

## SYMPLECTIC OPERATOR–EXTENSION TECHNIQUES AND ZERO–RANGE QUANTUM MODELS

BORIS S. PAVLOV AND VLADIMIR I. KRUGLOV

(Received June 2005)

Abstract. F. Berezin and L. Faddeev have interpreted the Fermi zero-range model as a self-adjoint extension of the Laplacian. Various modifications of this model possess rich spectral properties, but unavoidably have negative effective radius and contain numerous parameters which do not have a direct physical meaning. We suggest, for spherically-symmetric scattering, a generalization of the Fermi zero-range model supplied with an indefinite metric in the inner space and a Hamiltonian of the inner degrees of freedom. The effective radius of this model may be positive or negative. We also propose a general *principle of analyticity*, formulated in terms of  $k \cot \delta(k)$  as a function of the scattering phase shift  $\delta(k)$  depending on the wave-number  $k$ . This principle allows us to evaluate all parameters of the model, including the indefinite metric tensor of the inner space, once the basic parameters of the model: that is the spectrum of the inner Hamiltonian, the scattering length and the effective radius, are fixed, so that the sign of the effective radius is connected with the spectrum of the inner Hamiltonian by a consistency condition.

### 1. Introduction

In [1] a realistic description of quantum scattering by a point-wise object was interpreted, based on [2], in terms of von Neumann operator extensions. From the point of view developed in [1, 4], the solvable zero-range model suggested by Fermi in [2], is actually a self-adjoint extension, see [3], of the Laplace operator defined on smooth functions which have a singularity at the origin

$$\Psi(r) = \frac{\mathcal{A}^\psi}{4\pi r} + \mathcal{B}^\psi + o(1), \quad (1)$$

with the asymptotic boundary values  $\mathcal{A}$  and  $\mathcal{B}$  submitted to the real *boundary condition*,  $\gamma = \bar{\gamma}$

$$\gamma \mathcal{A} + \mathcal{B} = 0. \quad (2)$$

This operator has only trivial spectral structure (one negative eigenvalue  $-\kappa^2$ , for  $-i\kappa = 4\pi\gamma > 0$ , or one resonance  $\kappa' = i4\pi\gamma$  for  $\gamma < 0$ ). In attempt to extend the construction to solvable models with rich spectrum, Wigner suggested in [5] to calculate scattered waves for the Schrödinger operator with compactly supported potentials  $V(\mathbf{r})$ , ( $V(\mathbf{r}) = 0$  for  $r = |\mathbf{r}| > r_0$ ) imposing an appropriate energy-dependent boundary condition at  $r = r_0$ . From the modern point of view, the suggestion of Wigner is equivalent to matching the scattering Ansatz in the outer space  $r > r_0$  with solutions of the Schrödinger equation on the inner space  $r < r_0$  based on Dirichlet-to-Neumann map (DN-map)  $\Lambda$  of the Schrödinger operator

on the inner space, see [6]. In particular, for the spherically-symmetric potential  $V$ , when using the spherically symmetric Ansatz for the scattered waves (in “s-channel”):

$$\psi(r, k) = \frac{C}{r} \sin(kr + \delta(k)) := \frac{C}{r} \chi(r, k), \quad r > r_0 \quad (3)$$

in the outer space, one can present the matching condition in form

$$\left( \frac{\partial}{\partial r} \chi(r, k) + \alpha(k) \chi(r, k) \right) \Big|_{r=r_0} = 0. \quad (4)$$

Here  $\alpha$  is the corresponding Weyl-Titchmarsh function, see [22] - the component of DN-map in s-channel. Unfortunately, this condition contains the full spectral data ( encoded in  $\alpha(k)$  ) of the inner problem, thus reducing the scattering problem to the inner spectral problem. Nevertheless the suggestion of Wigner inspired numerous attempts to construct energy-dependent potentials characterized by the energy-dependent boundary conditions. In [7] the “zero-range potential with inner structure” was suggested based on replacement, for spherically-symmetric scattering, the Schrödinger operator in the inner space by a matrix which plays a role of the Hamiltonian of inner degrees of freedom. Then the role of the component  $\alpha$  of the DN-map is played by the analog  $F(k)$  of the corresponding Krein-function, see [8]. The operator extension procedure suggested in [7] is equivalent to the “asymptotic boundary condition” at the origin :

$$\left( \frac{1}{\chi(\mathbf{r}, k)} \frac{\partial}{\partial r} \chi(\mathbf{r}, k) \right) \Big|_{r=0} = k \cot \delta(k) := F(k). \quad (5)$$

Here the function  $F(k)$  is defined by the spectral structure of the Hamiltonian of the inner degrees of freedom and by the boundary parameters of the model, see [7]. The operator extensions approach allows to construct simple solvable models for few - channels and few - body scattering systems with interesting resonance properties, without solving sophisticate boundary problems, see [9, 10, 11, 12] and more references in [13]. The behavior of the function  $F(k)$  which corresponds to the zero-range model [7] looks similar to the low-energy behavior of the corresponding function in s-channel for conventional rapidly decreasing potentials, see [14]:

$$F(k) = k \cot \delta(k) = -\frac{1}{a} + \frac{r_0}{2} k^2 \dots \quad (6)$$

The parameters  $r_0$ ,  $a$  in (6) are called, respectively, the *effective radius* of the scatterer and the scattering length, see [14]. They are measured in experiment. Unfortunately the zero-range model [7] with inner structure and *positive metric in the inner space*

- (a) unavoidably has a negative effective radius, [15], because in that case  $-F$  is a Nevanlinna class function of  $\lambda = k^2$ , and
- (b) It contains numerous “extra parameters” which *have no straightforward physical interpretation* and hence are not *fitting parameters*. These are: the deficiency vector in the inner space and the boundary parameters.

We propose a solution of both problems a,b for a “special zero-range model”, with an *indefinite metric on the inner space*, see section 4, and the scattering phase

shift  $\delta(k)$  (we call it further just “the scattering phase”) if the function  $F$  is selected as an entire function of the parameter  $\lambda = k^2$ :

$$F(k) = \sum_{l=0}^{\infty} g_l k^{2l}. \quad (7)$$

In case of finite-dimensional “inner space”  $\dim E = n$  the function  $F(k)$  submitted to this *principle of analyticity* is just a polynomial degree  $2n$ . The condition (7) enables us to evaluate all “extra parameters” of the special zero-range model with inner structure, if the *fitting data*: the scattering length, the effective radius and the spectrum of the Hamiltonian of the inner degrees of freedom are connected by the appropriate consistency condition. The standard self-adjoint quantum Hamiltonian of the model is obtained via restriction of the model onto the absolutely continuous subspace which has the positive metric. In this paper we pay special attention to general fitting procedure, see Section 4 and to the description of the positive subspace of the constructed model, Theorem 3.4. More physical discussion and fitting of the model to experimental data can be found in [25].

## 2. Preliminaries: Symplectic Extension Procedure

Zero-range model with inner structure can be constructed via operator extension procedure applied to an orthogonal sum of a restricted bounded inner Hamiltonian and the restricted Laplacian in  $L_2(R_3)$ . But the classical version of the operator-extension procedure, [3], is inconvenient for differential operators. The extension procedure for them is usually reduced to choosing of a Lagrangian plane of the corresponding symplectic *boundary form* e.g.  $\frac{\partial u}{\partial r}(r_0)\bar{v}(r_0) - u(r_0)\frac{\partial \bar{v}}{\partial r}(r_0)$ , for Laplacian in the outer space  $r > r_0$ . I.M. Gelfand in sixties, [16], attracted attention of specialists to necessity of developing a symplectic version of the operator extension procedure for abstract operators. It was done in [7] for solvable quantum models with bounded inner Hamiltonian and positive metric in the inner spaces. In this preliminary section we describe the symplectic version of the extension procedure for operators in Pontryagin space, based on an abstract analog of the boundary form.

Consider the scattering problem for two quantum particles with masses  $m_1, m_2$  and the reduced mass  $\mu = m_1 m_2 (m_1 + m_2)^{-1}$ . Choosing some characteristic wave number  $k_0 > 0$  and the corresponding characteristic energy  $E_0 = (2\mu)^{-1} \hbar^2 k_0^2$ , we introduce the dimensionless coordinates and the spectral parameter, connected with the standard Euclidean coordinates  $\mathbf{r}$  in  $R_3$  and the conventional energy  $E = (2\mu)^{-1} \hbar^2 k^2$  as

$$x = k_0 \mathbf{r}, \quad \lambda = k^2 k_0^{-2} = E(E_0)^{-1}. \quad (8)$$

Then we re-write the conventional Schrödinger equation as

$$\left( \Delta + \lambda - \tilde{V}(x) \right) u(x) = 0. \quad (9)$$

Here  $\tilde{V}(x) = E_0^{-1} V(\mathbf{r})$  is the dimensionless interaction potential. The wave functions  $u(x)$  are dimensionless. We will replace the equation (9) by the zero-range model, assuming that the component of the model in the outer space is defined by the Schrödinger equation with zero potential. The “free” Green’s function which

corresponds to the equation with zero potential on the whole space  $R^3$ , satisfies the equation

$$-(\Delta + \lambda)G_\lambda(x, x') = \delta^{(3)}(x - x') \quad (10)$$

and has a form of an outgoing wave generated at the pole  $x'$ :

$$G_\lambda(x, x') = -(\Delta + \lambda)^{-1}\delta^{(3)}(x - x') = \frac{e^{i\sqrt{\lambda}|x-x'|}}{4\pi|x-x'|}. \quad (11)$$

We assume that the outer space is supplied with the standard  $L_2(R^3)$  - dot-product  $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle = \langle \bar{\alpha} u, v \rangle$  and the non-perturbed two-body Hamiltonian is defined by the three-dimensional self-adjoint Laplacian with respect to dimensionless coordinates

$$lu = -\Delta u, \quad (12)$$

on the Sobolev class  $W_2^2(R^3)$ . Variables  $\tilde{x}, \tilde{y}, \dots$  are reserved for the inner space  $E$  of the model.

Following [1] we restrict the Laplacian  $l$  onto the domain consisting of all smooth functions vanishing near the point  $x_0$ . The closure  $l_0$  of the restricted operator is defined on the domain  $D_0$  which consists of all  $W_2^2$ -functions vanishing just at  $x_0$ . This operator is symmetric and has deficiency indices  $(1, 1)$ :

$$\overline{(l_0 - iI)D_0} = L_2(R_3) \ominus N_i, \quad \overline{(l_0 + iI)D_0} = L_2(R_3) \ominus N_{-i},$$

$\dim N_i = \dim N_{-i} = 1$ . The adjoint operator  $l_0^+ = (-\Delta)^+$  is defined, due to the von Neumann theorem, see [3, 17], by the same differential expression (12) on the domain

$$D_0^+ = D_0 + N_i + N_{-i}. \quad (13)$$

The one-dimensional deficiency subspaces  $N_i, N_{-i}$  are spanned by the corresponding non-perturbed free Green functions

$$N_i = \{G_{-i}(*, x_0)\}, \quad N_{-i} = \{G_i(*, x_0)\},$$

$$G_{-i}(x, x_0) = \frac{e^{i\sqrt{-i}|x-x_0|}}{4\pi|x-x_0|}, \quad G_i(x, x_0) = \frac{e^{i\sqrt{i}|x-x_0|}}{4\pi|x-x_0|} = \frac{l+iI}{l-iI}G_{-i}(x, x_0), \quad (14)$$

with branches of square roots defined by the condition  $\Im m \sqrt{\lambda} > 0$ . We assume further that the restriction point is  $x_0 = 0$ , if another point is not selected.

The above representation (13) of the domain of the adjoint operator was used in [17, 1]. Another representation of the domain of the adjoint Laplacian as a set of singular elements described above (1) was suggested by Fermi [2] and is now commonly used in physical literature, see for instance [18, 4]. It characterizes each element  $u$  from the domain of the adjoint operator by *dimensionless asymptotic boundary values*  $A, B$  for  $|x| \rightarrow 0$ :

$$u(x) = \frac{A^u}{4\pi|x|} + B^u + o(1). \quad (15)$$

The boundary form of  $l_0^+$  on elements from the domain of the adjoint Laplacian is calculated via integration by parts as

$$\mathcal{J}(u, v) = \langle (l_0^+)^+ u, v \rangle - \langle u, (l_0^+)^+ v \rangle = \bar{B}^u A^v - \bar{A}^u B^v. \quad (16)$$

Due to the connection (8) between dimensional and dimensionless coordinates the corresponding asymptotic values are connected by the dimensional factor  $k_0$  :  $A = \mathcal{A}k_0$ ,  $B = \mathcal{B}$ .

It is well known that two descriptions (13,15) of the domain of the adjoint Laplacian are equivalent, [1]. They lie as a basement, respectively, for the classical von-Neumann and symplectic version of the extension procedure for Laplacian. Planning to develop the symplectic version for abstract operators we will derive the equivalence of both above representations based on decomposition of elements of the defect  $N = N_i + N_{-i}$  with respect to the *symplectic basis*

$$\begin{aligned} W_+(x) &= \frac{1}{2} [G_{-i}(x, 0) + G_i(x, 0)] = \frac{l}{l - iI} G_{-i}(x, 0), \\ W_-(x) &= \frac{1}{2i} [G_{-i}(x, 0) - G_i(x, 0)] = -\frac{I}{l - iI} G_{-i}(x, 0). \end{aligned} \quad (17)$$

We will present the von-Neumann formula for elements of the domain of the adjoint operator in terms of the decomposition with respect to the basis  $W_{\pm}$  :

$$u = u_0 + \eta_+^u W_+ + \eta_-^u W_-, \quad (18)$$

where  $u_0$  is an element from the domain of the closure  $l_0$  of the restricted operator and  $\eta_{\pm}^u$  are complex coefficients. Deficiency elements  $G_{\pm i}(*, 0)$  are eigenvectors of the adjoint operator with eigenvalues  $\mp i$  respectively. Then the elements  $W_{\pm}$  are transformed by the adjoint operator as:

$$l_0^+ W_+ = W_-, \quad l_0^+ W_- = -W_+. \quad (19)$$

**Lemma 2.1.** *The boundary form (16) of the adjoint operator  $l_0^+$  depends only on components  $u_d = \eta_+^u W_+ + \eta_-^u W_-$ ,  $v_d = \eta_+^v W_+ + \eta_-^v W_-$  of the elements  $u, v$  in the defect. It is an Hermitian symplectic form of the variables  $\eta_{\pm}$  and can be presented alternatively as*

$$\mathcal{J}(u, v) = \mathcal{J}_l(u_d, v_d) = \langle l_0^+ u_d, v_d \rangle - \langle u_d, l_0^+ v_d \rangle = \frac{1}{4\pi\sqrt{2}} (\bar{\eta}_+^u \eta_-^v - \bar{\eta}_-^u \eta_+^v). \quad (20)$$

The above variables  $\eta_{\pm}$  are connected with the asymptotic boundary values  $A, B$  via the transformation:

$$\begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -4\pi\sqrt{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \quad (21)$$

**Remark 2.2.** Both pairs of variables  $\eta_{\pm}$  and  $A, B$  are *symplectic coordinates* of the element  $u$  in representations (15,18). The representation (i7) of the boundary form in terms of the symplectic variables  $A, B$  is usually obtained, see for instance [2, 1, 18, 4] via straightforward integration by parts. An alternative calculation based on (19) goes in line with an abstract analog (Lemma II.2 below) of the above statement .

**Proof.** of the formula (20) is obtained via the direct application of the adjoint operator  $l_0^+$  to the above version (18) of the von-Neumann decomposition of elements from  $D_0^+$  in the boundary form of the adjoint operator:  $\langle l_0^+ u, v \rangle - \langle u, l_0^+ v \rangle := \mathcal{J}_l(u, v)$ . One can easily see that the boundary form depends only on the parts of the elements  $u, v$  in the sum of the deficiency subspaces (in the “defect”),

$u_d = \eta_+^u W_+ + \eta_-^u W_-$ ,  $v_d = \eta_+^v W_+ + \eta_-^v W_-$  and can be calculated using (19), since  $l_0^+ u_0 = l_0 u_0$ . Then due to  $\int_{R^3} |G_i(\xi, 0)|^2 d^3 \xi = 1/(4\pi\sqrt{2})$  we obtain:

$$\mathcal{J}(u, v) = \langle l_0^+ u_d, v_d \rangle - \langle u_d, l_0^+ v_d \rangle = (\bar{\eta}_+^u \eta_-^v - \bar{\eta}_-^u \eta_+^v) 1/(4\pi\sqrt{2}). \quad (22)$$

The second formula (21) is based on the asymptotic behavior of the non-perturbed Green function for  $x \rightarrow 0$ :  $W_+(x) = \frac{1}{4\pi|x|} - \frac{1}{4\pi\sqrt{2}} + o(1)$ ,  $W_-(x) = -\frac{1}{4\pi\sqrt{2}} + o(1)$ .

Self-adjoint extensions of  $l_0$  were obtained in [1] via submitting  $A$ ,  $B$  to the Fermi boundary conditions  $B = \gamma A$ . For general construction of extensions of densely defined operators in terms of boundary forms see also [21]. Though the boundary forms (16) and (20) are equivalent, our version (20) can be used for the whole class of all differential operators with square-integrable Green-function, in particular for the Schrödinger operators in a bounded domain with Robin boundary conditions, even restricted at the point  $x)_0$  on the boundary boundary.

We choose the non-perturbed Hamiltonian of inner degrees of freedom ( we call it further just inner Hamiltonian) as a  $J$ -self-adjoint operator in the inner Pontryagin space supplied with an indefinite metric. After introducing the dimensionless spectral parameter  $\lambda$  the zero-range model of two particles Hamiltonian will be obtained as an extension of the orthogonal sum  $l_0 \oplus \mathcal{H}_0$  of the restricted Laplacian and the restricted inner Hamiltonian  $\mathcal{H}$  in the Pontryagin space  $L_2(R^3) \oplus E_J$ . The inner component  $E_J$  is obtained via equipping a finite-dimensional Hilbert space  $E$ ,  $\dim E = N$ , with an indefinite metric. Consider two complementary orthogonal projections  $P_\pm$  with respect to the conventional dot-product  $\langle *, * \rangle$  in  $E$ . Then the indefinite dot product in  $E$  ( $J$ -dot product) is defined as :

$$\langle J\tilde{x}, \tilde{y} \rangle = [\tilde{x}, \tilde{y}] = \langle P_+ \tilde{x}, P_+ \tilde{y} \rangle - \langle P_- \tilde{x}, P_- \tilde{y} \rangle.$$

Obviously  $[\alpha\tilde{x}, \tilde{y}] = \alpha[\tilde{x}, \tilde{y}] = [\tilde{x}, \alpha\tilde{y}]$ . The space  $E$  supplied with the above  $J$ -dot product is denoted by  $E_J$ . For general properties of Pontryagin spaces see [19]. We sketch below a symplectic version of the operator-extension technique in the inner component  $E_J$ .

A bounded operator  $\mathcal{H}$  is called  $J$ -symmetric (symmetric with respect to the above  $J$ -dot product) if  $[\mathcal{H}\tilde{x}, \tilde{y}] = [\tilde{x}, \mathcal{H}\tilde{y}]$ . In finite-dimensional space each  $J$ -symmetric operator is  $J$ -self-adjoint. We will apply the symplectic scheme to the bounded operator which is also *symmetric with respect to the conventional dot-product in  $E$  and commutes with  $J$* . We assume that the spectrum of the  $J$ -symmetric operator  $\mathcal{H}$  consists of a finite number of simple *positive* eigenvalues  $\lambda_s = k_s^2 k_0^{-2}$ ,  $(\mathcal{H} - \lambda_s I) e_s = 0$  and  $\mathcal{H} - iI$  is invertible. Choose a normalized *generating* vector  $e \in E_J$ ,  $[e, e] = \rho = \pm 1$  ( which is non-orthogonal to each eigenvector  $e_s$  of the operator  $\mathcal{H}$ ,  $[e, e_s] := e^s \neq 0$ ,  $s = 1, 2, \dots, N$ ). For selected vector  $e$  define the domain  $D_{\mathcal{H}_0}$  of the restricted operator as  $D_{\mathcal{H}_0} = [\mathcal{H} - iI]^{-1} (E_J \ominus_J \{e\})$  and set  $\mathcal{H}_0 = \mathcal{H}|_{D_{\mathcal{H}_0}}$ , where the  $J$ -orthogonal difference is denoted by  $\ominus_J$ . The vectors  $e, e'$  play roles of deficiency vectors of the restricted operator at the spectral points  $\pm i$ :  $[(\mathcal{H} - iI)D_{\mathcal{H}_0}, e] = 0$ ,  $[(\mathcal{H} + iI)D_{\mathcal{H}_0}, e'] = 0$ . Then the vectors  $e$  and  $e' = \frac{\mathcal{H} + iI}{\mathcal{H} - iI} e$  form a linearly-independent pair, with a positive angle between them. The subspaces  $M_i := \bigvee_\alpha \{\alpha e\}$ ,  $M_{-i} := \bigvee_\alpha \{\alpha e'\}$  will play roles of the deficiency subspaces.

At any regular points  $\lambda, \bar{\lambda}$  of the operator  $\mathcal{H}$  the deficiency subspaces  $M_\lambda, M_{\bar{\lambda}}$  are also one-dimensional, the defect is two-dimensional and the deficiency vectors are calculated as

$$e_\lambda = \frac{\mathcal{H} + iI}{\mathcal{H} - \bar{\lambda}I}e, \quad e_{\bar{\lambda}} = \frac{\mathcal{H} + iI}{\mathcal{H} - \lambda I}e.$$

Generally the  $J$ -adjoint operator  $\mathcal{H}_0^+$  is defined by the formula  $[\mathcal{H}_0\tilde{x}, \tilde{y}] = [\tilde{x}, \mathcal{H}_0^+\tilde{y}]$  on elements  $\tilde{y}$  for which the  $J$ -dot product in the left hand side may be continued onto the whole space  $E$  as an “anti-linear” functional of  $\tilde{x}$ . For densely-defined operators this condition implies  $(\mathcal{H}_0^+ + iI)e = 0$ ,  $(\mathcal{H}_0^+ - iI)e' = 0$ . For a bounded operator  $\mathcal{H}$  we just define the formal adjoint operator  $\mathcal{H}_0^+$  on the defect  $M = M_i + M_{-i}$  by the above formulae. Then, for complex values of  $\lambda$ , the deficiency vectors are eigenvectors of the corresponding formal adjoint operator:

$$(\mathcal{H}_0^+ - \bar{\lambda}I)e_\lambda = 0, \quad (\mathcal{H}_0^+ - \lambda I)e_{\bar{\lambda}} = 0.$$

An analog of the von-Neumann representation of the domain of the adjoint operator remains true in Ponryagin space, see [19]. For elements from the domain of the formal adjoint operator we have the representation  $\tilde{x} = \alpha e + \beta e'$ ,  $\tilde{x}_0 \in D_0$  with

$$\mathcal{H}_0^+(\alpha e + \beta e') = -i\alpha e + i\beta e'.$$

M. Krasnosel'skii noticed, see [20], that the extension construction for non-densely defined closed operators is actually developed in the defect  $M = M_i + M_{-i}$ , and can be accomplished similarly to von-Neumann construction, if the deficiency subspaces do not overlap. We obtain a symmetric extension of the restricted operator  $\mathcal{H}_0$  on the defect via restriction of the formal adjoint  $\mathcal{H}_0^+|_{\mathcal{L}}$  onto some Lagrangian plane  $\mathcal{L} \subset M$  where the corresponding boundary form (25) vanishes. Thereafter an extension in the whole space is obtained as the a direct sum  $\mathcal{H}_0 + \mathcal{H}_0^+|_{\mathcal{L}}$ . We develop here the operator-extension construction with a pair of deficiency subspaces assuming that  $\dim E_j \geq 2$ . In case  $\dim E_j = 1$  the resulting formulae describing the extended operator also remain true and may be verified via direct calculation.

Consider a new basis in the defect  $M = M_i + M_{-i}$  which is similar to the above basis (17) combined of the Greens-functions:

$$w_+ = \frac{e + e'}{2} = \frac{\mathcal{H}}{\mathcal{H} - iI}e, \quad w_- = \frac{e - e'}{2i} = \frac{-I}{\mathcal{H} - iI}e. \quad (23)$$

Due to above definition of the formal adjoint operator we have:

$$\mathcal{H}_0^+w_+ = w_-, \quad \mathcal{H}_0^+w_- = -w_+. \quad (24)$$

We use the new basis  $\{w_+, w_-\}$  to represent elements from the domain of the formal adjoint operator via new dimensionless symplectic variables  $\xi_\pm$  which play a role similar to the above pair  $\eta_\pm$ :

$$\tilde{x} = \xi_+^x w_+ + \xi_-^x w_-.$$

□

**Lemma 2.3.** *The boundary form of the (formal) adjoint operator in terms of the variables  $\xi_\pm$  is given by an Hermitian symplectic form :*

$$\mathcal{K}(\tilde{x}, \tilde{y}) = [\mathcal{H}_0^+\tilde{x}, \tilde{y}] - [\tilde{x}, \mathcal{H}_0^+\tilde{y}] = [\bar{\xi}_+^x \xi_-^y - \bar{\xi}_-^x \xi_+^y]\rho. \quad (25)$$

**Proof.** exactly follows the pattern of the previous lemma (II.1). The vector  $e$  is normalized in  $E_J$ , hence the normalization constant similar to  $(4\pi\sqrt{2})^{-1}$  in (20) is equal to  $\rho = \pm 1$ .

Note that the formula (25) can be interpreted as an “abstract formula of integration by parts” for the operator  $\mathcal{H}_0^+$ . The symplectic version of the extension procedure allows to obtain the  $J$ -self-adjoint extension of the formal adjoint operator via restriction of it onto the Lagrangian plane  $\mathcal{L}_\gamma$  in the defect defined by the boundary conditions as  $\xi_- = \gamma\xi_+$ ,  $\gamma = \bar{\gamma}$ . Then the construction of the  $J$ -self-adjoint extension in  $E_J$  is accomplished via forming of the direct sum  $\mathcal{H}_0 + \mathcal{H}_0^+|_{\mathcal{L}_\gamma}$ .

Our aim is the construction of the joint extension  $\mathcal{A}_\Gamma$  of  $l_0 \oplus \mathcal{H}_0$ . Consider the orthogonal sum of operators  $l \oplus \mathcal{H}$  in the Pontryagin space  $L_2(R^3) \oplus E_J$  with elements  $U = (u, \tilde{x})$ ,  $u \in L_2(R^3)$ ,  $\tilde{x} \in E_J$ . Restricting both operators as above we obtain the symmetric operator  $l_0 \oplus \mathcal{H}_0$  with the defect  $N \oplus M$  and the deficiency index  $(2, 2)$ . We define the adjoint operator as an orthogonal sum of “adjoint” operators  $l_0^+ \oplus \mathcal{H}_0^+$  and calculate the boundary form of it as a sum of boundary forms of the outer and inner components:

$$\mathbf{J}(U, V) = \mathcal{J}(u, v) + \mathcal{K}(\tilde{x}, \tilde{y}) = (\bar{B}^u A^v - \bar{A}^u B^v) + (\bar{\xi}_+^x \xi_-^y - \bar{\xi}_-^x \xi_+^y),$$

where we choose the dimensionless symplectic variables  $A, B$  defined by (15) for the “outer” boundary form. The boundary form  $\mathbf{J}(U, V)$  of the orthogonal sum of operators  $l_0^+ \oplus \mathcal{H}_0^+$  in the Pontryagin space  $L_2(R^3) \oplus E_J$  is a symplectic Hermitian form. Hermitian extension of the operator  $\mathcal{A}_0 = l_0 \oplus \mathcal{H}_0$  (or Hermitian restriction of the adjoint operator  $\mathcal{A}_0 = l_0^+ \oplus \mathcal{H}_0^+$ ) is constructed via imposing proper boundary conditions on symplectic variables with an hermitian matrix  $\mathbf{\Gamma}$ :

$$\begin{pmatrix} B \\ -\xi_- \end{pmatrix} = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix} \begin{pmatrix} A \\ \xi_+ \end{pmatrix}. \quad (26)$$

The joint boundary form  $\mathbf{J}(U, V)$  vanishes on the corresponding Lagrangian plane  $\mathcal{L}_\Gamma \subset M + N$  defined by the above boundary condition. All Lagrangian planes of the joint boundary form may be constructed either with a help of various Hermitian matrices  $\mathbf{\Gamma}$ , or obtained from already constructed planes by proper  $J$ -unitary transformation. As previously, the extended operator  $\mathcal{A}_\Gamma$  is obtained as the direct sum of the constructed restriction of the formal adjoint  $\mathcal{A}_\gamma^+ = l_0^+ \oplus \mathcal{H}_0^+|_{\mathcal{L}_\Gamma}$  onto the Lagrangian plane  $\mathcal{L}_\Gamma$  in the defect and the closure of the restricted operator,  $\mathcal{A}_\Gamma = \mathcal{A}_\gamma^+ + \overline{(l_0 \oplus \mathcal{H}_0)}$ .  $\square$

### 3. The Spectral Structure of $\mathcal{A}_\Gamma$ and the Positive Subspace

One can show, combining [20, 12], that the constructed operator  $\mathcal{A}_\Gamma$  is a  $J$ -self-adjoint operator in the Pontryagin space  $L_2(R^3) \oplus E_J$ . According to general description of the spectral structure of self-adjoint operators in Pontryagin space, see [19], the operator  $\mathcal{A}_\Gamma$  has a *non-negative* invariant subspace  $H \subset L_2(R^3) \oplus E_J$  of a finite co-dimension in  $L_2(R^3) \oplus E_J$ . If this subspace is strictly positive, then the part of the operator  $\mathcal{A}_\Gamma$  in it is a conventional self-adjoint operator with respect to the positive metric  $P_H[I \oplus J]P_H$ . This self-adjoint operator may play a role of a conventional quantum mechanical Hamiltonian of the constructed model. Normally, description of the positive subspace is complicated task since the indefinite

metric can be degenerated. In our case we can provide an alternative description of the subspace by showing that it coincides with the subspace of the absolutely-continuous spectrum of  $\mathcal{A}_\Gamma$ .

Assume that the extension procedure is developed as described in the previous section based on a generating vector  $e = e_i$  with use of the boundary conditions (26). The spectrum of the constructed operator  $\mathcal{A}_\Gamma$  is defined by the classical Krein formula, see [8]. In our case it coincides, see for instance [12], with singularities (the cut and the poles) of the scattering matrix (33) and consists of a finite number of eigenvalues and Lebesgue absolutely continuous spectrum filling the positive semi-axis  $0 < \lambda < \infty$  with infinite multiplicity. The generalized eigenfunctions of the absolutely continuous spectrum of  $\mathcal{A}_\Gamma$  are obtained via fitting to the above boundary conditions (26) the Ansatz for scattered waves :

$$\Psi_\lambda = \begin{pmatrix} \psi_\lambda(x, \nu) \\ \tilde{\psi}_\lambda \end{pmatrix} = \begin{pmatrix} e^{i\sqrt{\lambda}(\nu, x)} + k_0 f(\sqrt{\lambda}) \frac{e^{i\sqrt{\lambda}|x|}}{|x|} \\ \xi_+ \frac{\mathcal{H} + iI}{\mathcal{H} - \lambda I} e \end{pmatrix}, \quad (27)$$

with components  $\psi_\lambda, \tilde{\psi}_\lambda$  in the outer and inner spaces. The symplectic variables  $\xi_\pm$  of the inner component of the solution of the adjoint homogeneous equation are connected  $\xi_- = -Q(\lambda)\xi_+$  via the Krein  $Q$ -function of the inner Hamiltonian - an abstract analog of the Weyl-Titchmarsh  $m$ -function, see [22]:

**Lemma 3.1.** *The  $Q$ -function for the operator  $\mathcal{H}_0^+$  has the form:*

$$Q(\lambda) = \left[ e, \frac{I + \lambda\mathcal{H}}{\mathcal{H} - \lambda I} e \right].$$

**Proof.** Proof of the statement, similarly to the proof of the corresponding statement in case of positive metric in [9], is reduced to the calculation of symplectic coordinates of the deficiency element  $e_{\bar{\lambda}}$ . Consider the  $J$ -orthogonal projections  $P_e = e][e$  and  $I - P_e$  onto the one-dimensional deficiency subspace  $N_i$  and onto the complementary subspace  $E \ominus_j N_i$ , respectively. Then the solution of the adjoint homogeneous equation  $e_{\bar{\lambda}}$  can be presented as:

$$e_{\bar{\lambda}} = \frac{\mathcal{H} + iI}{\mathcal{H} - \lambda I} e = \frac{\mathcal{H}}{\mathcal{H} - iI} e + \frac{I}{\mathcal{H} - iI} e \left[ e, \frac{I + \lambda\mathcal{H}}{\mathcal{H} - \lambda I} e \right] + \frac{I}{\mathcal{H} - iI} (I - P_e) \frac{I + \lambda\mathcal{H}}{\mathcal{H} - \lambda I} e. \quad (28)$$

The last term in the right-hand side is an element  $u_0$  from the domain of the restricted operator  $\mathcal{H}_0$ , but the first two terms belong to the defect  $M$  so the whole linear combination with use of the notation  $Q(\lambda) \equiv [e, \frac{I + \lambda\mathcal{H}}{\mathcal{H} - \lambda I} e]$  can be presented as

$$\frac{\mathcal{H} + iI}{\mathcal{H} - \lambda I} e = w_+ - Q(\lambda)w_- + u_0.$$

Hence the symplectic coordinates  $\xi_\pm$  of  $e_{\bar{\lambda}}$  are  $\xi_+ = 1, \xi_- = -Q(\lambda)$ .

We consider an indefinite metric tensor  $J$  defined by a diagonal matrix, for instance  $\{J_{ss}\} = \pm 1, s = 1, 2, \dots, N$ , commuting with a positive diagonal matrix  $\mathcal{H} = \{\lambda_s\}, s = 1, 2, \dots, \dim E$ . Recall that  $J$ -normalized deficiency vector  $e$  ( $[e, e] = \rho = \pm 1$ ) has non-zero components  $e^s$  with respect to the standard basis in  $E$ . Then the  $Q$ -function has the form

$$Q(\lambda) = \sum_{s=1}^N \frac{1 + \lambda\lambda_s}{\lambda_s - \lambda} J_{ss} |e^s|^2 = \sum_{s=1}^N \frac{1 + \lambda\lambda_s}{\lambda_s - \lambda} P_s, \quad (29)$$

with *real* weights  $P_s = J_{ss}|e^s|^2$  and poles of first order at the eigenvalues  $\lambda_s$  of  $\mathcal{H}$ . In case of positive metric  $J_{ss} > 0$  the  $Q$ -function belongs to Nevanlinna class.

The previous lemma allows us to solve the adjoint non-homogeneous equation and obtain the Krein formula [8] for the resolvent of the  $J$ -self-adjoint extension of the symmetric operator  $l_0 \oplus \mathcal{H}_0$  with the boundary condition (26). The scattered waves in  $s$ -channel may be derived from it in a rather standard way, see [12]. One may also show, see for instance [4, 13], that the boundary conditions (26) formulated for elements of the domain of the extension  $\mathcal{A}_\Gamma$  are fulfilled for the corresponding scattered waves. We focus now on the straightforward derivation of the expressions for the scattered waves  $\{\Psi\}$  based on the above Ansatz (27). The symplectic variables in the ansatz for components of the scattered wave in the outer and the inner spaces are:

$$B = \left(1 + i\sqrt{\lambda}k_0 f\right), \quad A = 4\pi k_0 f, \quad \xi_- = -Q \xi_+. \quad (30)$$

This gives, due to (26), the following equation for the amplitudes  $f, \xi_+$ :

$$\begin{pmatrix} 1 + i\sqrt{\lambda}k_0 f \\ Q\xi_+ \end{pmatrix} = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix} \begin{pmatrix} 4\pi k_0 f \\ \xi_+ \end{pmatrix}. \quad (31)$$

□

Solving this equation we obtain the expression for the amplitude  $f$ :

**Theorem 3.2.** *The amplitude  $f$ , for real positive  $\lambda = k^2 k_0^{-2}$ , is equal to*

$$f(k) = \left(4\pi k_0 \gamma_{00} - \frac{4\pi k_0 |\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)} - ik\right)^{-1}. \quad (32)$$

**Proof.** follows from the previous discussion if the dimension of the inner space is  $\geq 2$ , and  $\xi_+ = \gamma_{01} \frac{4\pi k_0 f}{Q - \gamma_{11}}$ . In case of the one-dimensional inner space the formulae (32) are verified by the direct calculation. □

**Corollary 3.3.** *The scattering matrix in  $s$ -channel is connected to the amplitude  $f(\sqrt{\lambda})$  via the formula, see [13]:*

$$S(\sqrt{\lambda}) = 1 + 2ik f(\sqrt{\lambda}) = \frac{\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q} + i\frac{\sqrt{\lambda}}{4\pi}}{\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q} - i\frac{\sqrt{\lambda}}{4\pi}}, \quad (33)$$

with the branch of  $\sqrt{\lambda}$  defined by the condition  $\sqrt{\lambda} > 0$  for  $\lambda > 0$ . The dimensionless eigenvalues  $\lambda_s$  of  $\mathcal{A}_\gamma$  are found as poles of the scattering matrix on the spectral sheet  $\Im m \sqrt{\lambda} > 0$ :

$$\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)} - i\frac{\sqrt{\lambda}}{4\pi} = 0.$$

When considering the non-stationary scattering problem, see for instance [23], it is convenient to use the complex conjugate scattering matrix, instead of (33). Non-pure-imaginary zeroes of our scattering matrix  $S(\sqrt{\lambda})$  sit in the upper half-plane  $\sqrt{\lambda} > 0$ , and the corresponding complex-conjugate poles - in the lower half-plane. Vice versa, the non-pure-imaginary zeroes of the complex conjugate matrix sit in the lower half-plane  $\sqrt{\lambda} < 0$ , and the corresponding poles sit in complex-conjugate points in the upper half-plane.

In the next section we consider zero-range models with scattering matrix approaching 1 at infinity. For these models the scattering matrix is presented as a ratio of two finite Blaschke products  $S(k) = S_+(k) S_-(k)$  with zeroes in upper and lower half-planes correspondingly:  $S_+(k) = \prod_l \frac{k-k_l}{k-\bar{k}_l}$ ,  $\Im m k_l > 0$ ,  $S_-(k) = \prod_n \frac{k-\kappa_n}{k-\bar{\kappa}_n}$ ,  $\Im m \kappa_n < 0$ . We proceed assuming that the Scattering matrix has this form.

In the remaining part of this section we describe the minimal positive subspace of the operator  $\mathcal{A}_\Gamma$ . In case of s-scattering, only eigenfunctions with spherically - symmetric outer part differ from the non-perturbed exponentials. The whole absolutely-continuous subspace is split orthogonally into the trivial part constituted by eigenfunctions with trivial inner components and the complementary part  $H_a$  characterized by spherically-symmetric outer components of the corresponding (generalized) eigenfunctions. Consider the restriction of  $\mathcal{A}_\Gamma$  onto subspace  $H_a$ . The scattered waves from  $H_a$  are obtained via fitting an appropriate ansatz to (26). This gives:

$$\Psi_k(r) = \begin{pmatrix} \frac{1}{4\pi r} \left[ e^{-ikr} - S(k) e^{ikr} \right] \\ \xi(k) \frac{\mathcal{H} + iI}{\mathcal{H} - \lambda I} e \end{pmatrix}, \tag{34}$$

$S(k)$  coincides with (33) and  $\xi(k) = \gamma_{01} \frac{2i\sqrt{\lambda}}{Q - \gamma_{11}} f(k)$ . Further construction of the positive subspace is developed in  $H_a$ .

**Theorem 3.4.** *The spherically - symmetric subspace  $H_a$  of the absolutely- continuous spectrum of  $\mathcal{A}_\Gamma$  is positive, and the restriction of the operator  $\mathcal{A}_\Gamma$  onto that subspace is a conventional self-adjoint operator which is equivalent to the multiplication by  $k^2$  in  $L_2(R)$ .*

**Proof.** We suggest an indirect proof of this statement based on analysis of the energy norm associated with the wave equation  $u_{tt} + \mathcal{A}_\Gamma u = 0$ . Introduce the corresponding energy norm on a dense linear set of all Cauchy data  $\mathbf{U} = \{u, u_t\}$  with both components from  $H_a$ :

$$\mathbf{U}(t) = \begin{pmatrix} u \\ u_t \end{pmatrix} = \int_{-\infty}^{\infty} \begin{pmatrix} \frac{1}{ik} \Psi \\ \Psi \end{pmatrix} e^{ikt} h(k) dk, \quad -\infty < t < \infty.$$

We can choose  $h = h_- \in S_+^{-1} m_- H_-^2$ , with a rapidly decreasing at infinity outer function  $m_-(k)$  in lower half-plane. Then both components of  $\mathbf{U}(t)$  belong to the domain of  $\mathcal{A}_\Gamma$ . The closure of  $\bigvee_{t>0} e^{ikt} S_+^{-1} m_- H_-^2$  in  $L_2$ -norm coincides with  $L_2(R)$  and gives the spectral representation of the “whole” absolutely-continuous subspace. As usual, see [23], the evolution of the above wave equation on the space of Cauchy data with the energy-norm  $\|\mathbf{U}\|^2 = \frac{1}{2} \{[Au, u] + [u_t, u_t]\}$  is energy-preserving. It can be presented as  $\exp(i\mathcal{L})$  with the corresponding matrix generator:

$$\mathcal{L} = i \begin{pmatrix} 0 & -1 \\ \mathcal{A}_\Gamma & 0 \end{pmatrix}.$$

The absolutely continuous spectrum of the operator  $\mathcal{L}$  is simple on the whole real axis  $k$ , and corresponding generalized eigenfunctions are two-component vectors  $(\frac{1}{ik} \Psi, \Psi)$ . To prove that the energy norm is positive, it suffice, due to the energy conservation, to verify the positivity on elements  $\mathbf{U}(t)$  for  $t \rightarrow -\infty$ . Note that, due

to the special choice of  $h_-$ , the parts of each component  $u, u_t$  in the inner subspace vanish for large negative  $t$ , so that the total energy norm coincides with one of the component of  $\mathbf{U}$  in the outer space and is equal to the standard energy norm for the wave equation. Hence it is positive for large negative  $t$ . Then due to the energy conservation, the energy norm is positive on elements  $\mathbf{U}(t)$  for each  $t$ . Hence the generator  $\mathcal{L}$  of the evolution is a self-adjoint operator in the space of Cauchy data  $\mathbf{U}$  with the positive energy. The square of it is defined on smooth elements by the diagonal matrix in the space of energy-normed Cauchy data

$$\mathcal{L}^2 = \begin{pmatrix} \mathcal{A}_r & 0 \\ 0 & \mathcal{A}_r \end{pmatrix}.$$

It can be extended by Friedrichs procedure onto the maximal domain in the energy normed space. The square norm of the first component of the corresponding decomposition of the Cauchy data coincides with the quadratic form of  $\mathcal{A}_r$ . The restriction of the energy norm onto the second component  $u_t$  of Cauchy data is still positive and equivalent to the  $L_2$  - norm on  $kh_-$ . Closure of the subspace of the second components of Cauchy data gives the subspace of the absolutely-continuous spectrum of  $\mathcal{A}_r$ . Hence the corresponding absolutely-continuous subspace is positive, and the part of  $\mathcal{A}_r$  in it is a conventional self-adjoint operator.  $\square$

#### 4. Fitting of Parameters for Zero-Range Model

In this section we define the metric of the inner space, and evaluate the boundary parameters and the moduli of components of the deficiency vector in the finite-dimensional case  $\dim E = N$ , assuming that the spectrum of the inner Hamiltonian and the real values of the scattering length and of the effective radius are given.

The scattering matrix  $S(k) = \exp 2i\delta(k)$  can be presented as the Cayley-transform of the function  $\cot \delta(k)$ :

$$S(k) = \frac{\cot \delta(k) + i}{\cot \delta(k) - i} = 1 + 2ikf(k). \quad (35)$$

Hence the scattering amplitude  $f(k)$  is presented as a function of the scattering phase  $\delta(k)$ :

$$f(k) = [k \cot \delta(k) - ik]^{-1}. \quad (36)$$

Combining Eqs.(32), (35) and (36) one can find the S-scattering matrix in the form:

$$S(k) = 1 + \frac{2ik}{4\pi k_0 [\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)}] - ik} = \frac{F(k) + ik}{F(k) - ik}, \quad (37)$$

with

$$F(k) = 4\pi k_0 \left[ \gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)} \right] = k \cot \delta(k). \quad (38)$$

Here we used the above notation (29) and the dimensionless energy  $\lambda = (k/k_0)^2$

In this section we consider a special class of zero-range models for which the scattering matrix tends to 1 at infinity. This implies

$$\gamma_{11} + \sum_{s=1}^N \lambda_s P_s = 0, \quad (39)$$

and the representation of the denominator in (38) as a ratio of two polynomials  $\mathcal{P}_M, \mathcal{P}_N$  with real coefficients,  $M < N$ :

$$D(\lambda) = \gamma_{11} - Q(\lambda) = - \sum_{s=1}^N \frac{1 + \lambda_s^2}{\lambda_s - \lambda} P_s = \Lambda_{NM} \frac{\prod_{r=1}^M (\lambda - h_r)}{\prod_{s=1}^N (\lambda - \lambda_s)} := \Lambda_{NM} \frac{\mathcal{P}_M}{\mathcal{P}_N}. \quad (40)$$

Points  $\{h_r\}$  sit in complex plane  $\lambda$  symmetrically with respect to complex conjugation. The real coefficient  $\Lambda_{NM}$  will be found later. Consider the Laurent series

$$D(\lambda) = \Lambda_{NM} \left[ \frac{1}{\lambda^{(N-M)}} + \frac{d_{N-M+1}}{\lambda^{(N-M+1)}} + \dots + \frac{d_N}{\lambda^N} + \dots \right], \quad (41)$$

with  $d_l = 0$  for  $l = 1, 2, \dots, N - M - 1$ ,  $d_{N-M} = 1$ , and

$$D^{-1}(\lambda) = \frac{1}{\Lambda_{NM}} \frac{\prod_{r=1}^M (\lambda - h_r)}{\prod_{s=1}^N (\lambda - \lambda_s)} = \frac{1}{\Lambda_{NM}} \left[ \lambda^{(N-M)} + \dots + q_1 \lambda + q_0 + \frac{q_{-1}}{\lambda} \dots \right]. \quad (42)$$

The real coefficients  $d_l, q_l$  are uniquely defined by the poles  $\lambda_s$  and zeroes  $h_s$  of  $D$ . For instance, integrating  $D^{-1}$  on a large circle with proper weights  $z^{-1}, z^{-2}$  we obtain

$$q_0 = \frac{\mathcal{P}_N(0)}{\mathcal{P}_M(0)} + \sum_{r=1}^M \frac{\prod_{s=1}^N (h_r - \lambda_s)}{h_r \prod_{t \neq r} (h_r - h_t)}, \quad q_1 = \frac{d}{d\lambda} \left( \frac{\mathcal{P}_N}{\mathcal{P}_M} \right) (0) + \sum_{r=1}^M \frac{\prod_{s=1}^N (h_r - \lambda_s)}{h_r^2 \prod_{t \neq r} (h_r - h_t)}, \quad (43)$$

Comparing coefficients in front of powers of  $\lambda$  in (40, 41) we obtain a linear system for the variables  $p_s = (1 + \lambda_s^2) P_s$ :

$$\sum_{s=1}^N \left( 1 + \lambda_s^2 \right) \lambda_s^{l-1} P_s = \Lambda_{NM} d_l, \quad l = 0, 1, \dots, N - 1. \quad (44)$$

. This system has Vandermonde determinant, which is positive if the eigenvalues are enumerated in increasing order  $W(\lambda_1, \lambda_2, \dots, \lambda_N) = \prod_{s>t} |\lambda_s - \lambda_t| := W$ . The solution  $\{p_s\}$  is unique and is presented by the Cramer formula. Denote by  $W_{NM}^s$  the determinant with the column  $(1, \lambda_s, \lambda_s^2, \lambda_s^3, \dots, \lambda_s^{N-1})$  replaced by the column  $(1, d_1, \dots, d_{N-1})$ . Then the parameter  $\Lambda_{NM}$  is defined from the normalization condition  $[e, e] = \sum_{s=1}^N P_s = \rho = \pm 1$ , see Lemma I2.3 :

$$\Lambda_{NM} = \rho \left( \sum_{s=1}^N (1 + \lambda_s^2)^{-1} W^{-1} W_{NM}^s \right)^{-1} \quad (45)$$

The role of fitting parameters will play :

1. The coefficients  $\frac{r_0}{2}, \frac{1}{a}$  in front of the powers  $k^2, k^0$  in the Laurent expansion

$$F(k) = 4\pi k_0 \left\{ \gamma_0 - \frac{|\gamma_{01}|^2}{D(\lambda)} \right\} = 4\pi k_0 \left\{ \gamma_0 - \frac{|\gamma_{01}|^2}{\Lambda_{NM}} \left[ \lambda^{(N-M)} + \dots + q_1 \lambda + q_0 + \frac{q_{-1}}{\lambda} \dots \right] \right\} \quad (46)$$

of the function  $F(k)$  at infinity, with  $\lambda = k^2 k_0^{-2}$ .

2. The eigenvalues  $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_N$  of the inner Hamiltonian.
3. The zeroes  $h_1, h_2, \dots h_M$  of  $D(\lambda)$ .

We assume that the above data are *consistent* in the following sense:

$$\text{sign } r_0 = -\text{sign } \Lambda_{NM} q_1 \quad (47)$$

**Lemma 4.1.** *If the consistency condition (47) is fulfilled, then the boundary parameters  $\gamma_{st}$  and weights  $P_s = J_{ss} |e^s|^2$  of the model  $\mathcal{A}_\Gamma$  are defined in several steps:*

1. Calculate the weights  $P_s = J_{ss} |e^s|^2 = \Lambda_{NM} W^s W^{-1} (1 + \lambda_s^2)^{-1}$  via solving the system (44).

2. Define the boundary parameters  $\gamma_{00}, |\gamma_{01}|^2$ , for given  $r_0, a$  from the equations:

$$\frac{r_0}{2} = -4\pi k_0^{-1} |\gamma_{01}|^2 \Lambda_{NM}^{-1} q_1 - \frac{1}{a} = 4\pi k_0 \left[ \gamma_{00} - |\gamma_{01}|^2 \Lambda_{NM}^{-1} q_0 \right]. \quad (48)$$

3. Set  $\gamma_{11} = -\Lambda_{NM} \sum_s \lambda_s W_{NM}^s W^{-1} (1 + \lambda_s^2)^{-1}$ .

Note that the poles  $h_r$  of  $F$  are the points where the scattering matrix is equal to 1. In real physical problems in 3-d space this is never observed for finite real energy. In [25] we suggested extending of this observation to complex values of energy in form of *the principle of analyticity*, assuming that  $F$  is analytic on the whole plane  $\lambda$ . In finite-dimensional case this means that  $F$  is just a polynomial degree  $N, M = 0$ . The corresponding operator  $\mathcal{A}_\Gamma$  is called “the special zero-range model”. Arranging the eigenvalues of the inner Hamiltonian in increasing order and denoting by  $W_s$  the Vandermond determinant with  $\lambda_s$  just omitted, we obtain via direct calculation an expression for  $P_s$ :

$$(1 + \lambda_s^2) P_s = \frac{(-1)^s}{\prod_{t(t \neq s)} |\lambda_t - \lambda_s|} \Lambda, \quad s = 1, 2, \dots N,$$

where

$$\Lambda^{-1} = \sum_{s=1}^N \frac{(-1)^s}{\prod_{t(t \neq s)} |\lambda_t - \lambda_s|} \frac{\rho}{1 + \lambda_s^2}. \quad (49)$$

If the non-degeneracy condition

$$\Xi := \sum_{s=1}^N \frac{(-1)^s}{\prod_{t(t \neq s)} |\lambda_t - \lambda_s|} \frac{1}{1 + \lambda_s^2} \neq 0$$

is fulfilled,  $\Lambda \Xi = \rho$ , then we can fit the model based on the above equations. Then returning to the dimensional wave-numbers  $k$  and spectrum  $k_s^2$  we obtain the following statement:

**Theorem 4.2.** *For scattering systems which can be modelled by special zero-range potentials the only fitting parameters are: the scattering length, the effective radius, and the spectrum of the inner Hamiltonian. The components of the metric tensor for those models are defined as  $J_{ss} = (-1)^s \text{sign } r_0$ . Other essential parameters of the model like: boundary parameters  $\gamma_{00}, \gamma_{11}, |\gamma_{01}|$ , and components  $|e^s|^2$  of the*

deficiency vector are defined as functions of the scattering length  $a$ , effective radius  $r_0$ , and  $\lambda_s$  ( $s = 1, 2, \dots, N$ )

$$a = \left( -4\pi\gamma_{00}k_0 + \frac{4\pi|\gamma_{01}|^2}{k_0^{2N-1}\Lambda} \prod_{s=1}^N k_s^2 \right)^{-1}, \quad r_0 = \frac{8\pi|\gamma_{01}|^2}{\Lambda k_0^{2N-1}} \left( \prod_{t=1}^N k_t^2 \right) \sum_{s=1}^N k_s^{-2}. \quad (50)$$

In particular because of  $\rho = \pm 1$  we can always choose  $\Lambda > 0$ . Hence in this case the effective radius is positive ( $r_0 > 0$ ) and  $g_{ss} = (-1)^s$ . Introducing the notations  $\varepsilon = 4\pi\gamma_{00}k_0$ ,  $\gamma = 4\pi|\gamma_{01}|^2 k_0^{-2N+1} \Lambda^{-1}$  we obtain for the function  $F(k)$  the following representation:

$$F(k) = k \cot \delta(k) = \varepsilon - \gamma \prod_{s=1}^N (k_s^2 - k^2). \quad (51)$$

Sign of the effective radius is the same as the sign  $\gamma$ :

$$a = \left( -\varepsilon + \gamma \prod_{s=1}^N k_s^2 \right)^{-1}, \quad r_0 = 2\gamma \sum_{s=1}^N \prod_{t(t \neq s)} k_t^2. \quad (52)$$

**Corollary 4.3.** *Using (37,51) and above parameters  $\varepsilon$ ,  $\gamma$ , we may obtain a convenient representation for the scattering matrix  $S$  in terms of the resonance parameters  $k_s$ :*

$$S(k) = 1 + \frac{2ik}{\varepsilon - ik - \gamma \prod_{s=1}^N (k_s^2 - k^2)}. \quad (53)$$

The total scattering cross-section is  $\sigma(k) = 4\pi|f(k)|^2$  or can be written in explicit form (see next section) using equations (8) and (51). This expression for the scattering matrix in  $s$ -channel describes the resonance scattering of particles with resonances defined by the spectral properties of the Hamiltonian  $\mathcal{H}$  of the inner degrees of freedom.

### Re-normalization.

We already proved that our resonance scattering model depends only the scattering length  $a$ , effective radius  $r_0$ , the spectrum  $k_s$  ( $s = 1, 2, \dots, N$ ) of the inner Hamiltonian and the typical wave number. Now we introduce instead of the scattering length and effective radius a new parameter  $a_0$  with dimension ( $[a_0] = cm$ ) and a dimensionless parameter  $\alpha$  by the formulae:

$$k_0 := -\frac{1}{4\pi\gamma_{00}a_0}, \quad \alpha := -\frac{4\pi|\gamma_{01}|^2}{\Lambda} (4\pi\gamma_{00})^{2N-1}, \quad (54)$$

Then the function  $F(k) = k \cot \delta(k)$  for our zero-range potential is:

$$F(k) = -\frac{1}{a_0} - \frac{\alpha}{a_0} \prod_{s=1}^N (a_0^2 k_s^2 - a_0^2 k^2). \quad (55)$$

The effective radius and the scattering length can be found from (55) as first coefficients of the polynomial  $F(k) = -a^{-1} + \frac{r_0}{2}k^2$ :

$$\frac{r_0}{2} = \alpha a_0 \sum_{n=1}^N \prod_{s(s \neq n)}^N a_0^2 k_s^2, \quad a = \frac{a_0}{1 + \alpha \prod_{s=1}^N a_0^2 k_s^2}. \quad (56)$$

Then the parameter  $a_0$  can be interpreted as a *non-re-normalized scattering length* for the zero-range potential without inner structure, which corresponds to  $\alpha = 0$ . The above equation connects the re-normalized scattering length  $a$  to the *non-re-normalized scattering length*  $a_0$ , taking into account resonance scattering.

Thus the function  $F(k)$  given by Eq. (55) depends on the typical wave-number and  $N + 2$  parameters:  $a_0, \alpha$  and  $k_s, s = 1, 2, \dots, N$ . Consequently, the non-re-normalized scattering length  $a_0$ , the dimensionless parameter  $\alpha$  and the spectrum  $k_s (s = 1, 2, \dots, N)$  of the Hamiltonian define the scattering length  $a$  and effective radius  $r_0$  of the model.

### Resonance cross-section.

The “total” cross-section for spherically-symmetric scattering is generally calculated as

$$\sigma(k) = 4\pi|f(k)|^2 = \frac{4\pi}{|F(k) - ik|^2}.$$

This gives due to (55) an explicit formula for total cross-section of the special zero-range model:

$$\sigma(k) = \frac{4\pi a_0^2}{1 + a_0^2 k^2 + 2\alpha \prod_{s=1}^N (a_0^2 k_s^2 - a_0^2 k^2) + \alpha^2 \prod_{s=1}^N (a_0^2 k_s^2 - a_0^2 k^2)^2}. \quad (57)$$

One can see from (57,55) that the maxima of the total cross-section  $\sigma(k)$  are shifted from the eigenvalues of the inner Hamiltonian. They can be interpreted again as *re-normalized* eigenvalues of the inner Hamiltonian. The re-normalization is caused by the interaction introduced via the boundary condition (26). Note that the final formula obtained via substitution of (55) into (57) is not a phenomenological formula, but an exact formula derived for certain Hamiltonian. The corresponding solvable model has equal rights with other quantum solvable models, but unlike them it may have resonances at positive energy, the sign  $\pm$  of the effective radius defined by the spectrum of the inner Hamiltonian via (49) and allows complete fitting of all parameters from the experimental data, thus prescribing them a certain physical meaning.

### References

1. F.A. Berezin and L.D. Faddeev, *A remark on Schrödinger equation with a singular potential*, Soviet Math. Dokl. **2** (1961), 372–376.
2. E. Fermi, *Sul motto dei neutroni nelle sostanze idrogenate*, (in Italian), Richerka Scientifica, **7** (1936), 13.
3. J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Twelfth printing, Princeton Landmarks in Mathematics, Princeton Paperbacks, Princeton University Press, Princeton, NJ, 1996.
4. S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, Springer-Verlag, New York, 1988.
5. E.P. Wigner, *On a class of analytic functions from the quantum theory of collisions*, Annals of mathematics, **2** (1951), N53, 36.
6. J. Sylvester and G. Uhlmann, *The Dirichlet to Neumann map and applications*, in Proceedings of the Conference “Inverse Problems in Partial Differential Equations (Arcata,1989)”, SIAM, Philadelphia, vol. 101, 1990.

7. B. Pavlov, *A model of zero-range potential with an internal structure*, (in Russian), *Teor. Mat. Fizika*, **59** (1984), 345–353. English Translation : *Theor. and Math. Physics* **59** (1984), 544–550.
8. M. Krein, *Concerning the resolvents of an Hermitian operator with deficiency index  $(m, m)$* , (in Russian), *Doklady AN USSR*, **52** (1946), 651.
9. B. Pavlov, *The theory of extensions and explicitly solvable models*, (in Russian), *Uspekhi Mat. Nauk*, **42** (1987), 99. English translation in : *Russian Math. Surveys* 42:6 (1987), 127–168.
10. S.P. Novikov, *Schrödinger operators on graphs and symplectic geometry*, Proceedings of the Fields Institute Conference in Honour of the 60th Birthday of Vladimir I. Arnol'd, (E. Bierstone, B. Khesin, A. Khovanskii and J. Marsden, eds), *Communications of Fields Institute*, AMS, 1999.
11. P. Exner and P. Sheba, *A new type of quantum interference transistor*, *Phys. Lett. A*, **129** (N:8,9) (1988), 477–480 .
12. V.M. Adamyn and B.S. Pavlov, *Zero-radius potentials and M.G.Krein's formula for generalized resolvents*, *Zap. Nauch. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, Issled. Linein. Teor. Functsii. XV, 186, **149** (1986), 7–23.
13. S. Albeverio and P. Kurasov, *Singular Perturbations of Differential Operators*, London Math. Society Lecture Note Series 271. Cambridge University Press, 2000.
14. L. Landau and E. Lifschitz, *Physique Theoretique ("Landau-Lifschitz")*, Tome III: *Mechanique Quantique. Theorie non Relativiste*, (French), Deuxieme edition. Mir, Moscow, 1967.
15. K.Makarov, V. Melezhik and A. Motovilov, *Point interactions in the problem of three quantum particles with internal structure*, *Teor. Mat. Fiz.* **102** (2), 258–282. Translation in : *Theor.and Math. Phys.* **102** (1995), 188–207.
16. Communicated by S.P. Novikov.
17. N.I. Akhiezer and I.M. Glazman, *Theory of Linear Operators in Hilbert Space*, Frederick Ungar Publ., New York, vol. 1, 1966.
18. Yu. N. Demkov, and V.N. Ostrovskij, *Zero-Range Potentials and Their Applications in Atomic Physics*, Plenum Press, NY-London, 1988 .
19. I.S. Iohvidov, M.G. Krein and H. Langer, *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric*, *Mathematical Research*, 9. Akademie-Verlag, Berlin, 1982.
20. M. Krasnosel'skij, *On selfadjoint extensions of Hermitian operators*, (in Russian), *Ukr. Matem. Journ*, **1** (1949), 21–38.
21. V.I. Gorbachuk, *Boundary problems for differential- operator equations*, (in Russian), Kiev, Naukova Dumka, (1984), 284.
22. E.C. Titchmarsh, *Eigenfunctions Expansions Associated with Second Order Differential Equations*, Vol. 1, Clarendon Press, Oxford, 1962.
23. P. Lax and R. Phillips, *Scattering Theory*, Academic press, New York, 1967.

24. B. Pavlov, *Boundary conditions on thin manifolds and the semiboundedness of the three-particle Schroedinger operator with pointwise potential*, Math. USSR Sbornik, **64** (1) (1989), 161–175.
25. V. Kruglov and B. Pavlov, *Operator extension techniques for resonance scattering of neutrons by nuclei*, Hadronic Journal, **26** (2005), 259–286.

Boris S. Pavlov  
Department of Mathematics  
The University of Auckland  
Private Bag 92019  
Auckland  
NEW ZEALAND  
pavlov@math.auckland.ac.nz

Vladimir I. Kruglov  
Department of Physics  
The University of Auckland  
Private Bag 92019  
Auckland  
NEW ZEALAND  
vik@phy.auckland.ac.nz