tau(R) \approxeq N/\big(N \cap Mt_\tau(R)\big) = N/Nt_\tau(R)$ and so $M/Mt_\tau(R) \approxeq N/Nt_\tau(R) \oplus (M/N)/\big((M/N)t_\tau(R)\big)$. Therefore $M/Mt_\tau(R)$ is a projective $R/t_\tau(R)$–module, so in view of Theorem 1.1, M is a τ–projective R–module. □

We are interested in conditions under which the class of τ–projective modules coincides with the class of projective modules. We now give two equivalent conditions each of which is necessary and sufficient for this to hold.

Theorem 1.3. *The following are equivalent for a torsion theory τ on* Mod–R.

(a) *Every τ–projective R–module is projective.*

(b) *For every τ–projective R–module M, $t_\tau(M)$ is a semisimple direct summand of M.*

(c) *The ideal $t_\tau(R) = eR$ for some idempotent $e \in R$ and $t_\tau(R)$ is a semisimple R–module.*

Proof. (a) $\Longrightarrow$ (b) If M is a τ–projective R–module, then $M/t_\tau(M)$ is τ–projective and therefore projective. Hence, the exact sequence $0 \to t_\tau(M) \to M \to M/t_\tau(M) \to 0$ splits. Consequently, $t_\tau(M)$ is a direct summand of M and since M is projective, $t_\tau(M)$ is also projective. If N is a submodule of $t_\tau(M)$, $t_\tau(M)/N$ is τ–projective and hence projective. Thus $0 \to N \to t_\tau(M) \to t_\tau(M)/N \to 0$ splits, so $t_\tau(M)$ is semisimple.

(b) $\Longrightarrow$ (c) is clear, so let's show (c) $\Longrightarrow$ (a). Suppose that (c) holds and let e be an idempotent element of R such that $t_\tau(R) = eR$. If M is a τ–projective

R–module, then $M/Mt_\tau(R)$ is a projective $R/t_\tau(R)$–module, so $M/Mt_\tau(R)$ is a direct summand of a free $R/t_\tau(R)$–module. It follows that $M/Mt_\tau(R)$ is a direct summand of a free R–module, so $M/Mt_\tau(R)$ is a projective R–module. Therefore, $0 \to Mt_\tau(R) \to M \to M/Mt_\tau(R) \to 0$ splits and so $M = Mt_\tau(R) \oplus N$ for some projective submodule N of M. Next, note that $Mt_\tau(R)$ is a homomorphic image of a direct sum of copies of $t_\tau(R)$. Since $t_\tau(R)$ is semisimple, there is a family $\{S_i\}_{i\in I}$ of simple and necessarily projective submodules of $t_\tau(R)$ such that $t_\tau(R) = \oplus_{i\in i} S_i$. This leads to $Mt_\tau(R) = \oplus_{j\in J} U_j$ where for each $j \in J$ there is an $i \in I$ such that $U_j \approxeq S_i$. Hence, $Mt_\tau(R)$ is a projective R–module, so M is projective. □

The set of torsion theories on Mod–R can be partially ordered. We write $\sigma \le \tau$ if $\mathsf{T}_\sigma \subseteq \mathsf{T}_\tau$ or equivalently if $\mathsf{F}_\tau \subseteq \mathsf{F}_\sigma$. If τ_0 denotes the torsion theory in which every module is considered to be torsion free and τ_1 is the torsion theory in which every module is viewed as being torsion, then $\tau_0 \le \tau \le \tau_1$ for any torsion theory τ on Mod–R. An interesting and often studied torsion theory on Mod–R is the Goldie torsion theory [**8**]. This torsion theory is the smallest torsion theory τ_G on Mod–R such that $\mathcal{F}_{\tau_G}$ contains all the essential right ideals of R. A module is τ_G–torsion free if and only if $Z(M) = 0$ where $Z(M)$ is the singular submodule [10] of M. $Z(R_R)$ is an ideal of R and a τ_G–torsion free module is said to be *nonsingular*.

A torsion theory τ is said to be *faithful* if $t_\tau(R) = 0$. For such a torsion theory τ on Mod-R, Theorem 1.1 shows that a module is τ–projective if and only if it is projective. The following corollary to Theorem 1.3 indicates that the converse holds for the Goldie torsion theory.

Corollary 1.4. *R is a right nonsingular ring if and only if every τ_G-projective R–module is projective.*

Proof. If $Z(R_R) = 0$, then R is τ_G–torsion free, so the result follows immediately from Theorem 1.1. For the converse, suppose that every τ_G–projective R–module is projective and let $r \in t_{\tau_G}(R)$. Then, because of Theorem 1.3, there is an idempotent $e \in t_{\tau_G}(R)$ such that $rR = eR$. If I is an essential right ideal I of R such that $eI = 0$, then $I \subseteq (1 - e)R$, so $eR \cap I = 0$. Hence $eR = 0$ and, consequently, $r = 0$. Therefore $Z(R_R) = 0$. □

An R–module M is said to be *semi-artinian* if $\mathrm{Soc}_R(M/N) \ne 0$ for each proper submodule N of M and R is a *right semi–artinian ring* if R is semi–artinian as a right R–module. One can prove that R is right semi–artinian if and only if every R–module is semi–artinian [**15**, VIII, 2.5]. If $\mathsf{T} = \{M \mid M \text{ is semi–artinian}\}$ and $\mathsf{F} = \{M \mid \mathrm{Soc}_R(M) = 0\}$, then $\tau_S = (\mathsf{T}, \mathsf{F})$ is a torsion theory on Mod–R. For this torsion theory, $t_{\tau_S}(M)$ is the largest and necessarily unique semi–artinian submodule of M. Theorem 1.3 takes a particularly nice form for semi–artinian torsion theories.

Corollary 1.5. *The following are equivalent for the semi–artinian torsion theory τ_S on* Mod–R.

(a) *Every τ_S–projective R–module is projective.*

(b) $\mathrm{Soc}_R(M)$ *is a direct summand of M for every τ_S–projective R–module M.*

(c) $\mathrm{Soc}_R(R) = eR$ *for some idempotent $e \in R$.*

Proof. (a) $\Longrightarrow$ (b). If every τ_S–projective R–module is projective, Theorem 1.3 indicates that $t_{\tau_S}(M)$ is a semisimple direct summand of M. But $\mathrm{Soc}_R(M) \subseteq t_{\tau_S}(M)$, so it follows that $\mathrm{Soc}_R(M) = t_{\tau_S}(M)$.

(b) $\Longrightarrow$ (c) is clear, so let's show (c) $\Longrightarrow$ (a). In light of Theorem 1.3, it suffices to show that $\mathrm{Soc}_R(R) = t_{\tau_S}(R)$. Now $t_{\tau_S}(R)$ is the unique, largest semi–artinian submodule of R and $\mathrm{Soc}_R(R)$ is a semi–artinian submodule of R, so $\mathrm{Soc}_R(R) \subseteq t_{\tau_S}(R)$. Hence it must be the case that $\mathrm{Soc}_R(R) = \mathrm{Soc}_R\big(t_{\tau_S}(R)\big)$. But any semi–artinian module is an essential extension of its socle [**15**, VIII, §2], so if $\mathrm{Soc}_R(R)$ is a direct summand of R, then $\mathrm{Soc}_R(R) = t_{\tau_S}(R)$. □

A somewhat "stronger" condition on the class of τ–projective modules is for it to coincide with the class of all R–modules. Proposition 4.2.2 in [**3**] shows that every R–module is τ–projective if and only if $R/t_\tau(R)$ is a semisimple ring. We conclude this section with an easily proved and similar result for right semi–artinian rings.

Theorem 1.6. *Every right R–module is τ_S–projective if and only if R is right semi–artinian.*

2. Relative Injective Modules

If R is a ring and τ is a torsion theory on Mod–R, then an R–module M is said to be *τ–injective* if for each short exact sequence $0 \to L \to X \to N \to 0$ of R–modules, where N is τ–torsion, $\mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(L, M) \to 0$ is exact.

It was pointed out in Section 1 that factor modules of τ–projective modules by τ–torsion submodules are τ–projective. A dual result holds for τ–injective modules: τ–pure submodules of τ–injective modules are τ–injective. This follows since if M is a τ–pure submodule of a τ–injective module E, then $E_\tau = \{x \in E \mid x + M \in t_\tau(E/M)\}$ is a τ–injective submodule of E containing M, E_τ/M is τ–torsion and E/M is τ–torsion free. Thus $E_\tau/M \subseteq E/M$ shows that $M = E_\tau$.

If $E(M)$ is the injective envelope [**5**] of M and we set $E = E(M)$ and $E_\tau = E_\tau(M)$, then $E_\tau(M)$ is the τ–injective envelope [**3**], [**7**] of M. In this case, $E_\tau(M)$ is a τ–dense essential extension of M that is unique up to isomorphism.

We saw in Theorem 1.3 that every τ–projective R–module is projective if and only if $t_\tau(R)$ is a semisimple direct summand of R. We now characterize the torsion theories for which every τ–injective R–module is injective.

Theorem 2.1. *If τ_G is the Goldie torsion theory on* Mod–R*, then for any torsion theory τ, every τ–injective R–module is injective if and only if $\tau_G \leq \tau$.*

Proof. Let $\tau_G \leq \tau$, and suppose that M is τ–injective. If I is a right ideal of R and $f : I \to M$ is an R–linear mapping, let K be a complement of I in R. Then $I \oplus K$ is an essential right ideal of R and so is in $\mathcal{F}_{\tau_G} \subseteq \mathcal{F}_\tau$. Extend f to $g : I \oplus K \to R$ by $g(r + k) = f(r)$, $r \in I$, $k \in K$, and then extend g to $h : R \to M$ by using the τ–injectivity of M. If $h(1) = m$, then $f(r) = mr$ for all $r \in I$, so Baer's Criterion [**10**], [**15**] shows that M is an injective R–module.

Conversely, suppose that every τ–injective module is injective and let M be a τ–torsion free R–module. If I is an essential right ideal of R and $i : I \to R$ and $j : I \to E(I)$ are canonical injections, then there is an injective R–linear mapping $g : R \to E(I)$ such that $g \circ i = j$. If R is identified with its image in $E(I)$, then we have $R/I \subseteq E(I)/I$. Since τ–injectives are injective, $E_\tau(I) = E(I)$, so

$E(I)/I$ is τ–torsion and therefore R/I is τ–torsion. But M is τ–torsion free, so $\mathrm{Hom}_R(R/I, M) = 0$. Next, suppose that m is a nonzero element of M such that $mI = 0$. If $f : R/I \to M$ is such that $r + I \longmapsto mr$, then $0 \neq f \in \mathrm{Hom}_R(R/I, M)$. Hence it must be the case that $mI \neq 0$ for all $m \in M$, $m \neq 0$. Consequently, the singular submodule of M is zero, so M is τ_G–torsion free. Therefore the τ–torsion free class of τ is contained in the τ_G–torsion free class of τ_G, so $\tau_G \leq \tau$. □

Corollary 2.2. *The following are equivalent for the semi-artinian torsion theory* τ_S *on* Mod–R.

(a) *Every* τ_S*–injective* R*–module is injective.*

(b) *The* R*–module* R/I *has nonzero socle for every proper essential right ideal* I *of* R.

Proof. (a) $\Longrightarrow$ (b) Since every τ_S–injective R–module is injective, $\tau_G \leq \tau_S$ and so $\mathcal{F}_{\tau_S}$ contains every essential right ideal of R. Thus if I is a proper essential right ideal of R, then R/I is semi–artinian and so $\mathrm{Soc}_R(R/I) \neq 0$.

(b) $\Longrightarrow$ (a) Suppose that $\mathrm{Soc}_R(R/I) \neq 0$ for every proper essential right ideal I of R. If I is an essential right ideal of R, suppose that R' is a nonzero homomorphic image of R/I. Then there is a proper and necessarily essential right ideal J of R containing I such that $R/J \approxeq R'$. Hence $\mathrm{Soc}_R(R') \neq 0$ which shows that R/I is a semi–artinian R–module. Therefore $I \in \mathcal{F}_{\tau_S}$ and we have $\tau_G \leq \tau_S$. Theorem 2.1 now indicates that every τ_S–injective module is injective. □

A torsion theory is said to *split* if the torsion submodule of each injective module is injective. Theorem 2.1 produces a condition sufficient for this always to happen.

Corollary 2.3. *If* τ *is a torsion theory on Mod–R such that* $\tau_G \leq \tau$*, then* τ *splits.*

Proof. The τ–torsion submodule of an injective module is τ–injective and Theorem 2.1 indicates that this submodule is injective. □

It is not necessary for $\tau_G \leq \tau$ in order for the τ–torsion submodule to split off from each injective module. This is illustrated by the following example.

Example 2.4. If R is a commutative Noetherian ring and I is an ideal of R, Matlis has shown in [**11**] that the torsion submodule of each injective module is injective for the I–adic torsion theory [**15**] on Mod–R. If $\mathbb{Z}$ is the ring of integers and p is a prime in $\mathbb{Z}$, consider the (p)–adic torsion theory $\tau_{(p)}$ on Mod–$\mathbb{Z}$ with filter $\mathcal{F}_{\tau_{(p)}} = \{(p^k) \mid k \text{ a non-negative integer}\}$. In light of what is known for commutative Noetherian rings, if M is an injective $\mathbb{Z}$–module with $\tau_{(p)}$–torsion submodule $t_{\tau_{(p)}}(M)$, $t_{\tau_{(p)}}(M)$ is a direct summand of M. But every nonzero ideal of $\mathbb{Z}$ is essential, so the filter of ideals for the Goldie torsion theory τ_G on Mod–$\mathbb{Z}$ is not contained in the filter of ideals of $\tau_{(p)}$.

A torsion theory τ is *cohereditary* if every homomorphic image of each τ–torsion free module is τ–torsion free. For this type of torsion theory, the condition that every τ–injective module is injective yields information about the ring.

Theorem 2.5. *If* τ *is a cohereditary torsion theory on Mod–R and every* τ*–injective* R*–module is injective, then* $R/t_\tau(R)$ *is a semisimple ring.*

Proof. Let τ be cohereditary and suppose that every τ–injective R–module is injective. If M is τ–torsion free, then $E_\tau(M)$ is also τ–torsion free, so that $E_\tau(M)/M$ is both τ–torsion and τ–torsion free. Therefore $M = E_\tau(M)$, so we see that every τ–torsion free R–module is τ–injective and hence injective. If $I/t_\tau(R)$ is a right ideal of $R/t_\tau(R)$, then $I/t_\tau(R)$ is a τ–torsion free R–module and so $I/t_\tau(R)$ is an injective R–module. It follows that $I/t_\tau(R)$ is an $R/t_\tau(R)$ direct summand of $R/t_\tau(R)$. □

Corollary 2.6. *If τ is a cohereditary torsion theory on Mod–R and every τ–injective R–module is injective, then every R–module is τ–projective.*

It was pointed out in Section 1 that the class of τ–projective modules is closed under extensions. Since this is the case for injective modules, one wonders if the class of τ–injective modules is closed under extensions. The following theorem shows that this is indeed the case.

Theorem 2.7. *If N is a submodule of an R–module M such that N and M/N are both τ–injective, then M is τ–injective.*

Proof. Suppose that M is a τ–dense submodule of X. Then the sequence $0 \to M/N \to X/N \to X/M \to 0$ splits since M/N is τ—injective. Hence there is a submodule Y of X containing N such that $X/N = (M/N) \oplus (Y/N)$ and $Y/N \approxeq X/M$. Thus N is a τ–dense submodule of Y and, since N is τ–injective, the sequence $0 \to N \to Y \to Y/N \to 0$ splits. Hence there is a submodule Z of Y such that $Y = N \oplus Z$. Now $X/N = (M/N) \oplus (Y/N)$ implies that $X = M + Y$ and $M \cap Y = N$, so it follows that $X = M \oplus Z$. This shows that when N and M/N are τ–injective, M is a direct summand of any module in which it is a τ–dense submodule. But M is a τ–dense essential submodule of $E_\tau(M)$, so $M = E_\tau(M)$. Therefore M is τ–injective. □

The Generalized Baer's Criterion [**3**], [**7**] for τ–injective modules indicates that an R–module M is τ–injective if and only if for each $I \in \mathcal{F}_\tau$ and every R–linear map $f : I \to M$ there is an $m \in M$ such that $f(r) = mr$ for all $r \in I$. If τ_S is the semi–artinian torsion theory on Mod–R, it follows that $\mathcal{F}_{\tau_S}$ contains every maximal right ideal of R. We now show that when investigating the τ_S–injectivity of a module, only R–linear mappings defined on maximal right ideals of R need be considered.

Theorem 2.8. *An R–module M is τ_S–injective if and only if for each maximal right ideal I of R and every R–homomorphism $f : I \to M$ there is an $m \in M$ such that $f(r) = mr$ for all $r \in I$.*

Proof. If M is a τ_S–injective R–module, the stated condition follows from the Generalized Baer's Condition. Conversely, suppose that the stated condition holds for an R–module M and let $I \in \mathcal{F}_{\tau_S}$. Suppose also that $f : I \to M$ is an R-linear mapping. Next let $\mathcal{S}$ be the non–empty set of all ordered pairs (K, g) where K is a right ideal of R containing I, $g : K \to M$ is an R–linear mapping and g restricted to I gives f. Partially order $\mathcal{S}$ by $(K_1, g_1) \le (K_2, g_2)$ if and only if $K_1 \subseteq K_2$ and g_2 restricted to K_1 produces g_1. Then Zorn's Lemma can be used to show that $\mathcal{S}$ has a maximal element, say $(\overline{K}, \overline{g})$. If $\overline{K} \neq R$, then since $I \in \mathcal{F}_{\tau_S}$ and $I \subseteq \overline{K}$, it follows that $\overline{K} \in \mathcal{F}_{\tau_S}$. Thus there is a right ideal J of R containing $\overline{K}$

such that $J/\overline{K}$ is a simple R–module. If $x+\overline{K}$ is a nonzero element of $J/\overline{K}$, then $R/(\overline{K}:x) \approxeq J/\overline{K}$ and $(\overline{K}:x)=\{r\in R \mid xr\in \overline{K}\}$ is a maximal right ideal of R. Hence if we let $\widehat{g}:(\overline{K}:x)\to M$ be defined by $\widehat{g}(r)=\overline{g}(xr)$, then there is an $m\in M$ such that $\widehat{g}(r)=mr$ for all $r\in(\overline{K}:x)$. Finally, define $\varphi:(\overline{K}+xR)\to M$ by $\varphi(k+xr)=\overline{g}(k)+mr$ and note that φ extends $\overline{g}$ to $\overline{K}+xR$. But this contradicts the maximality of $(\overline{K},\overline{g})$ and so it must be the case that $\overline{K}=R$. Invoking the Generalized Baer's Condition again gives the result. □

3. Right Hereditary Rings and Torsion Theory

A ring R is right hereditary if every right ideal of R is projective. Cartan and Eilenberg have shown in [**4**] that such rings are characterized (1) by the fact that submodules of projective modules are projective and (2) by the fact that factor modules of injective modules are injective. When R is a right hereditary ring, τ–projective and τ–injective modules also enjoy these properties.

Theorem 3.1. *If R is a right hereditary ring, then the following hold for any torsion theory τ on Mod-R.*

(a) *Every submodule of a τ–projective module is τ–projective.*

(b) *Every homomorphic image of a τ–injective module is τ–injective.*

Proof. (a) Let M be a τ–projective R–module. Then there is a free R–module F and a τ–torsion submodule N of F such that M is a direct summand of F/N [**13**]. Consequently, if L is any submodule of M, then there is a submodule X of F containing N such that $L\approxeq X/N$. Since R is hereditary, X is projective and so since N is τ–torsion, L is τ–projective.

(b) Suppose that M is a τ–injective R-module and let N be a submodule of M. If $E(M)$ is the injective envelope of M, then since R is right hereditary, $E(M)/N$ is an injective R–module. But an R–module X is τ–injective if and only if $E(X)/X$ is τ–torsion free, so $E(M)/M$ is τ–torsion free. Hence, $M/N\subseteq E(M/N)\subseteq E(M)/N$ gives $E(M/N)/(M/N)\subseteq \big(E(M)/N\big)/(M/N)\approxeq E(M)/M$ and so $E(M/N)/(M/N)$ is τ–torsion free. Thus M/N is τ–injective. □

Kaplansky has shown in [**9**] that for a right hereditary ring, submodules of free R–modules are isomorphic to a direct sum of right ideals of R. This theorem carries over to a torsion theoretical setting in a straightforward manner.

Theorem 3.2. *If R is a ring in which every right ideal is τ–projective, then every τ–torsion free submodule of a free R–module is isomorphic to a direct sum of* (*necessarily τ–torsion free*) *right ideals of R.*

Proof. The proof of Proposition 4.17 given in [**14**] for hereditary rings works with only minor changes required to accommodate the torsion theory. □

For a torsion theory τ on Mod–R, we call a ring R *right τ–hereditary* if every τ–dense right ideal of R is projective. Of course every right hereditary ring is right τ-hereditary for any torsion theory τ on Mod–R and any ring R is right τ_0–hereditary. A ring R is also right τ–hereditary for any torsion theory τ on Mod–R for which every cyclic τ –torsion R–module is projective.

Right hereditary rings are closely linked to the Goldie torsion theory. Indeed, a ring R is right hereditary if and only if it is right τ–hereditary for any torsion theory τ on Mod–R such that $\tau_G \leq \tau$. The necessity of this is obvious, so for the converse suppose that R is right τ–hereditary where τ is a torsion theory on Mod–R such that $\tau_G \leq \tau$. If I is a right ideal of R and J is a complement in R of I, then $I \oplus J$ is an essential right ideal of R. Hence, $I \oplus J \in \mathcal{F}_{\tau_G} \subseteq \mathcal{F}_\tau$ and so $I \oplus J$ is projective. Thus, I is projective, so R is right hereditary. The proof of the following theorem concerning right τ–hereditary rings is an easy adaptation of the proof of Proposition 4.23 given in [**14**] for hereditary rings.

Theorem 3.3. *The following are equivalent for a torsion theory τ on* Mod–R*.*

(a) *R is right τ–hereditary.*

(b) *τ–dense submodules of τ–projective R–modules are τ–projective.*

(c) *Factor modules of τ–injective R– modules by τ–torsion free submodules are τ–injective.*

Since hereditary rings are precisely those rings in which the τ_G –dense right ideals of R are projective, one wonders what class of rings is characterized by the "weaker" condition that the τ_G–dense right ideals of the ring are only τ_G–projective? Rings for which τ–torsion free submodules of τ–projective R–modules are τ–projective are also connected to right hereditary rings.

Theorem 3.4. *If τ is a torsion theory on* Mod–R*, then $R/t_\tau(R)$ is a right hereditary ring if and only if τ–torsion free submodules of τ–projective R–modules are τ–projective.*

Proof. Suppose that $R/t_\tau(R)$ is right hereditary and let M be a τ–projective R-module. Then Theorem 1.1 shows that $M/Mt_\tau(R)$ is projective as an $R/t_\tau(R)$–module. If N is a τ–torsion free submodule of M, then $\big(N + Mt_\tau(R)\big)/Mt_\tau(R)$ is a projective $R/t_\tau(R)$–module. Now $N \cap Mt_\tau(R) = 0$, so it follows that $\big(N + Mt_\tau(R)\big)/Mt_\tau(R) \approxeq N/\big(N \cap Mt_\tau(R)\big) \approxeq N$. Hence N is a projective $R/t_\tau(R)$–module. But $Nt_\tau(R) = 0$ and so it follows from Theorem 1.1 that N is a τ–projective R–module.

Conversely, let $I/t_\tau(R)$ be a right ideal of $R/t_\tau(R)$. Then $I/t_\tau(R)$ is a τ–torsion free submodule of the τ–projective R–module $R/t_\tau(R)$. Now $\big(I/t_\tau(R)\big)t_\tau(R) = 0$, so Theorem 1.1 shows that $I/t_\tau(R)$ is a projective $R/t_\tau(R)$–module. □

One immediate consequence of Theorem 3.4 is that if $R/t_\tau(R)$ is a right hereditary ring, then every τ–torsion free submodule of a free right R–module is τ–projective and this in turn implies that every τ–torsion free right ideal of R is τ–projective. However, neither of these conditions need imply that $R/t_\tau(R)$ is right hereditary. The following example shows that torsion theories exist for which τ–torsion free submodules of free R–modules are τ–projective and yet $R/t_\tau(R)$ is not a right hereditary ring.

Example 3.5. Let D be a commutative domain that is not a Dedekind domain, suppose that Q is the field of fractions of D and consider the matrix ring $R = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a \in D \text{ and } b, c \in Q\}$. If τ_S is the semi–artinian torsion theory on Mod–R, then $t_{\tau_S}(R)$ consists of all matrices in R with $a = 0$. Consequently, $R/t_{\tau_S}(R)$ is isomorphic to D and so $R/t_{\tau_S}(R)$ is not a right hereditary ring. However, $t_{\tau_S}(R)$ is an essential right ideal of R and it follows from this that if F is a free R–module, then $t_{\tau_S}(F)$ is an essential submodule of F. Hence, 0 is the only τ_S–torsion free submodule of F and so every τ_S–torsion free submodule of F is τ_S–projective (and in fact projective.) Note that this example also shows that even if τ–torsion free submodules of free R–modules are projective, $R/t_\tau(R)$ may not be right hereditary.

4. Relative Projective Dimension

Let τ be a torsion theory on Mod–R. If $0 \to K \to X \stackrel{\varphi}{\to} M \to 0$ is exact with X projective, then $0 \to K/t_\tau(K) \to X/t_\tau(K) \stackrel{\overline{\varphi}}{\to} M \to 0$ is exact with $X/t_\tau(K)$ τ–projective and $\ker\overline{\varphi}$ τ–torsion free. This observation allows us to construct an exact complex $\cdots \to P_2 \stackrel{\delta_2}{\to} P_1 \stackrel{\delta_1}{\to} P_0 \stackrel{\delta_0}{\to} M \to 0$ in which P_n is τ–projective and $\ker\delta_n$ is τ–torsion free for each integer $n \geq 0$. We refer to such a complex as a *τ–projective resolution* of M and say that a τ–projective resolution of M is *finite* and of *length* n if there is a non–negative integer n such that $P_k = 0$ for all $k > n$ and $P_k \neq 0$ for $0 \leq k \leq n$. If M has at least one finite τ–projective resolution, then the *τ–projective dimension* of M, written $\tau\text{–pd}_R(M)$, is defined to be the length n of the shortest such resolution of M. If no finite τ–projective resolutions of M exist, then $\tau\text{–pd}_R(M) = \infty$. The *right global τ–projective dimension* of R is defined by $\text{rgl }\tau\text{–pd}_R(R) = \sup\{\tau\text{–pd}_R(M) \mid M \text{ is an } R\text{–module}\}$. If the projective dimension of an R–module M and the right global dimension of R are denoted by $\text{pd}_R(M)$ and $\text{rgl pd}_R(R)$, respectively, then $\tau_1\text{–pd}_R(M) = 0$ and $\tau_0\text{–pd}_R(M) = \text{pd}_R(M)$ for every R–module M. Hence, $\text{rgl }\tau_1\text{–pd}_R(R) = 0$ and $\text{rgl }\tau_0\text{–pd}_R(R) = \text{rgl pd}_R(R)$.

Lemma 4.1. *If τ is a torsion theory on* Mod–R *and*

$$\cdots \to P_2 \stackrel{\delta_2}{\to} P_1 \stackrel{\delta_1}{\to} P_0 \stackrel{\delta_0}{\to} M \to 0 \tag{1}$$

is a τ–projective resolution of M, then $M/Mt_\tau(R)$ has an $R/t_\tau(R)$–projective resolution of the form

$$\cdots \to P_2 \stackrel{\delta_2}{\to} P_1 \stackrel{\delta_1}{\to} P_0/P_0t_\tau(R) \stackrel{\delta_0^*}{\to} M/Mt_\tau(R) \to 0 \tag{2}$$

where $\delta_0^*\big(x + P_0t_\tau(R)\big) = \delta_0(x) + M/Mt_\tau(R)$ for each $x + P_0t_\tau(R) \in P_0/P_0t_\tau(R)$.

Proof. From the construction of (1), we have that $0 \to K_0 \to P_0 \stackrel{\delta_0}{\to} M \to 0$ is exact where K_0 is the kernel of δ_0. It follows that $0 \to \big(K_0 + P_0t_\tau(R)\big)/P_0t_\tau(R) \to P_0/P_0t_\tau(R) \stackrel{\delta_0^*}{\to} M/Mt_\tau(R) \to 0$ is exact. But K_0 is τ–torsion free and $P_0t_\tau(R)$ is τ–torsion, so $\big(K_0 + P_0t_\tau(R)\big)/P_0t_\tau(R) \cong K_0$. Hence, we have an exact sequence $0 \to K_0 \to P_0/P_0t_\tau(R) \stackrel{\delta_0^*}{\to} M/Mt_\tau(R) \to 0$ and Theorem 1.1 shows that $P_0/P_0t_\tau(R)$ is projective as an $R/t_\tau(R)$–module. It also follows from the technique used in the construction of (1) that $0 \to K_1 \to P_1 \to K_0 \to 0$ is exact where K_0 and K_1 are τ–torsion free and K_1 is the kernel of δ_1. Hence the induced sequence $0 \to \big(K_1 + P_1t_\tau(R)\big)/P_1t_\tau(R) \to P_1/P_1t_\tau(R) \to K_0/K_0t_\tau(R) \to 0$ is exact. But the class of τ–torsion free modules is closed under extensions [**3**], so P_1 is τ–torsion free. Therefore $K_0t_\tau(R) = P_1t_\tau(R) = 0$, so $0 \to K_1 \to P_1 \to K_0 \to 0$ is an exact sequence of $R/t_\tau(R)$–modules and P_1 is a projective $R/t_\tau(R)$–module. Consequently, we have an exact sequence $0 \to K_1 \to P_1 \stackrel{\delta_1}{\to} P_0/P_0t_\tau(R) \stackrel{\delta_0^*}{\to} M/Mt_\tau(R) \to 0$ where

P_1 and $P_0/P_0t_\tau(R)$ are projective $R/t_\tau(R)$–modules. The existence of (2) now follows by induction. $\square$

Theorem 4.2. *If τ is any torsion theory on Mod–R, then*

$$\tau\text{–pd}_R(M) \;=\; \text{pd}_{R/t_\tau(R)}\big(M/Mt_\tau(R)\big)$$

for any R–module M.

Proof. If M is an R–module such that $\tau\text{–pd}_R(M) = n$, let $0 \to P_n \overset{\delta_n}{\to} P_{n-1} \to \cdots \to P_0 \overset{\delta_0}{\to} M \to 0$ be a τ–projective resolution of M. Then Lemma 4.1 indicates that we have an $R/t_\tau(R)$–projective resolution of $M/Mt_\tau(R)$ of the form $0 \to P_n \overset{\delta_n}{\to} P_{n-1} \to \cdots \to P_0/P_0t_\tau(R) \overset{\delta_0^*}{\to} M/Mt_\tau(R) \to 0$. Hence, $\text{pd}_{R/t_\tau(R)}\big(M/Mt_\tau(R)\big) \le \tau\text{–pd}_R(M)$.

Conversely, suppose that $\text{pd}_{R/t_\tau(R)}\big(M/Mt_\tau(R)\big) = n$ and let

$$0 \to Q_n \overset{\gamma_n}{\to} Q_{n-1} \to \cdots \to Q_0 \overset{\gamma_0}{\to} M/Mt_\tau(R) \to 0 \tag{3}$$

be an $R/t_\tau(R)$–projective resolution of $M/Mt_\tau(R)$. Next, construct an exact sequence $0 \to K_n \overset{\delta_n}{\to} P_{n-1} \to \cdots \to P_0 \overset{\delta_0}{\to} M \to 0$ of R–modules where P_k is τ–projective for $k = 0, 1, \ldots, n-1$ and K_n is τ–torsion free. Then arguments similar to those given in the proof of Lemma 4.1 can be used to obtain an exact sequence

$$0 \to K_n \overset{\delta_n}{\to} P_{n-1} \to \cdots \to P_0/P_0t_\tau(R) \overset{\delta_0^*}{\to} M/Mt_\tau(R) \to 0 \tag{4}$$

where $P_0/P_0t_\tau(R)$ and P_k, for $k = 1, \ldots, n-1$, are projective $R/t_\tau(R)$–modules. An application of the long version of Schanuel's Lemma [**12**, 7.1.2] to (3) and (4) shows that K_n is a projective $R/t_\tau(R)$-module. Since Theorem 1.1 indicates that K_n is a τ–projective R-module, we have $\tau\text{–pd}_R(M) \le \text{pd}_{R/t_\tau(R)}\big(M/Mt_\tau(R)\big)$. $\square$

Theorem 4.2 has several consequences, the proofs of which are left to the interested reader.

Corollary 4.3. *If τ is any torsion theory on* Mod–*R, then*

(a) rgl $\tau\text{–pd}_R(R) = \text{rgl pd}_{R/t_\tau(R)}\big(R/t_\tau(R)\big)$,

(b) *$R/t_\tau(R)$ is a right hereditary ring if and only if* rgl $\tau\text{–pd}_R(R) \le 1$, *and*

(c) rgl $\tau\text{–pd}_R(R) = \sup\{\tau\text{–pd}_R(M) \mid M$ *is a cyclic R–module* $\} = \sup\{\tau\text{–pd}_R(R/I) \mid I$ *is a right ideal of R such that* $I \supseteq t_\tau(R)\}$.

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