ON EXTENSION AND STRUCTURE OF GENERALIZED DERIVATIONS

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Abstract. We are concerned with the extension problem and the nature of generalized derivations of an underlying associative algebra \( A \) on a field \( k \) to the corresponding double centralizer algebra and to the unitized algebra. This investigation will be made in the frame of some general associative algebras. We shall also seek on the continuity of generalized derivations on normed algebras.

1. Introduction

In this article we are concerned with the extension problem of generalized derivations of an underlying associative algebra \( A \) on a field \( k \) to the corresponding double centralizer algebra \( D(A) \). Here \( D(A) \) consists of pairs of linear maps \((L, R)\) on \( A \) so that \( L(ab) = L(a)b, R(ab) = aR(b) \) and \( aL(b) = R(a)b \) for all \( a, b \in A \). It becomes a unital algebra if it is endowed with pointwise linear operations and with the multiplication \((L_1, R_1)(L_2, R_2) \triangleq (L_1L_2, R_2R_1)\) for all \((L_1, R_1), (L_2, R_2) \in D(A)\). There is a natural homomorphism \( h : A \rightarrow D(A) \) given as \( h(a) = (L_a, R_a) \), where \( L_a(b) \triangleq ab \) and \( R_a(b) \triangleq ba \) for all \( a, b \in A \). The kernel of \( h \) is the so called annihilator ideal of \( A \), usually written as \( \text{ann}(A) \). Thus, \( \text{ann}(A) = \{a \in A : aA \cap Aa = \{0\}\} \). The double centralizer algebra of \( A \) embraces relevant information of \( A \). For instance, the range of \( h \) is an ideal and it is surjective if and only if \( A \) is unital, in which case \( h \) is an isomorphism. If the annihilator ideal of \( A \) is zero then \( A \) is abelian if and only if \( D(A) \) is abelian (cf. [10], 1.2, p. 24). The commutative version of this idea was first introduced by S. Helgason under the name of the algebra of multipliers [7].

As an associative algebra \( A \) has a Lie algebra structure with respect to the Lie bracket defined as \([a, b] \triangleq ab - ba\) for all \( a, b \in A \). Consequently, the anticommutativity law and the Jacobi identity hold on \( A \), i.e. \([a, a] = 0\) and \([[b, c], a] + [[c, a], b] + [[a, b], c] = 0\) for all \( a, b, c \in A \), respectively. If \( A \) is also a Banach space, following [11] we will say that \( A \) is a Banach-Lie algebra if \( ||[a, b]\| \leq 2||a|| ||b|| \) for all \( a, b \in A \). As usual, by a derivation \( \delta : A \rightarrow A \) we mean a linear mapping satisfying \( \delta(ab) = \delta(a)b + a\delta(b) \) for all \( a, b \in A \). This concept has been generalized in many ways. In particular, in this article we will focus our attention in generalized derivations on \( A \), i.e. pairs \((f, d)\) of linear mappings on \( A \) so that \( f(ab) = f(a)b + ad(b) \) for all \( a, b \in A \). The subject in the context of prime rings and noncommutative Banach algebras was raised in [6] and [8]; for studies in the setting of semiprime rings the reader can see [1], [2], [4], [5]; for studies

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concerning generalized Jordan derivations see [9]. By $D(A)$ and $\Delta(A)$ we denote the vector space of derivations and generalized derivations on $A$. Thus $D(A)$ become a non associative algebra on $k$ with respect to the current Lie product of two derivations. Likewise $\Delta(A)$ is a Lie algebra if for $(f_1,d_1),(f_2,d_2) \in \Delta(A)$ we write $[(f_1,d_1),(f_2,d_2)] \equiv [([f_1,f_2],[d_1,d_2])]$. The mapping $\delta \rightarrow (\delta,\delta)$ defines a natural injection $D(A) \hookrightarrow \Delta(A)$. If $A$ is a complete norm algebra $D(A)$ will be the Banach-Lie algebra of bounded derivations on $A$, $\mathcal{D}(A)$ been also a Banach algebra if we set $\| (L,R) \| \equiv \max \{ \| L \|, \| R \| \}$ for each $(L,R) \in \mathcal{D}(A)$.

In the Section 2 the reader can see a few examples of generalized derivations on some associative algebras. Sections 3 and 4 are devoted to the extension of generalized derivations together with their structures to the unitization and to the double commutant of an associative algebra, respectively. Generalized derivations in normed algebras will be considered in Section 5.

2. A Few Examples

Example 1. Let $f_{a,b}(x) \equiv ax + xb$ for $a,b,x \in A$. So $f_{a,b}$ is called the gene-ralized inner derivation defined by $a$ and $b$. Indeed, $f_{a,b}(xy) = f_{a,b}(x)y + x[y,b]$ for all $x,y \in A$, where $[y,b] \equiv yb - by$ denotes the usual Lie product of $y$ and $b$. Thus $(f_{a,b},[a,b]) \in \Delta(A)$.

Example 2. Let us only demand additivity in the restricted context of a ring $A$ so that $A_A = \{0\}$ (i.e. if $Aa = \{0\}$ implies $a = 0$), for example if $A$ is a semiprime ring. Then, if $(f,d) \in \Delta(A)$ then $d$ is a derivation ([3], Remark 1).

Example 3. $(L,0) \in \Delta(A)$, where $L$ is a left multiplier $L$ of $A$, i.e. a linear mapping $L$ on $A$ such that $L(ab) = L(a)b$ for all $a,b \in A$.

Example 4. Let $A$ be a unital associative algebra. Given $a \in A$ and $\delta \in D(A)$ then $(\delta + L_a,\delta) \in \Delta(A)$. Further, by the Example 2 any generalized derivation on $A$ has that form.

Example 5. Let $A = k[X]$ be the algebra of polynomials on a field $k$, $n \in \mathbb{N}$ and $d \in D(A)$. Then $d(X^n) = nX^{n-1}d(X)$ and $d$ is uniquely determined by $d(X)$. Thus, by the Ex. 4 there is clearly a bijection $\Delta(k(X)) \approx k(X) \times k(X)$.

Example 6. Following the Example 5 let $B = \{ p \in k(X) : p(0) = 0 \}$. Now, $B$ is a non unital algebra over the field $k$. Given $(f,d) \in \Delta(B)$ then $f(p) = (f(X) - d(X))\frac{p}{X} + d(p)$ for all $p \in B$. It is easy to see that the mapping $(f,d) \rightarrow (f(X),d(X))$ defines a bijection between $\Delta(B)$ and $B \times B$.

Example 7. Let $A$ be an associative algebra so that $A_A = \{0\}$. If $f \in \mathcal{L}(A)$, there is $\delta \in \mathcal{L}(A)$ so that $(f,\delta) \in \Delta(A)$ if and only if for any $b \in A$ there is a unique $\theta \in A$ so that $[f,R_b] = R_b \theta$. For, if $(f,\delta) \in \Delta(A)$ clearly $[f,R_b] = R_b \delta(b)$ for all $b \in A$ and the condition is necessary because $A$ has null right annihilator. On the other hand, if the condition holds let $\delta(b) \equiv b$ for $b \in A$. It is easy to see that $\delta \in \mathcal{L}(A)$ and $(f,\delta) \in \Delta(A)$.
3. GD’s on Unitized Algebras

**Theorem 8.** Let $\mathcal{A}$ be an associative algebra on a field $k$ with null annihilator such that $\mathcal{A}^2 = \mathcal{A}$. We shall consider the cartesian product $\mathcal{A}^2 \triangleq \mathcal{A} \times k$ endowed with the usual algebraic structure that realizes it as the unitization of $\mathcal{A}$. Let also $j : \mathcal{A} \rightarrow \mathcal{A}^2$, $p_1 : \mathcal{A}^2 \rightarrow \mathcal{A}$ and $p_2 : \mathcal{A}^2 \rightarrow k$ so that $j(a) \triangleq (a,0)$, $p_1(a,\lambda) \triangleq a$ and $p_2(a,\lambda) \triangleq \lambda$ respectively, where $a \in \mathcal{A}$ and $\lambda \in k$.

(i): The mapping

$$G : \Delta(\mathcal{A}^2) \rightarrow \Delta(\mathcal{A}), G(D + L_{(a,\lambda)}, D) \triangleq (p_1Dj + L_{a+\lambda}, p_1Dj)$$

is well defined and it is a monomorphism of algebras.

(ii): Let $(f, d) \in \Delta(\mathcal{A})$. The following assertions are equivalent:

(a): $(f, d)$ admits a natural extension to an element of $\Delta(\mathcal{A}^2)$, i.e. there is $(f^\sharp, d^\sharp) \in \Delta(\mathcal{A}^2)$ such that $p_1f^\sharp j = f$ and $p_1d^\sharp j = d$.

(b): There exist $a_0 \in \mathcal{A}, \lambda_0 \in k$ and $d \in D(\mathcal{A})$ so that $f = L_{a_0+\lambda_0} + d$.

(c): $(f, d) \in \text{Im}(G)$.

**Proof.** (i): Given $(D + L_{(a,\lambda)}, D) \in \Delta(\mathcal{A}^2)$ we know that $D \in D(\mathcal{A}^2)$.

Since $p_2$ becomes an algebraic homomorphism if $(x,\alpha), (y, \beta) \in \mathcal{A}^2$ we have

$$(p_2D)(xy + \beta x + \alpha y, \alpha\beta) = (p_2D)((x,\alpha)(y,\beta))$$

$$= p_2(D(x,\alpha)(y,\beta) + (x,\alpha)D(y,\beta))$$

$$= (p_2D)(x,\alpha)\beta + \alpha(p_2D)(y,\beta)$$

$$= (p_2D)(\beta x + \alpha y, 2\alpha\beta),$$

i.e. $(xy,-\alpha\beta) \in \ker(p_2D)$ for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in k$. Since $\mathcal{A}^2 = \mathcal{A}$ and $k$ is a field we conclude that $p_2D = 0$. Thus, if $x, y \in \mathcal{A}$ we see that

$$(p_1Dj)(xy) = (p_1Dj)(x) y + (p_2D)(x,0) y + x(p_1Dj)(y) + x(p_2D)(y,0)$$

$$= (p_1Dj)(x) y + x(p_1Dj)(y),$$

i.e. $p_1Dj \in D(\mathcal{A})$. It is immediate that $G$ is well defined, as well as that $G$ is a linear injective mapping. The verification that $G$ is a homomorphism is direct once one observe the following facts: If $D \in D(\mathcal{A}^2)$ then $Djp_1 = D$, $p_2Dj = 0_{L(\mathcal{A})}$ and $p_1Dj \in D(\mathcal{A})$. For, the first claim follows because $D(0,1) = (0,0)$. As $j$ is a homomorphism if $y, z \in \mathcal{A}$ we obtain

$$(p_1(D(yz,0)), p_2(D(yz,0))) = D(yz,0)$$

$$= (Dj)(yz)$$

$$= D(j(y)j(z))$$

$$= D(j(y))j(z) + j(y)D(j(z)).$$

Since $j(\mathcal{A})$ is an ideal in $\mathcal{A}^2$ by (1) we get that $(p_2Dj)(yz) = 0$. As $\mathcal{A}^2 = \mathcal{A}$ then $p_2Dj = 0_{L(\mathcal{A})}$ and (1) again yields $p_1Dj \in D(\mathcal{A})$.

(ii): (a $\Rightarrow$ b) Let $f^\sharp(0,1) \triangleq (a_0,\lambda_0)$ in $\mathcal{A}^2$, $p_2f^\sharp \triangleq s$ and $p_2d^\sharp \triangleq t$ in $L(\mathcal{A}, k)$.

Hence, if $(a, \lambda) \in \mathcal{A}^2$ we have

$$f^\sharp(a, \lambda) = (f(a) + \lambda a_0, s(a) + \lambda \lambda_0) \quad \text{and} \quad d^\sharp(a, \lambda) = (d(a), t(a)).$$
As \((f^2, d^2) \in \Delta (\mathcal{A}^2)\) the following equations hold if \((a, \lambda), (b, \mu) \in \mathcal{A}^2:\)

\[
\lambda f (b) = \lambda a_0 b + (s (a) + \lambda \lambda_0) b + at (b) + \lambda d (b),
\]

(2)

\[
\lambda t (b) = s (ab) + \mu s (a) + \lambda s (b).
\]

(3)

Letting \(b = 0\) in (3) then \(\mu s (a) = 0\) for any \(\mu \in k\) and \(a \in \mathcal{A}\), i.e. \(s = 0\). So, by (3) again we see that \(t = 0\). Now the claim follows from (2).

(a \Rightarrow b): Let \(f = L_{a_0 + \lambda_0} + d\) for some \(a_0 \in \mathcal{A}\) and \(\lambda_0 \in k\). If \((x, \lambda) \in \mathcal{A}^2\) let us write

\[
f^2 (x, \lambda) \equiv ((a_0 + \lambda_0) x + d(x) + \lambda a_0, \lambda_0),
\]

\[
d^2 (x, \lambda) = (d(x), 0).
\]

It is straightforward to see that \((f^2, d^2) \in \Delta (\mathcal{A}^2)\) and \((f, d) = G (f^2, d^2)\).

(b \Rightarrow c): If \((f, d) = G (D + L_{(a, \lambda)}, D)\) for some \((D + L_{(a, \lambda)}, D) \in \Delta (\mathcal{A}^2)\) it is easy to see that \(D + L_{(a, \lambda)}\) and \(D\) are natural extensions of \(f\) and \(d\) respectively. \(\square\)

4. GD’s on Double Algebras

**Theorem 9.** Let \(\mathcal{A}\) be an associative algebra with null annihilator such that \(\mathcal{A}^2 = \mathcal{A}\). Then there a monomorphism of algebras \(F : \Delta (\mathcal{D} (\mathcal{A})) \to \Delta (\mathcal{A})\), which is surjective if \(\mathcal{A}\) is unital.

**Proof.** By Ex. 4 any \(r \in \Delta (\mathcal{D} (\mathcal{A}))\) has the form \(r = (\delta + L_{(L_0, R_0)}, \delta)\), where \(\delta \in D (\mathcal{D} (\mathcal{A}))\) and \((L_0, R_0) \in \mathcal{D} (\mathcal{A})\) are uniquely determined by \(r\). Since \(\mathcal{A}^2 = \mathcal{A}\) and \(h (\mathcal{A})\) is a bilateral ideal in \(\mathcal{D} (\mathcal{A})\) then \((h (\mathcal{A})) \cup L_{(L_0, R_0)} (h (\mathcal{A})) \subseteq h (\mathcal{A})\).

Further, as \(\mathcal{A}\) has null annihilator we can define

\[
F (r) \equiv (h^{-1} (\delta + L_{(L_0, R_0)}) h, h^{-1} \delta h).
\]

Since \((L_0, R_0) h (a) = h (L_0 (a))\) for all \(a \in \mathcal{A}\) it is straightforward to see that \(F (r) \in \Delta (\mathcal{A})\) and \(F\) is a monomorphism of algebras. With the above notation let \(\delta (L, R) = ((\delta_L (L, R), \delta_R (L, R)))\) for \((L, R) \in \mathcal{D} (\mathcal{A})\). If \(F (r) = (0, 0)\) and \(a \in \mathcal{A}\) we see that

\[
0 = \delta (L, R) h (a) = h (L_\delta (L, R) (a))
\]

and so \(L_\delta (L, R) (a) = 0\). Thus \(L_\delta (L, R) = 0\) for all \((L, R) \in \mathcal{D} (\mathcal{A})\), i.e. \(L_\delta = 0\).

Analogously, \(R_\delta = 0\), i.e. \(\delta = 0\). Thus \(L_{(L_0, R_0)} = 0\) and hence \((L_0, R_0) = (0, 0)\).

Consequently, \(r\) is injective.

Let us assume that \(\mathcal{A}\) has a unit \(e\) to prove that then \(F\) becomes surjective. For, let \((f, d) \in \Delta (\mathcal{A})\) and given \((L, R) \in \mathcal{D} (\mathcal{A})\) let \(D (L, R) \equiv ([d, L], [d, R])\). Clearly
[\mathcal{D}(\mathbb{A})]. \text{ Since } d \text{ becomes a derivation on } \mathbb{A} \text{ if } a, b \in \mathbb{A} \text{ we have}
\begin{align*}
a [d, L] (b) &= a d (L(b)) - aL (d(b)) \\
    &= d (a L(b)) - d(a) L(b) - aL(d(b)) \\
    &= d (R(a)b) - R(d(a))b - R(a)d(b) \\
    &= [d, R] (a)b,
\end{align*}
\begin{align*}
[d, L] (ab) &= d (L (a) b) - L (d (ab)) \\
    &= d (L (a)) b + L (a) d(b) - L (d (a)) b - L (a) d(b) \\
    &= [d, L] (a) b,
\end{align*}
\begin{align*}
[d, R] (ab) &= d (aR(b)) - R (d (ab)) \\
    &= d (a) R (b) + ad (R (b)) - d (a) R (b) - aR (d (b)) \\
    &= a [d, R] (b),
\end{align*}
i.e. \( D (L, R) \in \mathcal{D}(\mathbb{A}) \). \text{ Thus, } D \text{ defines a mapping of } \mathcal{D}(\mathbb{A}) \text{ into } \mathcal{D}(\mathbb{A}) \text{ which is clearly linear. Further, if } (L_1, R_1), (L_2, R_2) \in \mathcal{D}(\mathbb{A}) \text{ then}
\begin{align*}
D ((L_1, R_1), (L_2, R_2)) &= ([d, L_1] L_2 + [d, R_2] R_1 + R_2 [d, R_1]) \\
    &= ([d, L_1] L_2 + [d, R_2] R_1) + (L_1 [d, L_2], [d, R_2] R_1) \\
    &= D ((L_1, R_1)) (L_2, R_2) + (L_1, R_1) (L_2, R_2),
\end{align*}
i.e. \( D \) is a derivation on \( \mathcal{D}(\mathbb{A}) \). \text{ It is readily seen that } h^{-1} Dh = d. \text{ Now, let us look for a solution } (L_0, R_0) \in \mathcal{D}(\mathbb{A}) \text{ of the equation } f = h^{-1} (D + L(L_0, R_0)) h. \text{ For, let } L_0 \triangleq f - d \text{ and } R_0 \triangleq R_{f(e)} \text{ in } \mathcal{L}(\mathbb{A}). \text{ Certainly, if } a, b \in \mathbb{A} \text{ then}
\begin{align*}
al L_0 (b) &= a (f (b) - d(b)) = a (f (e) b - ed (b) - d (b)) = R_{f(e)} (a) b = R_0 (a) b, \\
L_0 (ab) &= f(a)b + ad (b) - d(a)b - ad(b) = L_0 (a) b, \\
R_0 (ab) &= (ab) f(e) = a (bf(e)) = aR_0 (b)
\end{align*}
i.e. \((L_0, R_0) \in \mathcal{D}(\mathbb{A})\). \text{ Finally, if } a \in \mathbb{A} \text{ we have}
\begin{align*}
(h^{-1} (D + L(L_0, R_0)) h) (a) &= (d + L_0) (a) = f(a),
\end{align*}
and \( F (D + L(L_0, R_0), D) = (f, d) \).
\begin{proof}
(i): \text{ Let } \mathbb{A} \text{ be an associative Banach algebra on a field } k \text{ of complex or real numbers with null right annihilator and let } (f, d) \in \Delta(\mathbb{A}). \text{ If } f \in \mathcal{B}(\mathbb{A}) \text{ then } d \in \mathcal{B}(\mathbb{A}).
\begin{enumerate}[i)]
\item \text{ Let } \Delta_0 (\mathbb{A}) \text{ be the set of elements } (f, d) \in \Delta(\mathbb{A}) \text{ so that } f \text{ is bounded.}
\item \text{ Then, } \Delta_0 (\mathbb{A}) \text{ is a Banach-Lie subalgebra of } \mathcal{B}(\mathbb{A}) \oplus \mathcal{B}(\mathbb{A}).
\item \text{ If besides } \mathbb{A}^2 = \mathbb{A} \text{ the monomorphism } F \text{ of Theorem 9 is continuous.}
\item \text{ If besides } \mathbb{A}^2 = \mathbb{A} \text{ the monomorphism } G \text{ of Theorem 8 is continuous.}
\end{enumerate}
\begin{proof}
(i): \text{ If } (a_n, d(a_n)) \to (0, b) \text{ in } \mathbb{A} \times \mathbb{A} \text{ and } x \in \mathbb{A} \text{ then}
\begin{align*}
x b &= \lim_{n \to \infty} x d(a_n) = \lim_{n \to \infty} \lfloor f (xa_n) - f (x)a_n \rfloor = 0,
\end{align*}
\end{proof}
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\end{document}
i.e. \( b \in A \). Thus \( b = 0 \), the separating space of \( d \) becomes trivial and so \( d \in B(A) \).

(ii): It is straightforward.

(iii): First, \( F[\Delta_b(D(A))] \subseteq \Delta_b(A) \). For, if \( (\delta + L_{(L_n, R_n)}, \delta) \in \Delta_b(D(A)) \) then \( \delta \) becomes a bounded derivation on \( D(A) \). By the closed graph theorem \( h^{-1} \delta h \in B(A) \). So \( h^{-1} \delta h + L_0 \in B(A) \) and as

\[
h^{-1}(\delta + L_{(L_n, R_n)})h = h^{-1} \delta h + L_0
\]

then \( F((\delta + L_{(L_n, R_n)}, \delta)) \in \Delta_b(A) \). Now, let \( (r_n, F(r_n)) \to (0_{\Delta_b(D(A))}, s) \) in \( \Delta_b(D(A)) \times \Delta_b(A) \). If \( n \in N \) we can represent \( r_n = (\delta_n + L_{(L_n, R_n)}, \delta_n) \), with \( \delta_n \in D(D(A)) \) and \( (L_n, R_n) \in D(A) \). Besides let us set \( s = (f, d) \) in \( \Delta_b(A) \). As convergence in \( \Delta_b(A) \) implies convergence in each coordinate and \( F(r_n) \to s \), \( h^{-1} \delta_n h + L_n \to f \) and \( h^{-1} \delta_n h \to d \). Consequently, \( L_n \to f - d \) in \( B(A) \). But the uniform convergence implies strong convergence and \( h \) becomes bounded. Thus, if \( x \in A \) we see that

\[
\lim_{n \to \infty} L_{(L_n, R_n)}(h(x)) = \lim_{n \to \infty} h(L_n(x)) = h(f(x) - d(x)).
\]

Since \( r_n \to 0_{\Delta_b(D(A))} \) then \( \delta_n \to 0 \) and \( L_{(L_n, R_n)} \to 0 \) in \( B(D(A)) \). So, by (4) we obtain that \( h(f(x) - d(x)) = 0 \). Since \( h \) is injective and \( x \) is arbitrary then \( f = d \). Moreover,

\[
0 = \lim_{n \to \infty} \delta_n(h(x)) = \lim_{n \to \infty} h((h^{-1} \delta_n h)(x)) = h(d(x))
\]

for all \( x \in A \) and so \( s = 0_{\Delta_b(A)} \). Thus \( F \in B(\Delta_b(D(A)), \Delta_b(A)) \) since its separating space is trivial.

(iv): It is immediate that \( G[\Delta_b(A^2)] \subseteq \Delta_b(A) \). Let \( (\eta_n, G(\eta_n)) \to (0_{\Delta_b(A^2)}, \mu) \) in \( \Delta_b(A^2) \times \Delta_b(A) \). Given \( n \in N \) let \( \eta_n = (D_n + L_{(a_n, \lambda_n)}, D_n) \), where \( D_n \in D(A^2) \) and \( (a_n, \lambda_n) \in A^2 \). As \( \{D_n + L_{(a_n, \lambda_n)}\}_{n \in N} \subseteq B(A^2) \) certainly \( \{D_n\}_{n \in N} \subseteq B(A^2) \). If \( \mu = (f, d) \) then \( d = \lim_{n \to \infty} p_1 D_n/\mu = 0_{B(A)} \) and \( f(x) = \lim_{n \to \infty} (a_n + \lambda_n) x \) for all \( x \in A \). Further, we see that \( L_{(a_n, \lambda_n)} \to 0_{B(A^2)} \) and so

\[
0 = \lim_{n \to \infty} \|L_{(a_n, \lambda_n)}\|_{B(A^2)} = \lim_{n \to \infty} \|(a_n, \lambda_n)\|_{A^2} = \lim_{n \to \infty} (\|a_n\| + |\lambda_n|).
\]

So \( f = 0_{B(A)} \), i.e. \( \mu = 0_{\Delta_b(A)} \). Since the separating ideal of \( G \) becomes trivial the claim follows.

\( \square \)

Remark 11. The result of Th.10(i) is the best we can assert. In general, it is not true that if \((f, d) \in \Delta(A) \) and \( d \in B(A) \) then \( f \) must be bounded, even in the context of commutative Banach algebras with null annihilator. For, consider the Banach algebra \( L^1(\mathbb{R}) \) with the convolution product and define \( f(x) = x \), where the domain \( D(f) \) of \( f \) is the linear subspace of \( L^1(\mathbb{R}) \) of absolutely continuous functions on \( \mathbb{R} \). Moreover, \( D(f) \) is an ideal of \( L^1(\mathbb{R}) \). For, let \( x \in D(f) \) and \( y \in L^1(\mathbb{R}) \). There is \( x \in L^1(\mathbb{R}) \) so that \( x(t) = \int_{-\infty}^t x \) almost everywhere \( t \in \mathbb{R} \). In particular,
if \( t \in \mathbb{R} \) by Fubini’s Theorem we have

\[
\int \int_{(-\infty,t) \times (-\infty, +\infty)} |\chi(u) y(u+v)| \, du \times dv = \int_{-\infty}^{t} |\chi(u)| \int_{-\infty}^{+\infty} |y(u+v)| \, dvdu \\
\leq \int_{-\infty}^{t} |\chi(u)| \, du \|y\|_1 \\
\leq \|\chi\|_1 \|y\|_1 < +\infty.
\]

Therefore, the following identity

\[
(x \ast y)(t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} \chi(u+v) \tilde{y}(u) \, dudv
\]

holds for almost all \( t \in \mathbb{R} \), where \( \tilde{y}(t) \triangleq y(-t) \) for \( t \in \mathbb{R} \). As the function \( v \to \int_{-\infty}^{+\infty} \chi(u+v) \tilde{y}(u) \, du \) belongs to \( L^1(\mathbb{R}) \) then \( x \ast y \in D(f) \). Now we see that \( f(x \ast y) = \chi \ast y = f(x) \ast y \) in \( L^1(\mathbb{R}) \). So we may consider \((f, 0) \in \Delta (L^1(\mathbb{R}))\) but \( f \) is not bounded although it is closed.

References

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