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ANY PSEUDO–pG–COMPACT GROUP G IS FINE

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Abstract. A topological group G is called *fine* if every continuous real-valued function on G is left uniformly continuous. If G is a non-discrete group, the cardinal pG is the minimal number of open subsets of G whose intersection is not open. Given a cardinal κ, a space X is *pseudo-κ-compact* if every discrete family of non-empty open subsets of X has cardinality $< \kappa$. We prove that a group G is fine whenever it is pseudo-pG-compact; we apply this fact to give a complete answer to several questions of Comfort and Hager.
A topological group G is called *left uniformly $\mathbb{R}$-factorizable* if, for every left uniformly continuous function $f : G \to \mathbb{R}$, there exists a continuous homomorphism $\pi : G \to M$ of the group G onto a second countable group M such that $f = g \circ \pi$ for some left uniformly continuous function $g : M \to \mathbb{R}$. We prove that a topological group is left uniformly $\mathbb{R}$-factorizable if and only if it is ω-narrow ($\equiv \aleph_0$-bounded).

1. Introduction

It is a classical result of Calculus that any continuous real-valued function on a compact space is uniformly continuous. This result formally involves metrics but it is in fact topological because there is only one uniformity on any compact space. Thus the fact that every real-valued continuous function is uniformly continuous on a space X could mean that X has some compactness property. In the general case we would need to specify a uniformity on X so the result would not be a topological one.

The situation is different in topological groups because they have natural left-sided and right-sided uniform structures. Since these structures carry essential information about G, left/right uniformly continuous real-valued mappings on topological groups were intensively studied (see [**10**], [**11**], [**5**], [**1**], [**2**]) as well as topological groups on which every continuous real-valued function is uniformly continuous (it turns out that it does not matter which uniformity we consider for the definition of this class [**5**]). The groups on which every real-valued continuous function is uniformly continuous were called *groups with property U* in [**5**] and [**12**], and *groups with property* (K) in [**1**]. Adopting terminology well-established in the theory of uniform spaces Comfort and Hager later renamed them *fine* in [**6**] and we will use this terminology.

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It is easy to see that every pseudocompact group as well as every discrete group is fine [**5**]. It was shown in [**5**] that every fine group is either pseudocompact or a P-space so if we are interested in studying fine groups, we must study P-groups, i.e. the groups which are P-spaces. Given a non-discrete topological group G, let $pG = \min\{|\mathcal{U}| : \mathcal{U}$ is a family of open subsets of G and $\bigcap \mathcal{U}$ is not open$\}$. Comfort and Hager discovered in [**6**] that the cardinal function pG is crucial for deciding if a group G is fine. They conjectured that a non-discrete group G is fine if and only if it is pseudo-pG-compact which means $|\mathcal{U}| < pG$ for any locally finite (or, equivalently, discrete) family of non-empty open subsets of G.

Comfort and Hager proved in [**6**] that pseudo-pG-compactness is sufficient for the group G to be fine if the identity of G is an intersection of not more than pG open sets and found some other properties, which, taken together with pseudo-pG-compactness, imply that G is a fine group. They also gave a proof that every fine group G is pseudo-pG-compact but it has a gap[1] so no implication has been established for the general case so far. Alas proved in [**2**] that if a group is fine then it is pseudo-pG-compact provided the cardinal pG is not strongly inaccessible. She also established that pseudo-pG-compactness is equivalent to the group G being fine if G is paracompact (see [**3**]).

The main result of our paper is to show that any pseudo-pG-compact topological group G is fine. This answers Question (1) in [**6**] and shows that a group G is fine if and only if it is pseudo-pG-compact in case pG is not strongly inaccessible. We also show that some examples of Tkachenko [**17**] give answers to Questions (2), (3) and (5) from [**6**]. Tkachenko asked in [**17**, Problem 4.21] whether every Lindelöf P-group G is κ-monolithic for each cardinal κ, i.e. $nw(\overline{A}) \leq \kappa$ for every $A \subset G$ with $|A| \leq \kappa$. We show that there are models of ZFC in which not every Lindelöf P-group is ω_1-monolithic.

A group G is called left uniformly $\mathbb{R}$-factorizable if for every left uniformly continuous real-valued function $f : G \to \mathbb{R}$ there exists a second countable group M, a continuous homomorphism $p : G \to M$ and a left uniformly continuous map $g : M \to \mathbb{R}$ such that $f = g \circ p$. If we omit the uniformity of all maps in this definition, we obtain the concept of an $\mathbb{R}$-factorizable group introduced by Tkachenko [**15**]. While it is known that every $\mathbb{R}$-factorizable group is $\aleph_0$-bounded and many other results are obtained about $\mathbb{R}$-factorizable groups, there is no complete internal description of this property. We give a complete characterization of left uniformly $\mathbb{R}$-factorizable groups proving that a group is uniformly $\mathbb{R}$-factorizable if and only if it is $\aleph_0$-bounded, i.e., for any open neighbourhood U of the identity e of the group G, there exists a countable set $A \subset G$ such that $A \cdot U = G$. This shows, in particular, that if we define right uniformly $\mathbb{R}$-factorizable groups in an analogous way, then we obtain the same class.

To avoid confusion with the topological concept of ω-bounded space (a topological space X is called ω-bounded if the closure of any countable subset of X is compact), we call $\aleph_0$-bounded groups *ω-narrow*. This change of terminology has been introduced by leading specialists on topological groups among which are Arhangel'skii and Tkachenko.

[1]the authors were informed about this gap by Professor W.W. Comfort

2. Notation and Terminology

All topological spaces (as well as the spaces of topological groups) are assumed to be Tychonoff. The word "group" means "a topological group". Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\varnothing\}$; if $x \in X$ then $\tau(x, X) = \{U \in \tau(X) : x \in U\}$. Given a group G, the symbol e_G denotes the identity of G. If κ is an infinite cardinal, a space X is called *P_κ-space* if any intersection of $\leq \kappa$ open subsets of X is open in X. The space X is a $P_{<\kappa}$-space if $\bigcap \mathcal{U}$ is open in X for any collection $\mathcal{U} \subset \tau(X)$ with $|\mathcal{U}| < \kappa$. The P_ω-spaces have traditionally been called P-spaces. Given an infinite cardinal κ, a space X is *pseudo-κ-compact* if $|\mathcal{U}| < \kappa$ for any discrete family $\mathcal{U} \subset \tau^*(X)$. The space X is *κ-compact* if every open cover of X has a subcover of cardinality $< \kappa$.

Given a space X and $x \in X$, let $\chi(x, X)$ be the minimal cardinality of the local bases of X at the point x; *The character* $\chi(X)$ of a space X is the cardinal $\sup\{\chi(x, X) : x \in X\}$. If X is a set and $A \subset X$ then *the characteristic function* χ_A of the set A is defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ for all $x \in X\backslash A$. A space X is *left-separated* if there exists a well-ordering $<$ on X such that $\{y \in X : y < x\}$ is closed in X for any $x \in X$.

Given a topological group G, a function $f : G \to \mathbb{R}$ is called *left (right) uniformly continuous* if, for any $\varepsilon > 0$, there exists an open neighbourhood U of e_G such that $x^{-1}y \in U$ ($xy^{-1} \in U$) implies $|f(x) - f(y)| < \varepsilon$ for any $x, y \in G$. A group G is $\mathbb{R}$-factorizable if, for any continuous function $f : G \to \mathbb{R}$, there exists a second countable group H, a continuous homomorphism $h : G \to H$ and a continuous function $g : H \to \mathbb{R}$ such that $f = g \circ h$. A group G is called *ω-narrow* if, for any open neighbourhood U of e_G, there exists a countable set $A \subset G$ such that $A \cdot U = G$. The rest of our terminology is standard and can be found in [**7**].

3. Fine Groups and Pseudo-κ-Compactness

Definition 3.1. A group G is called *fine* if every continuous real-valued function on G is left uniformly continuous.

Fine groups were first studied by Kister [**12**] who called them *groups with property U*. Later Comfort and Ross [**5**] defined the property BU: a group G has property BU if every *bounded* real-valued continuous function on G is left uniformly continuous. In the same paper Comfort and Ross proved that the properties U and BU coincide. It is worth mentioning that, equivalently, we can define a fine group G by requiring that every continuous real-valued function be right uniformly continuous on G (see [**5**]). We will need several easy facts about pseudo-κ-compact groups and fine groups. The following Fact was proved in [**6**] and [**1**].

Fact 3.2.

(1) The cardinal pG is regular for any group G;

(2) if a P-group G is fine and $pG = \kappa$ then $|G/H| < \kappa$ for any open subgroup H of the group G;

(3) if X is a pseudo-κ-compact $P_{<\kappa}$-space for some infinite regular cardinal κ then any decreasing family $\{U_\alpha : \alpha < \kappa\}$ of non-empty clopen subsets of X has non-empty intersection.

The proof of the following proposition can be found in [**6**, Lemma 2.6(a)].

Proposition 3.3. *Let G be a non-discrete pseudo-κ-compact P-group. Then $|G/H| < \kappa$ for every open subgroup H of the group G.*

Proposition 3.4. *Let G be a non-discrete group. If there is an infinite cardinal κ such that G is a pseudo-κ-compact $P_{<\kappa}$-space then $pG = \kappa$. As a consequence, a non-discrete group G is pseudo-pG-compact if and only if there exists an infinite cardinal κ such that G is a pseudo-κ-compact $P_{<\kappa}$-space.*

Proof. If G is pseudo-pG-compact then, for $\kappa = pG$, the group G is a $P_{<\kappa}$-space. Now assume that G is a pseudo-κ-compact $P_{<\kappa}$-space for some infinite cardinal κ. Since it is evident that $pG \geq \kappa$, to show that $pG = \kappa$ it suffices to prove that G can not be a P_κ-space.

Suppose that G is a P_κ-space. Then we can construct a family $\{U_\alpha : \alpha < \kappa\}$ of clopen neighbourhoods of the identity e_G such that $U_\alpha \subset U_\beta$ and $U_\alpha \neq U_\beta$ whenever $\beta < \alpha < \kappa$. It follows easily from the fact that $U = \bigcap\{U_\alpha : \alpha < \kappa\}$ is an open set that the family $\mathcal{U} = \{U_\alpha \backslash U_{\alpha+1} : \alpha < \kappa\} \subset \tau^*(G)$ is discrete; since $|\mathcal{U}| = \kappa$, this contradicts the pseudo-κ-compactness of G. $\square$

The proof of the first item of the following proposition can be found in [**6**, Lemma 2.6(b)]; the second one was established in [**1**, Theorem 4].

Proposition 3.5. (1) *Any topological pseudo-pG-compact P-group G has a local base at e_G which consists of normal subgroups of G;*

(2) *Any fine P-group G has a local base at e_G which consists of normal subgroups of G.*

The proof of [**6**, Proposition 4.6(a)] can be easily adapted to obtain the following simple fact. We give its proof here for the sake of completeness.

Proposition 3.6. *Let G be a pseudo-κ-compact group such that $pG = \kappa > \omega$. Suppose that we have a family $\mathcal{H} = \{H_\alpha : \alpha < \kappa\}$ of open subgroups of G such that $\alpha < \beta$ implies $H_\beta \subset H_\alpha$ for all $\alpha, \beta < \kappa$. Then, for any clopen $W \supset H = \bigcap \mathcal{H}$ there is $\alpha < \kappa$ such that $H_\alpha \subset W$.*

Proof. If $P_\alpha = H_\alpha \bigcap (G\backslash W) \neq \varnothing$ for all $\alpha < \kappa$ then we obtain a decreasing family $\mathcal{P} = \{P_\alpha : \alpha < \kappa\}$ of non-empty clopen sets in a pseudo-κ-compact $P_{<\kappa}$-space $G\backslash W$ with $\bigcap \mathcal{P} = \varnothing$ which contradicts Fact 3.2. $\square$

The following theorem shows that there are strong dependencies between cardinal functions of fine groups.

Theorem 3.7. *We have $w(G) = \chi(G)$ for any non-discrete fine group.*

Proof. It is a well-known fact that this equality is true for any totally bounded group (and hence for any pseudocompact group); any fine group is either pseudocompact or a P-group [**5**] so we only need to prove that $w(G) = \chi(G)$ for any fine P-group G. If the density of G is strictly greater than $\kappa = \chi(G)$ then there is a left-separated subspace $L \subset G$ with $|L| = \kappa^+$. The group G has a local base $\mathcal{B}$ at the identity such that $|\mathcal{B}| \leq \kappa$ and each $U \in \mathcal{B}$ is an open subgroup of G (see Proposition 3.5). If $<$ is the well-order on L which shows that L is left-separated, then, for each $g \in L$ there is $U_g \in \mathcal{B}$ such that $gU_g \bigcap \{x \in L : x < g\} = \varnothing$. Since

$|L| = \kappa^+$, there is $U \in \mathcal{B}$ and a set $L' \subset L$ such that $|L'| = \kappa^+$ and $U_g = U$ for every $g \in L'$. Observe that $g, h \in L'$ and $g < h$ implies that $g \notin hU$; since gU and hU are cosets of the subgroup U, we have $gU \cap hU = \varnothing$. As a consequence, the family $\{gU : g \in L'\}$ has cardinality κ^+. Thus U is an open subgroup of G such that $|G/U| = \kappa^+ > \chi(G)$ which contradicts [**5**, Theorem 3.3]. This contradiction shows that $d(G) \leq \kappa$, which, together with $\chi(G) \leq \kappa$ implies $w(G) = \chi(G) \cdot d(G) \leq \kappa$ (see [**4**]). □

The following theorem is a key fact in the proof of our main result. It also seems to be interesting in itself. Recall that, given a space X, the cardinal $c(X)$ (called the Souslin number of X) is the supremum of cardinalities of disjoint families of non-empty open subsets of X. If $c(X) = \omega$ we say that the space X *has the Souslin property*.

Theorem 3.8. *If a topological group G is a pseudo-κ-compact $P_{<\kappa}$-space for some infinite cardinal κ then κ is regular and $c(G) \leq \kappa$.*

Proof. The regularity of κ follows from Proposition 3.4 and Fact 3.2. If $\kappa = \omega$ then the group G is pseudocompact so it suffices to observe every pseudocompact (and even every totally bounded) group G has the Souslin property. This follows from the fact that every totally bounded group is a dense subgroup of a compact group [**18**] and every compact group is a dyadic compact space and hence has the Souslin property [**14**].

Now assume that $\kappa > \omega$. For each family $\mathcal{W} \subset \tau^*(G)$ such that each $W \in \mathcal{W}$ is a coset of an open subgroup H_W, we let $\mathcal{W}' = \{H_W : W \in \mathcal{W}\}$. Take any disjoint family $\mathcal{U} \subset \tau^*(G)$; Proposition 1.5 implies that we can assume, without loss of generality, that every $U \in \mathcal{U}$ is a coset of an open normal subgroup H_U of the group G.

Take any $H_0 \in \mathcal{U}'$; then $|G/H_0| < \kappa$ by Proposition 3.3 so there exists a family $\mathcal{U}_0 \subset \mathcal{U}$ such that $|\mathcal{U}_0| < \kappa$ and $\bigcup\{H_0 V : V \in \mathcal{U}_0\} \supset$
$bigcup\mathcal{U}$. Suppose that $\alpha < \kappa$ and we have families $\{\mathcal{U}_\beta : \beta < \alpha\}$ with the following properties:

(i) $\mathcal{U}_\beta \subset \mathcal{U}$ and $|\mathcal{U}_\beta| < \kappa$ for each $\beta < \alpha$;

(ii) if $0 < \beta < \alpha$ and $H_\beta = \bigcap\{\bigcap \mathcal{U}'_\gamma : \gamma < \beta\}$ then $\bigcup\{H_\beta V : V \in \mathcal{U}_\beta\} \supset \bigcup \mathcal{U}$.

It is clear that the set $H_\alpha = \bigcap\{\bigcap \mathcal{U}'_\beta : \beta < \alpha\}$ is an open subgroup of G and therefore $|G/H_\alpha| < \kappa$. Thus there is a family $\mathcal{U}_\alpha \subset \mathcal{U}$ such that $|\mathcal{U}_\alpha| < \kappa$ and $\bigcup\{H_\alpha V : V \in \mathcal{U}_\alpha\} \supset \bigcup \mathcal{U}$. This shows that our inductive construction can go on providing us families $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ with the properties (i) and (ii).

The family $\mathcal{V} = \bigcup\{\mathcal{U}_\alpha : \alpha < \kappa\}$ has cardinality $\leq \kappa$. We claim that its union is dense in $\bigcup \mathcal{U}$. Indeed, let $H = \bigcap\{H_\alpha : \alpha < \kappa\}$; take any $W \in \mathcal{U}$ and pick any $x \in W$. Since $\bigcup\{H_\alpha V : V \in \mathcal{U}_\alpha\} \supset \bigcup \mathcal{U}$ for all $\alpha < \kappa$, there exists $x_\alpha \in \bigcup \mathcal{U}_\alpha$ such that $x \in W_\alpha = W \cap \bigcap\{x_\beta H_\beta : \beta \leq \alpha\})$ for all $\alpha < \kappa$.

We have $x \in P = \bigcap\{W_\alpha : \alpha < \kappa\} \neq \varnothing$; it is clear that $xH \subset WH$. The set W being a coset of an open subgroup H_W, the set WH is clopen in G so we can apply Proposition 3.6 to the family $\{H_\alpha : \alpha < \kappa\}$ to conclude that there exists $\alpha < \kappa$ such that $xH_\alpha \subset WH$. Since $x \in x_\alpha H_\alpha$, the cosets xH_α and $x_\alpha H_\alpha$ must coincide which shows that $x_\alpha H_\alpha \subset WH$ and therefore $x_\alpha H \subset WH$. The last inclusion implies that $x_\alpha H \cap W \neq \varnothing$; since $x_\alpha H \subset \bigcup \mathcal{U}_\alpha \subset \bigcup \mathcal{V}$, we have $(\bigcup \mathcal{V}) \cap W \neq \varnothing$. As a consequence, we found a family $\mathcal{V} \subset \mathcal{U}$ of cardinality $\leq \kappa$ such that the union of $\mathcal{V}$ intersects every element of $\mathcal{U}$; this implies $\mathcal{U} = \mathcal{V}$ and hence $|\mathcal{U}| \leq \kappa$. □

Comfort and Hager conjectured in [**6**] that every pseudo-pG-compact group is fine. They also asked whether this conjecture is true formulating this as an open question [**6**, Question (1)]. The following theorem proves the conjecture of Comfort and Hager and hence answers positively Question (1) from [**6**]. We will use in our proof the following criterion of Alas [**1**, Theorem 7]: a P-group G is fine if and only if, for any clopen $U \subset G$, there is $V \in \tau(e_G, G)$ such that $VU = U$.

Theorem 3.9. *Any pseudo-pG-compact group G is fine.*

Proof. If $pG = \omega$ then the group G is pseudocompact and hence fine [**5**]. If $pG = \kappa > \omega$ then G is a P-group; take any clopen $U \subset G$. Choose a maximal disjoint family $\mathcal{W} \subset \tau^*(G)$ such that, for each $W \in \mathcal{W}$, we have $W \subset U$ and W is a coset of an open normal subgroup H_W of the group G. Since $c(G) \leq \kappa$ by Theorem 3.8, we have $|\mathcal{W}| \leq \kappa$. Let $H = \bigcap\{H_W : W \in \mathcal{W}\}$.

Observe that $HU = U$; indeed, the quotient map $\varphi : G \to G/H$ is open so $\varphi(G \backslash U)$ is an open subset of $G' = G/H$. Hence $P = \varphi^{-1}\big(G' \backslash \varphi(G \backslash U)\big) \subset U$ is closed in G. Since $P \supset H \cdot (\bigcup \mathcal{W}) = \bigcup \mathcal{W}$ and $\bigcup \mathcal{W}$ is dense in U, the set P coincides with U so $P = HP = HU = U$. This shows that $U = \varphi^{-1}\big(\varphi(U)\big)$.

If G' is discrete then it is fine; if G' is not discrete then $pG' = \kappa$. Note that the group G' is pseudo-pG'-compact and the identity $e_{G'}$ is an intersection of $\leq \kappa$-many open sets. Therefore it is again fine by [**6**, Proposition 4.6]. The set $\varphi(U)$ being clopen in G' there is an open subgroup V' of the group G' such that $V'\varphi(U) = \varphi(U)$. It is clear that $V = \varphi^{-1}(V')$ is an open subgroup of G such that $VU = U$. Therefore the group G is fine by [**1**, Theorem 7]. □

Recall that a cardinal κ is called *strongly inaccessible* if $2^\mu < \kappa$ for any $\mu < \kappa$.

Corollary 3.10. *Suppose that G is a topological group such that pG is not strongly inaccessible. Then G is fine if and only if it is pseudo-pG-compact.*

Proof. Alas proved [**2**, Theorem 3] that any fine group is pseudo-pG-compact if pG is not strongly inaccessible. Now apply Theorem 3.9. □

Corollary 3.11. *Any pseudo-ω_1-compact P-group G is fine. In particular, if G is a P-group and all closed discrete subspaces of G are countable then G is fine.*

Proof. Any pseudo-ω_1-compact group is pseudo-κ-compact for any $\kappa \geq \omega_1$; since $\kappa = pG \geq \omega_1$, the group G is pseudo-pG-compact. Now apply Theorem 3.9. □

Corollary 3.12. *Any $\mathbb{R}$-factorizable P-group is fine.*

Proof. Any $\mathbb{R}$-factorizable P-group is pseudo-ω_1-compact [**17**]. Now apply Corollary 3.11. □

Definition 3.13. Say that a group G is *discretely factorizable* if for any continuous function $f : G \to \mathbb{R}$, there exists a discrete group D, a continuous homomorphism $h : G \to D$ and a (continuous) map $g : D \to \mathbb{R}$ such that $g \circ h = f$.

Theorem 3.14. *A P-group is fine if and only if it is discretely factorizable.*

Proof. If G is discretely factorizable then every real-valued continuous function on G is a composition of a continuous function on a discrete group and a continuous homomorphism. Every real-valued continuous function on a discrete group is uniformly continuous as well as any continuous homomorphism. Since the composition of uniformly continuous maps is a uniformly continuous map, the group G is fine.

Now assume that G is fine. The family $\mathcal{B} = \{H : H$ is an open normal subgroup of $G\}$ is a base at e_G by Proposition 3.5. Take any continuous function $f : G \to \mathbb{R}$; since f is uniformly continuous, there is a family $\{U_n : n \in \mathbb{N}\} \subset \mathcal{B}$ such that $x^{-1}y \in U_n$ implies $|f(x) - f(y)| < \frac{1}{n}$ for all $x, y \in G$ and $n \in \mathbb{N}$. It is immediate that $U = \bigcap\{U_n : n \in \mathbb{N}\} \in \mathcal{B}$ and we have $f(x) = f(y)$ whenever $x^{-1}y \in U$ for any $x, y \in G$.

Since U is a normal subgroup of G, the group $D = G/U$ is a well-defined group with $h : G \to D$ the relevant quotient homomorphism. The function f is constant on all cosets of G and hence there exists a map $g : D \to \mathbb{R}$ such that $g \circ h = f$. The group D is discrete because U is open in G so the map g is continuous. □

The proof of the following proposition was given in [**6**]. Although it uses unproven implication mentioned in our Introduction, the ideas of [**6**] are sufficient for considering that this result was obtained by Comfort and Hager. It can also be deduced from the results of Alas who proved that any group G with $pG = \chi(G)$ is paracompact and that fineness, pseudo-pG-compactness and pG-compactness are equivalent for paracompact groups (see [**3**]). For the reader's convenience, we present here a direct proof of this result.

Proposition 3.15. *If G is a non-discrete P-group with $\chi(G) = pG$ then the following conditions are equivalent:*

(1) *G is fine;*

(2) *G is pseudo-pG-compact;*

(3) *G is pG-compact.*

Proof. Observe that the implication (3) $\Longrightarrow$ (2) is trivial and (2) $\Longrightarrow$ (1) was proved in Theorem 3.9. Let $pG = \kappa$; then $w(G) \leq \kappa$ by Theorem 3.7. It is straightforward to show that any pseudo-κ-compact space of weight $\leq \kappa$ is κ-compact so (2) $\Longrightarrow$ (3) and hence we only have to show that (1) $\Longrightarrow$ (2).

Assume that G is fine and choose a local base $\mathcal{B} = \{O_\alpha : \alpha < \kappa\}$ of the identity e of the group G such that each O_α is an open normal subgroup of G and $O_\alpha \subset O_\beta$ for all $\beta < \alpha < \kappa$. If $\mathcal{U} \subset \tau^*(G)$ is a discrete family of cardinality κ we can assume that $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ where each U_α is a coset of O_α. Since G is non-discrete, we can pick distinct points $x_\alpha, y_\alpha \in U_\alpha$ for all $\alpha < \kappa$. There exists a clopen $W_\alpha \subset G$ such that $W_\alpha \subset U_\alpha$ and $x_\alpha \in W_\alpha$ while $y_\alpha \notin W_\alpha$. The function $f_\alpha = \chi_{W_\alpha}$ is continuous on G and it follows from the discreteness of $\mathcal{U}$ that the function $f = \sum\{f_\alpha : \alpha < \kappa\}$ is continuous on G. The group G being fine, the function f has to be left uniformly continuous. Therefore there is $\alpha < \kappa$ such that $x^{-1}y \in O_\alpha$ implies $|f(x) - f(y)| < \frac{1}{2}$, i.e., $f(x) = f(y)$. We have $|G/O_\alpha| < \kappa$ which implies that there is some $x \in G$ such that xO_α intersects κ-many elements of $\mathcal{U}$. However, the base $\mathcal{B}$ is nested, so that $yO_\beta \cap xO_\alpha \neq \varnothing$ implies $yO_\beta \subset xO_\alpha$ for any $\beta > \alpha$. This shows that $U_\beta = yO_\beta \subset xO_\alpha$ for some $\beta < \kappa$. Since $f|U_\beta = f_\beta|U_\beta$, the function f takes distinct values on $U_\beta \subset xO_\alpha$; however $f|xO_\alpha$ has to be constant by the choice of O_α. This contradiction shows that f is not uniformly continuous and hence G is pseudo-κ-compact. $\square$

In [**17**] Tkachenko asked whether any Lindelöf P-group G is monolithic, i.e., whether $nw(\overline{A}) \le |A|$ for any $A \subset G$. We give a partial answer to this question showing that a positive answer can not be given in ZFC.

Example 3.16. It is consistent with ZFC that there exists a Lindelöf P-group which is not ω_1-monolithic.

Proof. It is sufficient to prove that there exists a Lindelöf P-space which is not ω_1-monolithic because its Markoff free topological group will then be the required example (see [**17**]). Koszmider and Tall [**13**] showed that there are models of ZFC in which there exists a Lindelöf P-space Z which has no Lindelöf subspaces of cardinality ω_1. Take any set $Y \subset Z$ with $|Y| = \omega_1$. The space $X = \overline{Y}$ is the required Lindelöf P-space. Indeed, if X is ω_1-monolithic then $nw(X) = \omega_1$ and hence $\psi(X) = \omega_1$. It is a standard fact that $\psi(X) = \omega_1$ implies $\chi(X) = \omega_1$ in a Lindelöf P-space X (see e.g. [**17**, Corollary 4.11]) so we have $\chi(X) = \omega_1$. Now it is a standard construction (see [**6**]) to show the existence of a convergent ω_1-sequence in X which is a Lindelöf subspace of the space X of cardinality ω_1, a contradiction. $\square$

4. Uniformly $\mathbb{R}$-Factorizable Groups

We saw in the previous section that the class of $\mathbb{R}$-factorizable P-groups is contained in the class of fine groups. This makes it natural to consider a notion of uniformly $\mathbb{R}$-factorizable group defined in a similar manner as was done with $\mathbb{R}$-factorizable ones. The situation with uniformly $\mathbb{R}$-factorizable groups is very different from the case of $\mathbb{R}$-factorizable groups. While the latter are contained in the class of ω-narrow groups, it is known that there are ω-narrow P-groups which are not $\mathbb{R}$-factorizable [**16**].

Definition 4.1. Call a topological group G left uniformly $\mathbb{R}$-factorizable if, for any left uniformly continuous map $f : G \to \mathbb{R}$, there exists a second countable group H and a continuous onto homomorphism $\pi : G \to H$ such that $f = g \circ \pi$ for some left uniformly continuous map $g : H \to \mathbb{R}$.

Theorem 4.2. *A group G is left uniformly $\mathbb{R}$-factorizable if and only if it is ω-narrow.*

Proof. Assume that G is left uniformly $\mathbb{R}$-factorizable. Recall that a function $N : G \to \mathbb{R}$ is called *a seminorm on G* if $N(e_G) = 0$ and $N(xy^{-1}) \le N(x) + N(y)$ for any $x, y \in G$. It is easy to see that $N(x) \ge 0$ for any $x \in G$ and $|N(x) - N(y)| \le N(x^{-1}y)$ for any $x, y \in G$ (see [**8**]). Let $\mathcal{M}$ be the class of all second countable groups. Given any set $U \in \tau(e_G, G)$ there exists a continuous seminorm $N : G \to \mathbb{R}$ such that $\{x \in G : N(x) < 1\} \subset U$ [**8**]. The inequality $|N(x) - N(y)| \le N(x^{-1}y)$ which holds for all $x, y \in G$ easily implies that the seminorm N is left uniformly continuous on G. Therefore we can use left uniform $\mathbb{R}$-factorizability of G to find a group $M \in \mathcal{M}$, a continuous homomorphism $h : G \to M$ and a left uniformly continuous map $g : M \to \mathbb{R}$ such that $N = g \circ h$.

Since $h(e_G) = e_M$, we have $g(e_M) = N(e_G) = 0$ so $W = \{z \in M : |g(z)| < 1\}$ is an open neighbourhood of e_M. It is immediate that $h^{-1}(W) \subset U$ so we have shown that our group G has the following property

($*$) for any $U \in \tau(e_G, G)$ there exists a group $M \in \mathcal{M}$ and a continuous homomorphism $h : G \to M$ such that $h^{-1}(W) \subset U$ for some $W \in \tau(e_M, M)$.

It is a standard fact that ($*$) implies that G is topologically isomorphic to a subgroup of a direct product $\prod\{M_t : t \in T\}$ where $M_t \in \mathcal{M}$ for all $t \in T$ (see e.g. [**11**]). Recall that a group is ω-narrow if and only if it is topologically isomorphic to a subgroup of a direct product of second countable groups [**9**] so G is ω-narrow. This establishes the necessity.

To prove the sufficiency, take an arbitrary ω-narrow group G and any left uniformly continuous map $f : G \to \mathbb{R}$. Applying again the cited Guran's characterization of ω-narrow groups (see [**9**]) we can consider that G is a subgroup of a product $\prod\{M_\alpha : \alpha \in A\}$ of second countable groups. Fix a decreasing base $\{U^\alpha_n : n \in \mathbb{N}\}$ at the identity e_α in each group M_α and let $\pi_\alpha : G \to M_\alpha$ be the natural projection.

Given $\alpha_1, \dots, \alpha_n \in A$ and an open $V_i \subset M_{\alpha_i}$ for each $i \le n$, we will need the set $[\alpha_1, \dots, \alpha_n; V_1, \dots, V_n] = \{x \in G : \pi_{\alpha_i}(x) \in V_i \text{ for all } i \le n\}$. For each $n \in \mathbb{N}$ fix an open symmetric neighbourhood U_n of the identity e of the group G such that $x^{-1}y \in U_n$ implies $|f(x) - f(y)| < \frac{1}{n}$ for any $x, y \in G$.

It is easy to construct by induction a sequence of integers $\{k_j : j \in \mathbb{N}\}$, a sequence of indices $\{\alpha_i : i \in \mathbb{N}\} \subset A$ and sets $\{O^j_i : j \in \mathbb{N}; i = 1, \dots, k_j\}$ such that

(i) $k_j \le k_{j+1}$ for all $j \in \mathbb{N}$;

(ii) $O^j_i \in \tau(e_{\alpha_i}, M_{\alpha_i})$ for each $i \le k_j$ and $j \in \mathbb{N}$;

(iii) $O^{j+1}_i \subset O^j_i \subset U^{\alpha_i}_j$ for each $j \in \mathbb{N}$, $i \le k_j$;

(iv) $[\alpha_1, \dots, \alpha_{k_j}; O^j_1, \dots, O^j_{k_j}] \subset U_j$ for every $j \in \mathbb{N}$.

Let $\pi = \Delta\{\pi_{\alpha_i} : i \in \mathbb{N}\} : G \to \prod\{M_\alpha : \alpha \in B\}$ be the natural projection of G to the face defined by the set $B = \{\alpha_i : i \in \mathbb{N}\}$. The group $H = \pi(G)$ is second countable and $\pi : G \to H$ is a continuous homomorphism. We show that $\pi(x) = \pi(y)$ implies $f(x) = f(y)$. Indeed, if $\pi(x) = \pi(y)$ then we have $\pi_{\alpha_i}(x) = \pi_{\alpha_i}(y)$ and hence $\pi_{\alpha_i}(x^{-1}y) = e_{\alpha_i}$ for any $i \in \mathbb{N}$. This shows that $x^{-1}y \in [\alpha_1, \dots \alpha_{k_j}; O^j_1, \dots, O^j_{k_j}] \subset U_j$ for each $j \in \mathbb{N}$ whence $|f(x) - f(y)| < \frac{1}{j}$. Since this holds for all natural numbers j, we must have $f(x) = f(y)$. Thus there exists a map $g : H \to \mathbb{R}$ such that $g \circ \pi = f$, so the last thing we have to prove is that g is uniformly continuous.

Let $p_i : H \to M_{\alpha_i}$ be the natural projection for each $i \in \mathbb{N}$. Fix any $n \in \mathbb{N}$ and observe that $[\alpha_1, \dots, \alpha_{k_n}; O^n_1, \dots, O^n_{k_n}] \subset U_n$ by (iv) which implies that the set $V_n = \{z \in H : p_i(z) \in O^n_i \text{ for all } i \le k_n\}$ is open in H and $H \subset \pi(U_n)$. Take any $x, y \in H$ such that $x^{-1}y \in V_n$; choose $x', y' \in G$ such that $\pi(x') = x$ and $\pi(y') = y$. Observe that

$$\pi_{\alpha_i}\big((x')^{-1}y'\big) = \big(\pi_{\alpha_i}(x')\big)^{-1} \cdot \pi_{\alpha_i}(y') = \big(p_i(x)\big)^{-1} \cdot p_i(y) = p_i(x^{-1}y) \in O^n_i$$

for all $i \le k_n$. This shows that $(x')^{-1}y' \in [\alpha_1, \dots, \alpha_{k_n}; O^n_1, \dots, O^n_{k_n}] \subset U_n$ and therefore $|g(x) - g(y)| = |g\big(\pi(x')\big) - g(\pi(y'))| = |f(x) - f(y)| < \frac{1}{n}$ which proves that g is left uniformly continuous. Hence G is left uniformly $\mathbb{R}$-factorizable. $\square$

Remark 4.3. We could introduce in an analogous way the concept of right uniform $\mathbb{R}$-factorizability. After evident modifications in the proof of Theorem 4.2 we could observe that a group is right uniformly $\mathbb{R}$-factorizable if and only if it is ω-narrow.

Corollary 4.4. *Every $\mathbb{R}$-factorizable group is left uniformly $\mathbb{R}$-factorizable while a left uniformly $\mathbb{R}$-factorizable group is not necessarily $\mathbb{R}$-factorizable.*

Proof. It is known that every $\mathbb{R}$-factorizable group is ω-narrow [**15**] and hence left uniformly $\mathbb{R}$-factorizable by Theorem 4.2. It is also Tkachenko's result [**16**] that there exist ω-narrow groups which are not $\mathbb{R}$-factorizable. □

Corollary 4.5. *A fine P-group G is $\mathbb{R}$-factorizable if and only if G is ω-narrow.*

Corollary 4.6. *A fine P-group is pseudo-ω_1-compact if and only if it is ω-narrow.*

Proof. Tkachenko proved in [**17**] that pseudo-ω_1-compactness in P-groups coincides with $\mathbb{R}$-factorizability so Corollary 4.5 can be applied. □

Example 4.7. There exists an ω-narrow P-group which is not fine. Indeed, Tkachenko constructed in [**16**] an ω-narrow P-group G which is not $\mathbb{R}$-factorizable. Applying Corollary 4.5 we can see that the group G is not fine. It follows from the fact that G is ω-narrow that G/H is countable for any any open subgroup H of the group G. Since G is not pseudo-pG-compact by Theorem 3.9, this answers Question (5) of [**6**].

Remark 4.8. Alas proved in [**3**] that, for any strongly inaccessible non-weakly compact cardinal κ, there exists a paracompact group G which is not fine while $pG = \kappa$ and G is κ-narrow. This also answers Question (5) of [**6**] (for a strongly inaccessible non-weakly compact κ) if we observe that, in the class of paracompact groups G, the fineness of G is equivalent to pseudo-pG-compactness.

Example 4.9. Tkachenko constructed [**17**, Example 4.18] an example of a dense non-Lindelöf $\mathbb{R}$-factorizable subgroup G of a Lindelöf P-group L. In the same paper he proved that every $\mathbb{R}$-factorizable group is pseudo-ω_1-compact; thus the group G is pseudo-pG-compact. However, G is not complete because it is a proper dense subgroup of L. Every pG-compact group is complete [**6**, Corollary 4.2]; this shows that G is an example of a pseudo-pG-compact non-complete group which also fails to be pG-compact. This answers Questions (2) and (3) of [**6**].

5. Open Problems

The question that follows seems to be the most interesting one in this topic. Of course, it must be solved only for the case when pG is a strongly inaccessible (regular) cardinal.

Problem 5.1. Must every fine group G be pseudo-pG-compact?

The following question was formulated in [**6**]. It seems to us that it is worth repeating 27 years afterwards.

Problem 5.2. Let G be a fine P-group. Must $G \times G$ be fine?

The two problems that follow suggest that $\mathbb{R}$-factorizable P-groups can have the Lindelöf property under some rather weak assumptions.

Problem 5.3. Let G be a P-group with all its closed discrete subspaces countable (this, evidently, implies that G is $\mathbb{R}$-factorizable). Is it true that G is Lindelöf?

Problem 5.4. Let G be a normal $\mathbb{R}$-factorizable P-group. Must G be Lindelöf?

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