Approximation of Functions Belonging to the Weighted $L(\alpha, M, \omega)$-Class by Trigonometric Polynomials

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Abstract. In this work the approximation of the functions by means $t_n(f; x)$, $N^\alpha_n(f; x)$ and $R^\alpha_n(f, x)$ of the trigonometric Fourier series in weighted Orlicz spaces with Muckenhoupt weights are studied.

1. Introduction, Some Auxiliary Results and Main Results

Let $T$ denote the interval $[-\pi, \pi]$, $C$ the complex plane, and $L_p(T)$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $T$. A convex and continuous function $M : [0, \infty) \to [0, \infty)$ which satisfies the conditions

$$M(0) = 0, \quad M(x) > 0 \text{ for } x > 0,$$

$$\lim_{x \to 0} \frac{M(x)}{x} = 0; \quad \lim_{x \to \infty} \frac{M(x)}{x} = \infty,$$

is called a Young function. We will say that $M$ satisfies the $\Delta_2$-condition if $M(2u) \leq cM(u)$ for any $u \geq u_0 \geq 0$ with some constant $c$ independent of $u$.

We can consider a right continuous, monotone increasing function $\rho : [0, \infty) \to [0, \infty)$ with

$$\rho(0) = 0; \quad \lim_{t \to \infty} \rho(t) = \infty \quad \text{and} \quad \rho(t) > 0 \text{ for } t > 0,$$

then the function defined by

$$N(x) = \int_0^{[x]} \rho(t) \, dt$$

is called an $N$-function. For a given Young function $M$, let $\tilde{L}_M(T)$ denote the set of all Lebesgue measurable functions $f : T \to C$ for which

$$\int_T M(|f(x)|) \, dx < \infty.$$

The complementary $N$-function to $M$ is defined by

$$N(y) := \max_{x \geq 0} (xy - M(x)), \quad \text{for } y \geq 0.$$

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Let $N$ be the complementary Young function of $M$. It is well-known [22, p. 60], [39, p. 52-68] that the linear span of $\tilde{L}_M(\mathbb{T})$ equipped with the Orlicz norm

$$\|f\|_{L_M(\mathbb{T})} := \sup \left\{ \int_{\mathbb{T}} |f(x)g(x)| \, dx : g \in \tilde{L}_N(\mathbb{T}), \int_{\mathbb{T}} N(|g(x)|) \, dx \leq 1 \right\},$$

or with the Luxemburg norm

$$\|f\|_{L_M(\mathbb{T})}^* := \inf \left\{ k > 0 : \int_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right) \, dx \leq 1 \right\},$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an Orlicz space [22, p. 26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T}), 1 < p < \infty$. If $M(x) = M(x, p) := x^p$, $1 < p < \infty$, then Orlicz spaces $L_M(\mathbb{T})$ coincide with the usual Lebesgue spaces $L_p(\mathbb{T}), 1 < p < \infty$. Note that the Orlicz spaces play an important role in many areas such as applied mathematics, mechanics, regularity theory, fluid dynamics and statistical physics (e.g., [1], [9], [31] and [41]). Therefore, of investigation the approximation of the functions by means of Fourier trigonometric series in Orlicz spaces is also important in these areas of research.

The Luxemburg norm is equivalent to the Orlicz norm. The inequalities

$$\|f\|_{L_M(\mathbb{T})}^* \leq \|f\|_{L_M(\mathbb{T})} \leq 2 \|f\|_{L_M(\mathbb{T})}^*, \quad f \in L_M(\mathbb{T}),$$

hold [22, p. 80].

If we choose $M(u) = u^p/p, 1 < p < \infty$ then the complementary function is $N(u) = u^{p/q}$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p} \|u\|_{L_p(\mathbb{T})} = \|u\|_{L_M(\mathbb{T})}^* \leq \|u\|_{L_M(\mathbb{T})} \leq q^{1/q} \|u\|_{L_{p'}(\mathbb{T})},$$

where $\|u\|_{L_{p'}(\mathbb{T})} = \left( \int_{\mathbb{T}} |u(x)|^{p'} \, dx \right)^{1/p}$ stands for the usual norm of the $L_{p'}(\mathbb{T})$ space.

If $N$ is complementary to $M$ in Young’s sense and $f \in L_M(\mathbb{T}), g \in L_N(\mathbb{T})$ then the so-called strong H"older inequalities [22, p. 80]

$$\int_{\mathbb{T}} |f(x)g(x)| \, dx \leq \|f\|_{L_M(\mathbb{T})} \|g\|_{L_N(\mathbb{T})},$$

are satisfied.

The Orlicz space $L_M(\mathbb{T})$ is reflexive if and only if the $N$–function $M$ and its complementary function $N$ both satisfy the $\Delta_2$–condition [39, p. 113].

Let $M^{-1} : [0, \infty) \rightarrow [0, \infty)$ be the inverse function of the $N$–function $M$. The lower and upper indices [4, p. 350]

$$\alpha_M := \lim_{t \rightarrow +\infty} \frac{-\log h(t)}{\log t}, \quad \beta_M := \lim_{t \rightarrow 0^+} \frac{-\log h(t)}{\log t},$$

of the function
\[ h : (0, \infty) \to (0, \infty], \quad h(t) := \lim_{y \to \infty} \sup_{y \geq t} \frac{M^{-1}(y)}{M^{-1}(ty)}, \quad t > 0, \]
first considered by Matuszewska and Orlicz [29], are called the Boyd indices of the Orlicz spaces \( L_M(T) \).

It is known that the indices \( \alpha_M \) and \( \beta_M \) satisfy \( 0 \leq \alpha_M \leq \beta_M \leq 1 \), \( \alpha_M + \beta_M = 1 \), \( \alpha_M + \beta_N = 1 \) and the space \( L_M(T) \) is reflexive if and only if \( 0 < \alpha_M \leq \beta_M < 1 \).

The detailed information about the Boyd indices can be found in [5-8], [30].

A measurable function \( \omega : \mathbb{T} \to [0, \infty] \) is called a weight function if the set \( \omega^{-1}([0, \infty)) \) has Lebesgue measure zero. With any given weight \( \omega \) we associate the \( \omega \)-weighted Orlicz space \( L_M(T, \omega) \) consisting of all measurable functions \( f \) on \( \mathbb{T} \) such that
\[ \|f\|_{L_M(T, \omega)} := \|f\omega\|_{L_M(T)}. \]

Let \( 1 < p < \infty \), \( 1/p + 1/p' = 1 \) and let \( \omega \) be a weight function on \( \mathbb{T} \). \( \omega \) is said to satisfy Muckenhoupt’s \( A_p \)-condition on \( \mathbb{T} \) if
\[ \sup_{J} \left( \frac{1}{|J|} \int_{J} \omega^p(t) \, dt \right)^{1/p} \left( \frac{1}{|J|} \int_{J} \omega^{-p'}(t) \, dt \right)^{1/p'} < \infty, \]
where \( J \) is any subinterval of \( \mathbb{T} \) and \( |J| \) denotes its length [32].

Let us indicate by \( A_p(\mathbb{T}) \) the set of all weight functions satisfying Muckenhoupt’s \( A_p \)-condition on \( \mathbb{T} \).

Let further \( t_1, t_2, \ldots, t_n \) be distinct points on \( \mathbb{T} \) and let \( \lambda_1, \ldots, \lambda_n \) be real numbers. If \( 1 < p < \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( -\frac{1}{p} < \lambda_j < \frac{1}{q} \), \( j = 1, \ldots, n \) then the weight function
\[ \omega(\tau) := \prod_{j=1}^{n} |\tau - t_j|^\lambda, \quad (\tau \in \mathbb{T}) \]
belongs to \( A_p(\mathbb{T}) \).

According to [27], [28, Lemma 3.3], and [28, Section 2.3] if \( L_M(\mathbb{T}) \) is reflexive and the weighted function \( \omega \) satisfies the condition \( \omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T}) \), then the space \( L_M(\mathbb{T}, \omega) \) is also reflexive.

Let \( L_M(T, \omega) \) be a weighted Orlicz space, let \( 0 < \alpha_M \leq \beta_M < 1 \) and let \( \omega \in A_{1/\alpha_M}(\mathbb{T}) \cap A_{1/\beta_M}(\mathbb{T}) \). For \( f \in L_M(\mathbb{T}, \omega) \) we set
\[ (\nu_h f)(x) := \frac{1}{2h} \int_{-h}^{h} f(x + t) \, dt, \quad 0 < h < \pi, \quad x \in \mathbb{T}. \]

By reference [16, Lemma 1], the shift operator \( \nu_h \) is a bounded linear operator on \( L_M(\mathbb{T}, \omega) \):
\[ \|\nu_h (f)\|_{L_M(\mathbb{T}, \omega)} \leq c \|f\|_{L_M(\mathbb{T}, \omega)}. \]

The function
\[ \Omega_{M, \omega}(\delta, f) := \sup_{0 < h \leq \delta} \|f - (\nu_h f)\|_{L_M(\mathbb{T}, \omega)}, \quad \delta > 0, \]
is called the modulus of continuity of \( f \in L_M(\mathbb{T}, \omega) \).
It can easily be shown that $\Omega_{M, \omega}(\cdot, f)$ is a continuous, nonnegative and non-decreasing function satisfying the conditions
\[
\lim_{\delta \to 0} \Omega_{M, \omega}(\delta, f) = 0, \quad \Omega_{M, \omega}(\delta, f + g) \leq \Omega_{M, \omega}(\delta, f) + \Omega_{M, \omega}(\delta, g)
\]
for $f, g \in L_M(\mathbb{T}, \omega)$.

Let $0 < \alpha \leq 1$. The set of functions $f \in L_M(\mathbb{T}, \omega)$ such that
\[
\Omega_{M, \omega}(f, \delta) = O(\delta^\alpha), \quad \delta > 0
\]
is called the \textit{weighted Lipschitz class} $\text{Lip}(\alpha, M, \omega)$. Let
\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx)
\]
be the Fourier series of the function $f \in L^1(\mathbb{T})$, where $a_k(f)$ are the Fourier coefficients of the function $f$. The $n$-th partial sums, Cesaro means of the series (1) are defined, respectively, as
\[
s_n(x, f) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k(f) \cos kx + b_k(f) \sin kx)
\]
\[
\sigma_n(x, f) = \frac{1}{n+1} \sum_{m=0}^{n} s_m(x, f).
\]
Let $\sum_{n=0}^{\infty} a_n$ be an infinite series and let $\{s_n\}$ its $n$th partial sum. Let $\{t_n\}$ be a sequence of $(N, p, q)$ means of the sequence $\{s_n\}$. We define transform $(N, p, q)$ of $\{s_n(f; x)\}$ by [42]
\[
t_n(x) := t_n(f; x) := \frac{1}{r_n} \sum_{m=0}^{n} p_{n-m} q_m s_m(f; x),
\]
where
\[
r_n := \sum_{m=0}^{n} p_m q_{n-m} \neq 0, n \geq 0, \text{ and } p_{-1} = q_{-1} = r_{-1} = 0.
\]
Let $\{p_n\}$ be a real sequence, where $p_0 > 0$, $p_n \geq 0$ for $n > 0$. As in [2] we define
\[
p^\beta_m = \sum_{\nu=0}^{m} A_{m-\nu} P_\nu; \quad P^\beta_n = \sum_{m=0}^{n} p^\beta_m, \quad P^\beta_{-1} = p_{-1} = 0, \quad i \geq 1,
\]
where
\[
A^\beta_0 = 1; \quad A^\beta_n = \frac{(\beta + 1)(\beta + 2)(\beta + 3)\ldots(\beta + n)}{n!}, \quad \beta > -1, \quad n = 1, 2, 3, \ldots.
\]
In the proof of the main result we will use the notations
\[
\Delta \beta_n := \beta_n - \beta_{n+1}, \quad \Delta_m \beta(n, m) := \beta(n, m) - \beta(n, m + 1).
\]
We define the sequence
\[ p_n^\beta \] and the conditions
\[ \Delta \nu \] defines the
\[ \nu \] of the functions by trigonometric polynomials in the different spaces have
\[ \alpha \] approximations of the functions by trigonometric polynomials in the different spaces have
\[ \beta \] been investigated by several authors (see, for example, [42]).
\[ 42-45 \]

In the present paper we study the approximation of the functions by trigonometric polynomials, \( t_n(f;x) \), \( R_n^\beta(f;x) \) and \( N_n^\beta(f;x) \) in weighted Orlicz spaces. The results obtained in this work, are generalization of the results [13] and [42] to more general summability and weighted Orlicz spaces. Similar problems about approximations of the functions by trigonometric polynomials in the different spaces have been investigated by several authors (see, for example, [2], [3], [10-21], [23-26], [33-38], [40] and [42-45]).

Note that, in the proof of the main results we use the method as in the proof of [42].

Our main results are the following.

**Theorem 1.** Let \( L_M(T) \) be a reflexive Orlicz space and \( \omega \in A_{1/\alpha,M}(T) \cap A_{1/\beta,M} \) and the conditions
\[ (i) \quad n^2q_n = O(r_n), \]
\[ (ii) \quad \sum_{m=0}^{n-1} m^{2-\alpha} |\Delta_m(p_{n-m} q_m)| = O(r_n n^{-\alpha}), \]
are satisfied, then if \( f \in \text{Lip}(\alpha,M,\omega) \), \( 0 < \alpha \leq 1 \) the estimate
\[ \|t_n(\cdot,f) - f\|_{L_M(T,\omega)} = O(n^{-\alpha}), \]
holds.

**Theorem 2.** Let \( L_M(T) \) be a reflexive Orlicz space and \( \omega \in A_{1/\alpha,M}(T) \cap A_{1/\beta,M} \), and let \( \{p_n^\beta\} \) be a monotonic sequence such that
\[ (n+1)p_n^\beta = O(p_n^\beta). \]
Then for every \( f \in \text{Lip}(\alpha,M,\omega) \), \( 0 < \alpha \leq 1 \) the estimate
\[ \|f - R_n^\beta(\cdot,f)\|_{L_M(T,\omega)} = O(n^{-\alpha}), \quad n = 1, 2, \ldots \]
holds.

**Theorem 3.** Let \( L_M(T) \) be a reflexive Orlicz space and \( \omega \in A_{1/\alpha,M}(T) \cap A_{1/\beta,M} \), and let \( \{p_n^\beta\} \) be a sequence of positive real numbers such that
\[ \sum_{m=0}^{n-1} \frac{P_m^\beta}{m+1} - \frac{P_m^\beta}{m+2} = O\left(\frac{P_n^\beta}{n+1}\right). \]
Then for every \( f \in \text{Lip}(\alpha, M, \omega) \), \( 0 < \alpha \leq 1 \) the estimate
\[
\| f - N_n^2(\cdot, f) \|_{L_M(T,\omega)} = O(n^{-\alpha}), \; n = 1, 2, ...
\]
holds.

In the proof of the main result we need the following lemmas.

**Lemma 1 ([14]).** Let \( L_M(T) \) be a reflexive Orlicz space and let \( \omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M} \). Then for \( f \in L_M(T,\omega) \), the estimate
\[
\| f - \sigma_n(\cdot, f) \|_{L_M(T,\omega)} = O(n\Omega_M,\omega\left(\frac{1}{n}, f\right)), \; n = 1, 2, ...
\]
holds.

**Lemma 2 ([42]).** Let \( \{p_n^3\} \) be a monotonic sequence of positive numbers. Then,
\[
\sum_{m=1}^{n} m^{-\alpha}p_{n-m} = O\left(n^{-\alpha}P_n^3\right)
\]
for \( 0 < \alpha < 1 \).

2. Proofs of the Main Results

**Proof of Theorem 1.** By definition of \( t_n(f; x) \) and \( \sigma_n(f; x) \) we have [42, p. 1581]
\[
t_n(x,f) - f(x) = \frac{1}{r_n} \sum_{m=0}^{n} p_{n-m}q_m(s_m(x,f) - f(x))
= \frac{1}{r_n} \left\{ \sum_{m=0}^{n-1} \Delta_m(p_{n-m}q_m) \sum_{k=0}^{m} (s_k(x,f) - f(x)) \right\}
+ \frac{1}{r_n} \left\{ p_0q_0 \sum_{k=0}^{n} (s_k(x,f) - f(x)) \right\}
= \frac{1}{r_n} \left\{ \sum_{m=0}^{n-1} (m+1)\Delta_m(p_{n-m}q_m)(\sigma_m(x,f) - f(x)) \right\}
+ \frac{1}{r_n} \{(n+1)p_0q_n(\sigma_n(x,f) - f(x))\}.
\] (6)
By (6), conditions (2), (3) and Lemma 1

\[
\|t_n(\cdot, f) - f\|_{L_M(\mathbb{T}, \omega)} = O \left( \frac{1}{r_n} \sum_{m=0}^{n-1} (m+1)|\Delta_m(p_n-mq_m)| \|\sigma_m(\cdot, f) - f\|_{L_M(\mathbb{T}, \omega)} \right)
\]

which completes the proof.

**Proof of Theorem 2.** Case 1. We suppose that $0 < \alpha < 1$. It is clear that

\[
f(x) - R_n^\beta(x, f) = \frac{1}{P_n^\beta} \sum_{m=0}^{n} p_n^\beta f - s_m(x, f)\]

By [14, p. 8, relation (13)] the relation

\[
\|f - s_n(\cdot, f)\|_{L_M(\mathbb{T}, \omega)} = O \left( \Omega_{M, \omega} \left( \frac{1}{n}, f \right) \right)
\]

holds. Using (7), Lemma 2 and condition (4) we find

\[
\|f - R_n^\beta(\cdot, f)\|_{L_M(\mathbb{T}, \omega)} \leq \frac{1}{P_n^\beta} \sum_{m=0}^{\lambda(n)} p_n^\beta \|f - s_m(\cdot, f)\|_{L_M(\mathbb{T}, \omega)}
\]

\[
= \frac{1}{P_n^\beta} \sum_{m=1}^{n} p_n^\beta O \left( m^{-\alpha} \right) \|f - s_m(\cdot, f)\|_{L_M(\mathbb{T}, \omega)}
\]

\[
+ \frac{1}{P_n^\beta} \|f - s_0(\cdot, f)\|_{L_M(\mathbb{T}, \omega)}
\]

\[
= \frac{1}{P_n^\beta} O \left( n^{-\alpha} P_n^\beta \right) + O \left( \frac{1}{n+1} \right) = O \left( n^{-\alpha} \right).
\]

Case 2. Let $\alpha = 1$. Since

\[
R_n^\beta(x, f) = \frac{1}{P_n^\beta} \sum_{m=0}^{n} p_n^\beta A_m(x, f)
\]
using Abel’s transformation, we have

\[ s_n(x, f) - R_\beta^n(x, f) = \frac{1}{P^n} \sum_{m=1}^{n} \left( P_\beta^n - P_{n-m} \right) A_m(x, f) \]

\[ = \frac{1}{P^n} \sum_{m=1}^{n} \left( \frac{P_\beta^n - P_{n-m}}{m} - \frac{P_\beta^n - P_{n-(m+1)}}{m} \right) \left( \sum_{k=1}^{m} kA_k(x, f) \right) \]

\[ + \frac{1}{n+1} \sum_{k=1}^{n} kA_k(x, f). \] (8)

Then taking account of (8)

\[ \| s_n(\cdot, f) - R_\beta^n(\cdot, f) \|_{L_M(T, \omega)} \leq \frac{1}{P^n} \sum_{m=1}^{n} \left| \frac{P_\beta^n - P_{n-m}}{m} - \frac{P_\beta^n - P_{n-(m+1)}}{m} \right| \left( \sum_{k=1}^{m} kA_k(\cdot, f) \right)_{L_M(\cdot, \omega)} \]

\[ + \frac{1}{n+1} \left( \sum_{k=1}^{n} kA_k(\cdot, f) \right)_{L_M(\cdot, \omega)}. \] (9)

It is clear that if the Fourier series of \( f \) is

\[ f(x) \sim \sum_{k=0}^{m} A_k(x, f), \]

then \( \tilde{f}' \) has the Fourier series

\[ \tilde{f}'(x) \sim \sum_{k=0}^{m} kA_k(x, f), \]

where \( \tilde{f}' \) is the conjugate function of \( f' \in L_M(T, \omega)[16, p.166] \). Using boundedness of the partial sums and the conjugation operator in the space \( L_M(T, \omega)[16, p.155] \), we get

\[ \frac{1}{n+1} \left( \sum_{k=1}^{n} kA_k(\cdot, f) \right)_{L_M(\cdot, \omega)} = \frac{1}{n+1} \| s_n(\cdot, \tilde{f}') \|_{L_M(\cdot, \omega)} = O(n^{-1}). \] (10)

Thus, (9) and (10) yield

\[ \| s_n(\cdot, f) - R_\beta^n(\cdot, f) \|_{L_M(\cdot, \omega)} \leq \frac{1}{P^n} \sum_{m=1}^{n} \left| \frac{P_\beta^n - P_{n-m}}{m} - \frac{P_\beta^n - P_{n-(m+1)}}{m} \right| O(1) + O(n^{-1}) \]

\[ = O\left( \frac{1}{P^n} \sum_{m=1}^{n} \left| \frac{P_\beta^n - P_{n-m}}{m} - \frac{P_\beta^n - P_{n-(m+1)}}{m} \right| \right) + O(n^{-1}). \] (11)
According to [42] the following relations holds:
\[
\sum_{m=1}^{n} \left| \frac{P_n^\beta - P_{n-m}^\beta}{m} - \frac{P_n^\beta - P_{n-(m+1)}^\beta}{m} \right| = \frac{1}{n+1} O(P_n^\beta).
\]

The last inequality and (11) imply that
\[
\|s_n(\cdot,f) - R_n^\beta(\cdot,f)\|_{L_M(T,\omega)} = O(n^{-1}). \tag{12}
\]

By (12) and (7)
\[
\|f - N_n^\beta(\cdot,f)\|_{L_M(T,\omega)} \leq \|f - s_n(\cdot,f)\|_{L_M(T,\omega)} + \|s_n(\cdot,f) - R_n^\beta(\cdot,f)\|_{L_M(T,\omega)} = O(n^{-1}),
\]
which completes the proof. \[\square\]

**Proof of Theorem 3.** Case 1. We suppose that \(0 < \alpha < 1\). Since
\[
N_n^\beta(f;x) = \frac{1}{P_n^\beta} \sum_{m=0}^{n} p_m^\beta s_m(f;x),
\]
we can write
\[
f(x) - N_n^\beta(f;x) = \frac{1}{P_n^\beta} \sum_{m=0}^{n} p_m^\beta \{f(x) - s_m(f;x)\}. \tag{13}
\]

Use of (13) and (7) gives us
\[
\|f - N_n^\beta(\cdot,f)\|_{L_M(T,\omega)} \leq \frac{1}{P_n^\beta} \sum_{m=0}^{n} p_m^\beta \|f - s_m(\cdot,f)\|_{L_M(T,\omega)}
\]
\[
= O\left(\frac{1}{P_n^\beta}\right) \sum_{m=1}^{n} p_m^\beta m^{-\alpha} + \frac{P_0^\beta}{P_n^\beta} \|f - s_0(\cdot,f)\|_{L_M(T,\omega)}
\]
\[
= O\left(\frac{1}{P_n^\beta}\right) \sum_{m=1}^{n} p_m^\beta m^{-\alpha}. \tag{14}
\]

Considering [42, p. 1583]
\[
\sum_{m=1}^{n} p_m^\beta m^{-\alpha} = O(n^{-\alpha} P_n^\beta). \tag{15}
\]

Taking into account the relations (14) and (15) we have
\[
\|f - N_n^\beta(\cdot,f)\|_{L_M(T,\omega)} = O(n^{-\alpha}).
\]

Case 2. Let \(\alpha = 1\). Note that by Abel’s transformation
\[
N_n^\beta(f;x) = \frac{1}{P_n^\beta} \sum_{m=0}^{n-1} p_m^\beta \left\{s_m(x,f) - s_{m+1}(x,f) + P_n^\beta s_n(x,f)\right\}
\]
\[
= \frac{1}{P_n^\beta} \sum_{m=0}^{n-1} p_m^\beta (-A_{m+1}(x,f)) + s_n(x,f). \tag{16}
\]
Taking account of (16)

\[ N_\beta^n(x, f) - s_n(x, f) = - \frac{1}{P_\beta^n} \sum_{m=0}^{n-1} P_\beta^m A_{m+1}(x, f). \]  \hspace{1cm} (17)

On the other hand by Abel’s transformation [42, p. 1584] the equality

\[
\sum_{m=0}^{n-1} P_\beta^m A_{m+1}(x, f) \\
= \sum_{m=0}^{n-1} \frac{P_\beta^m}{m+1}(m+1)A_{m+1}(x, f) \\
= \sum_{m=0}^{n-1} \left[ \frac{P_\beta^m}{m+1} - \frac{P_\beta^{m-1}}{m+2} \right] \sum_{k=0}^{m}(k+1)A_{k+1}(x, f) \\
+ \frac{P_\beta^n}{n+1} \sum_{k=0}^{n-1} (k+1)A_{k+1}(x, f),
\]

holds. Using the last equality, condition (5) and (10) we reach

\[
\left\| \sum_{m=0}^{n-1} P_\beta^m A_{m+1}(\cdot, f) \right\|_{L_M(T, \omega)} \\
\leq \sum_{m=0}^{n-1} \left| \frac{P_\beta^m}{m+1} - \frac{P_\beta^{m-1}}{m+2} \right| \left\| \sum_{k=0}^{m}(k+1)A_{k+1}(\cdot, f) \right\|_{L_M(T, \omega)} \\
+ \frac{P_\beta^n}{n+1} \left\| \sum_{k=0}^{n-1} (k+1)A_{k+1}(\cdot, f) \right\|_{L_M(T, \omega)} \\
= O(1) \sum_{m=0}^{n-1} \left| \frac{P_\beta^m}{m+1} - \frac{P_\beta^{m-1}}{m+2} \right| + O\left( \frac{P_\beta^n}{n} \right) = O\left( \frac{P_\beta^n}{n} \right).
\]  \hspace{1cm} (18)

Consideration of (17) and (18) gives us

\[
\left\| N_\beta^n(\cdot, f) - s_n(\cdot, f) \right\|_{L_M(T, \omega)} \\
= \frac{1}{P_\beta^n} \left\| \sum_{m=0}^{n-1} P_\beta^m A_{m+1}(\cdot, f) \right\|_{L_M(T, \omega)} = \frac{1}{P_\beta^n} O\left( \frac{P_\beta^n}{n} \right) = O(n^{-1}).
\]  \hspace{1cm} (19)

Taking the relation (19) and (7) into account we have

\[
\left\| f - N_\beta^n(\cdot, f) \right\|_{L_M(T, \omega)} \\
\leq \left\| f - s_n(\cdot, f) \right\|_{L_M(T, \omega)} + \left\| N_\beta^n(\cdot, f) - s_n(\cdot, f) \right\|_{L_M(T, \omega)} = O(n^{-1}).
\]

The theorem is proved. \[\Box\]

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