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UNIQUENESS OF MEROMORPHIC FUNCTIONS AND SHARING OF THREE VALUES WITH SOME WEIGHT

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Abstract. We prove some uniqueness theorems for meromorphic functions sharing three values with some weight which improve some known results.

1. Introduction, Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $b \in \mathbb{C} \cup \{\infty\}$ we say that f and g share the value b CM (counting multiplicities) if f and g have the same b–points with the same multiplicities. If the multiplicities are ignored, we say that f and g share the value b IM (ignoring multiplicities). We do not explain the standard notations and definitions of the value distribution theory because those are available in [**2**]. However, we now explain some notations and definitions which will be needed in the sequel.

Definition 1.1 ([**3**])**.** If s is a positive integer, we denote by $\overline{N}(r, a; f \mid\geq s)$ the counting function of those a–points of f whose multiplicities are greater than or equal to s, where each a–point is counted only once.

Definition 1.2 ([**3, 10**])**.** If s is a positive integer, we denote by $N_s(r, a; f)$ the counting function of a–points of f, where an a–point with multiplicity m is counted m times if $m \leq s$ and s times if $m > s$. We put $N_\infty(r, a; f) \equiv N(r, a; f)$.

Definition 1.3. Let f, g share a value a IM. Let z be an a–point of f and g with multiplicities $p_f(z)$ and $p_g(z)$ respectively. If z is not an a–point of f and g then we suppose that $p_f(z) = p_g(z) = 0$. We put

$$\begin{aligned} \nu_f(z) = {} & p_f(z) \quad \text{if} \quad p_f(z) = p_g(z) \neq 0 \\ = {} & 0 \quad \text{if} \quad p_f(z) = p_g(z) = 0 \\ = {} & 0 \quad \text{if} \quad p_f(z) \neq p_g(z); \end{aligned}$$

and

$$\begin{aligned} \overline{\nu}_f(z) &= 1 \quad \text{if} \quad p_f(z) = p_g(z) \neq 0 \\ &= 0 \quad \text{if} \quad p_f(z) = g_g(z) = 0 \\ &= 0 \quad \text{if} \quad p_f(z) \neq p_g(z). \end{aligned}$$

Clearly $\nu_f(z) = \nu_g(z)$ and $\overline{\nu}_f(z) = \overline{\nu}_g(z)$.
Now we put $n_E(r, a; f, g) = \sum_{|z|\leq r} \nu_f(z)$ and $\overline{n}_E(r, a; f, g) = \sum_{|z|\leq r} \overline{\nu}_f(z)$.

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Also we denote by $N_E(r, a; f, g)$ and $\overline{N}_E(r, a; f, g)$ the integrated counting functions obtained from $n_E(r, a; f, g)$ and $\overline{n}_E(r, a; f, g)$ respectively.

Finally we define $N_*(r, a; f, g)$ and $\overline{N}_*(r, a; f, g)$ as follows

$$N_*(r, a; f, g) = N(r, a; f) - N_E(r, a; f, g)$$

and

$$\overline{N}_*(r, a; f, g) = \overline{N}(r, a; f) - \overline{N}_E(r, a; f, g).$$

Clearly $N_E(r, a; f, g) \equiv N_E(r, a; g, f)$, $\overline{N}_E(r, a; f, g) \equiv \overline{N}_E(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$.

Definition 1.4 ([**3**])**.** Let f, g share a value a IM. We denote by $\overline{N}(r, a; f < g)$ $\big(\overline{N}(r, a; f > g)\big)$ the counting function of those a–points of f whose multiplicities are less (greater) than the multiplicities of the corresponding a–points of g, where each a–point is counted only once.

Definition 1.5. We put

$$\delta_s(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N_s(r, a; f)}{T(r, f)}$$

where s is a positive integer.

Clearly $\delta(a; f) \le \delta_s(a; f) \le \delta_{s-1}(a; f) \le \cdots \le \delta_1(a; f) = \Theta(a; f) \le 1$.

H. Ueda [**6**] proved the following result.

Theorem A ([**6**])**.** *Let f and g be two distinct nonconstant entire functions sharing 0, 1 CM and let $a (\ne 0, 1)$ be a finite complex number. If a is lacunary for f then $1 - a$ is lacunary for g and $(f - a)(g + a - 1) \equiv a(1 - a)$.*

Improving *Theorem* A H.X. Yi [**8**] proved the following theorem.

Theorem B ([**8**])**.** *Let f and g be two distinct nonconstant entire functions sharing 0, 1 CM and let $a (\ne 0, 1)$ be a finite complex number. If $\delta(a; f) > \frac{1}{3}$ then a and $1-a$ are Picard exceptional values of f and g respectively and $(f-a)(g+a-1) \equiv a(1-a)$.*

S.Z. Ye [**7**] extended *Theorem* B to meromorphic functions and proved the following result.

Theorem C ([**7**])**.** *Let f and g be two distinct nonconstant meromorphic functions such that f and g share 0, 1, ∞ CM. Let $a (\ne 0, 1)$ be a finite complex number. If $\delta(a; f) + \delta(\infty; f) > \frac{4}{3}$ then a and $1 - a$ are Picard exceptional values of f and g respectively and also ∞ is a Picard exceptional value of both f and g and $(f - a)(g + a - 1) \equiv a(1 - a)$.*

The following two examples show that in the above theorems the sharing of 0 and 1 can not be relaxed from **CM** to **IM**.

Example 1.6. Let $f = e^z - 1$, $g = (e^z - 1)^2$ and $a = -1$. Then f, g share 0 **IM** and $1, \infty$ **CM**. Also $N(r, \infty; f) \equiv 0$ and $N(r, a; f) \equiv 0$ but $(f - a)(g + a - 1) \not\equiv a(1 - a)$.

Example 1.7. Let $f = 2 - e^z$, $g = e^z(2 - e^z)$ and $a = 2$. Then f, g share 1 **IM** and $0, \infty$ **CM**. Also $N(r, \infty; f) \equiv 0$ and $N(r, a; f) \equiv 0$ but $(f - a)(g + a - 1) \not\equiv a(1 - a)$.

Now one may ask the following question: ***Is it possible in any way to relax the nature of sharing of values in the theorems stated above?***

In the paper we study this problem. To this end we now explain a gradation of sharing values which measures how close a shared value is to being shared **IM** or to being shared **CM** and is called weight of sharing.

Definition 1.8 ([**3, 4**])**.** Let k be a nonnegative integer or infinity. For $a \in C \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a–points of f where an a–point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_o is an a–point of f with multiplicity $m(\le k)$ if and only if it is an a–point of g with multiplicity $m(\le k)$ and z_o is an a–point of f with multiplicity $m(> k)$ if and only if it is an a–point of g with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Now we state the main results of the paper.

Theorem 1.9. *Let f and g be two distinct meromorphic functions sharing $(0, 1)$, $(1, \infty)$, $(\infty, 11)$. If $a(\ne 0, 1, \infty)$ is a complex number such that $3\delta_2(a; f) + 3\delta(\infty; f) > 4$ then a and $1-a$ are Picard exceptional values of f and g respectively and also ∞ is a Picard exceptional value of both f and g and $(f-a)(g+a-1) \equiv a(1-a)$.*

Theorem 1.10. *Let f and g be two distinct meromorphic functions sharing $(0, 1)$, $(1, \infty)$, $(\infty, 0)$. If $a(\ne 0, 1, \infty)$ is a complex number such that $3\delta_2(a; f) + 14\delta(\infty; f) > 15$ then a and $1-a$ are Picard exceptional values of f and g respectively and also ∞ is a Picard exceptional value of both f and g and $(f-a)(g+a-1) \equiv a(1-a)$.*

Corollary 1.11. *Let f and g be two distinct meromorphic functions sharing $(0, \infty)$, $(1, 1)$, $(\infty, 11)$. If $a(\ne 0, 1, \infty)$ is a complex number such that $3\delta_2(a; f) + 3\delta(\infty; f) > 4$ then a and $1-a$ are Picard exceptional values of f and g respectively and also ∞ is a Picard exceptional value of both f and g and $(f-a)(g+a-1) \equiv a(1-a)$.*

Corollary 1.12. *Let f and g be two distinct meromorphic functions sharing $(0, \infty)$, $(1, 1)$, $(\infty, 0)$. If $a(\ne 0, 1, \infty)$ is a complex number such that $3\delta_2(a; f) + 14\delta(\infty; f) > 15$ then a and $1-a$ are Picard exceptional values of f and g respectively and also ∞ is a Picard exceptional value of both f and g and $(f-a)(g+a-1) \equiv a(1-a)$.*

Example 1.6 shows that in *Theorem* 1.9 and *Theorem* 1.10 sharing $(0, 1)$ cannot be relaxed to sharing $(0, 0)$. Also *Example* 1.7 shows that in *Corollary* 1.11 and *Corollary* 1.12 sharing $(1, 1)$ cannot be relaxed to sharing $(1, 0)$.

Throughout the paper we denote by f, g two nonconstant meromorphic functions defined in $\mathbb{C}$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([**3**])**.** *Let* f, g *share* $(0,1)$, $(1,\infty)$, $(\infty,0)$ *and* $f \not\equiv g$. *Then*

$$\overline{N}(r,0;f\,|\geq 2) \;=\; \overline{N}(r,0;g\,|\geq 2) \leq \overline{N}_*(r,\infty;f,g) + S(r,f).$$

Lemma 2.2 (cf. [**1, 3**])**.** *Let* f, g *share* $(0,0)$, $(1,0)$, $(\infty,0)$. *Then*

(i) $T(r,g) \leq 3T(r,f) + S(r,g)$,

(ii) $T(r,f) \leq 3T(r,g) + S(r,f)$.

Lemma 2.3. *Let* f, g *share* $(0,0)$, $(1,\infty)$, $(\infty,0)$ *and* $f \not\equiv g$. *If* $\alpha = (f-1)/(g-1)$ *and* $h = g/f$ *then*

(i) $\overline{N}(r,0;\alpha) = \overline{N}(r,\infty;f<g)$,

(ii) $\overline{N}(r,\infty;\alpha) = \overline{N}(r,\infty;f>g)$,

(iii) $\overline{N}(r,0;h) = \overline{N}(r,0;f<g) + \overline{N}(r,\infty;f>g)$,

(iv) $\overline{N}(r,\infty;h) = \overline{N}(r,0;f>g) + \overline{N}(r,\infty;f<g)$.

Proof. Let z_o be a pole of f and g with multiplicities m and n respectively. Since $h = g/f$, it follows that z_o will be a pole of h if $m < n$.

Again let z_1 be a zero of f and g with multiplicities p and q respectively. It is easy to verify that z_1 will be a pole of h if $p > q$.

Therefore

$$\overline{N}(r,\infty;h) \;=\; \overline{N}(r,0;f>g) + \overline{N}(r,\infty;f<g).$$

By similar arguments we can prove (i), (ii) and (iii). This proves the lemma. □

Lemma 2.4. *Let* $af + bg \equiv c$, *where* a, b, c *be nonzero constants. Then*

$$T(r,f) \;<\; \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + S(r,f).$$

Proof. The lemma follows as a consequence of the second fundamental theorem. □

Lemma 2.5. *Let* f, g *share* $(0,1)$, $(1,\infty)$, $(\infty,0)$ *and* α, h *be defined as in* Lemma 2.3. *If* $a\alpha h + b\alpha \equiv c$ *for nonzero constants* a, b, c *then*

$$T(r,f) \leq 6\overline{N}(r,\infty;f<g) + S(r,f).$$

Proof. If one of α and αh is constant then from the given condition we see that the other is also constant. Since $f = (1-\alpha)/(1-\alpha h)$, it follows that f becomes a constant, which is a contradiction. So α and αh are nonconstant.

Let z_o be a pole of f and g with multiplicities p and q respectively. If $p > q$ then z_o is a zero of $1/\alpha$ and h. So from $ah + b \equiv c/\alpha$, it follows that $b = 0$, which is a contradiction. So $N(r,\infty;f>g) \equiv 0$.

Again let z_1 be a zero of f and g with multiplicities m and n respectively. If $m > n$ then z_o is a pole of h but z_o is a regular point of α with $\alpha(z_o) = 1$. Since $ah \equiv (c/\alpha) - b$, this implies a contradiction. So $N(r,0;f>g) \equiv 0$.

By *Lemmas* 2.2, 2.3 and 2.4 we get in view of the first fundamental theorem

$$\begin{aligned} T(r,\alpha) &\leq \overline{N}(r,0;\alpha) + \overline{N}(r,\infty;\alpha) + \overline{N}(r,0;h) + S(r,\alpha) \\ &\leq \overline{N}(r,0;f<g) + \overline{N}(r,\infty;f<g) + S(r,f). \end{aligned} \tag{2.1}$$

Again since $ah + b \equiv c/\alpha$, it follows from *Lemma* 2.3 that

$$\overline{N}(r, \frac{-b}{a}; h) = \overline{N}(r, \infty; \alpha) = \overline{N}(r, \infty; f > g) \equiv 0,$$

$$\overline{N}(r, 0; h) = \overline{N}(r, 0; f < g),$$

and

$$\overline{N}(r, \infty; h) = \overline{N}(r, 0; \alpha) = \overline{N}(r, \infty; f < g).$$

By the second fundamental theorem we get

$$\begin{aligned} T(r,h) &\le \overline{N}(r,\infty;h) + \overline{N}(r,0;h) + \overline{N}(r,-b/a;h) + S(r,h) \\ &\le \overline{N}(r,0;f<g) + \overline{N}(r,\infty;f<g) + S(r,h). \end{aligned} \tag{2.2}$$

From the definitions of α and h it follows in view of *Lemma* 2.2 and the first fundamental theorem that $S(r, \alpha) = S(r, f)$ and $S(r, h) = S(r, f)$.

Since f, g share $(0, 1)$, $(1, \infty)$, $(\infty, 0)$, it follows from *Lemma* 2.1 that

$$\begin{aligned} \overline{N}(r,0;f<g) &\le \overline{N}(r,0;f \mid\ge 2) \\ &\le \overline{N}_*(r,\infty;f,g) + S(r,f) \\ &= \overline{N}(r,\infty;f<g) + S(r,f). \end{aligned} \tag{2.3}$$

Again since $f = (1-\alpha)/(1-\alpha h)$, it follows from (2.1), (2.2), (2.3) and the first fundamental theorem that

$$\begin{aligned} T(r,f) &\le 2T(r,\alpha) + T(r,h) + O(1) \\ &\le 3\overline{N}(r,0;f<g) + 3\overline{N}(r,\infty;f<g) + S(r,f) \\ &\le 6\overline{N}(r,\infty;f<g) + S(r,f). \end{aligned}$$

This proves the lemma. □

Lemma 2.6 ([**5**])**.** *Let f_1, f_2, f_3 be nonconstant meromorphic functions such that $f_1 + f_2 + f_3 \equiv 1$. If f_1, f_2, f_3 are linearly independent then*

$$T(r, f_1) \le \sum_{i=1}^{3} N_2(r, 0; f_i) + \sum_{i=1}^{3} \overline{N}(r, \infty; f_i) + S(r),$$

where $S(r) = \sum_{i=1}^{3} S(r, f_i)$.

Lemma 2.7 ([**9**])**.** *Let f_1, f_2, f_3 be nonconstant meromorphic functions such that $f_1 + f_2 + f_3 \equiv 1$ and let $g_1 = -f_1/f_3$, $g_2 = 1/f_3$, $g_3 = -f_2/f_3$. If f_1, f_2, f_3 are linearly independent then g_1, g_2, g_3 are linearly independent.*

Lemma 2.8. *Let f, g share $(0, 1)$, $(1, \infty)$, $(\infty, 0)$ and $f \not\equiv g$. Let*

$$f_1 = \frac{(f-a)(1-\alpha h)}{1-a}, \quad f_2 = \frac{-a\alpha h}{1-a}, \quad \text{and} \quad f_3 = \frac{\alpha}{1-a},$$

where $a(\ne 0, 1, \infty)$ be a complex number and α, h are defined as in Lemma 2.3. If f_1, f_2, f_3 are linearly independent then

(i) $\overline{N}(r, 0; f) \le N_2(r, a; f) + 5\overline{N}_*(r, \infty; f, g) + N_*(r, \infty; f, g) + S(r, f)$,

(ii) $\overline{N}(r, 1; f) \le N_2(r, a; f) + 6\overline{N}_*(r, \infty; f, g) + N_*(r, \infty; f, g) + S(r, f)$.

Proof. Since $(1-a)f_1 = (1-\alpha) - a(1-\alpha h)$, it follows that

$$\overline{N}(r,\infty;f_1) \;\le\; \overline{N}(r,\infty;f>g) + \overline{N}(r,0;f>g)$$

because if z_o is a pole of f and g with multiplicities m and n respectively then at z_o, $\alpha h = \frac{g(f-1)}{f(g-1)}$ has no pole and at z_o, α has a pole if $m > n$.

Also $\overline{N}(r,\infty;f_2) \le \overline{N}(r,0;f>g)$ and $\overline{N}(r,\infty;f_3) \le \overline{N}(r,\infty;f>g)$. First we suppose that α is nonconstant. Since by *Lemma* 2.2 $\sum_{i=1}^{3} S(r,f_i) = S(r,f)$, we get by *Lemma* 2.6

$$\begin{aligned} T(r,\alpha) &\le N_2(r,0;f_1) + N_2(r,0;f_2) + N_2(r,0;f_3) \\ &\quad + \overline{N}(r,\infty;f_1) + \overline{N}(r,\infty;f_2) + \overline{N}(r,\infty;f_3) + S(r,f) \\ &\le N_2(r,0;f_1) + 2\overline{N}(r,0;f_2) + 2\overline{N}(r,0;f_3) \\ &\quad + \overline{N}(r,\infty;f_1) + \overline{N}(r,\infty;f_2) + \overline{N}(r,\infty;f_3) + S(r,f). \end{aligned}$$

Since $\overline{N}(r,0;f_2) \le \overline{N}(r,0;f<g)$, it follows from above

$$\begin{aligned} T(r,\alpha) &\le N_2(r,0;f_1) + 2\overline{N}(r,0;f<g) + 2\overline{N}(r,\infty;f<g) \\ &\quad + \overline{N}(r,\infty;f>g) + \overline{N}(r,0;f>g) + \overline{N}(r,0;f>g) \\ &\quad + \overline{N}(r,\infty;f>g) + S(r,f) \\ &= N_2(r,0;f_1) + 2\overline{N}_*(r,0;f,g) + 2\overline{N}_*(r,\infty;f,g) + S(r,f). \end{aligned}$$

We see that $(1-a)f_1 \equiv (f-a)(1-\alpha h) \equiv (1-\alpha) - a(1-\alpha h)$ and $f \equiv (1-\alpha)/(1-\alpha h)$. So z_o is a possible zero of f_1 if either z_o is a zero of $f-a$ or z_o is a common zero of $1-\alpha$ and $1-\alpha h$. Therefore,

$$N_2(r,0;f_1) \;\le\; N_2(r,a;f) + N(r,0;1-\alpha h) - N(r,\infty;f \mid \alpha \ne \infty),$$

where $N(r,\infty;f \mid \alpha \ne \infty)$ denotes the counting function of those poles of f, counted with proper multiplicities, which are not the poles of α.

Since f, g share $(0,1)$, $(1,\infty)$, $(\infty,0)$, by *Lemma* 2.1 we obtain

$$\overline{N}_*(r,0;f,g) \le \overline{N}(r,0;f \mid \ge 2) \le \overline{N}_*(r,\infty;f,g) + S(r,f).$$

Therefore,

$$\begin{aligned} T(r,\alpha) &\le N_2(r,a;f) + 4\overline{N}_*(r,\infty;f,g) + N(r,0;1-\alpha h) \\ &\quad - N(r,\infty;f \mid \alpha \ne \infty) + S(r,f). \end{aligned}$$

Now we note that

$$\begin{aligned} N(r,\infty;f) - N(r,\infty;f \mid \alpha \ne \infty) &= N(r,\infty;f \mid \alpha = \infty) \\ &= N(r,\infty;f>g) \le N_*(r,\infty;f,g), \end{aligned}$$

where $N(r,\infty;f \mid \alpha = \infty)$ denotes the counting function of those poles of f, counted with proper multiplicities, which are also poles of α.

Since $f \equiv (1-\alpha)/(1-\alpha h)$, it follows that

$$\begin{aligned}\overline{N}(r,0;f) &\le N(r,0;1-\alpha) - N(r,0;1-\alpha h) + N(r,\infty;f) + \overline{N}(r,\infty;\alpha h)\\ &\le T(r,\alpha) - N(r,0;1-\alpha h) + N(r,\infty;f) + \overline{N}(r,0;f>g) + O(1)\\ &\le N_2(r,a;f) + 4\overline{N}_*(r,\infty;f,g) + \overline{N}(r,0;f\mid\ge 2)\\ &\quad + N_*(r,\infty;f,g) + S(r,f).\end{aligned}$$

Hence by *Lemma* 2.1 we get

$$\overline{N}(r,0;f) \le N_2(r,a;f) + 5\overline{N}_*(r,\infty;f,g) + N_*(r,\infty;f,g) + S(r,f).$$

If α is a constant, it follows that $\overline{N}(r,0;f) \equiv 0$ because $f-1 \equiv \alpha(g-1)$, $f \not\equiv g$ and f, g share $(0,1)$.

If h is a constant then $\overline{N}(r,1;f) \equiv 0$ because $g \equiv hf$, $f \not\equiv g$ and f, g share $(1,\infty)$. So we suppose that h is nonconstant. Let

$$g_1 = \frac{-f_1}{f_3} = \frac{-(f-a)(1-\alpha h)}{\alpha},\quad g_2 = \frac{1}{f_3} = \frac{1-a}{\alpha}\quad\text{and}\quad g_3 = \frac{-f_2}{f_3} = ah.$$

Then $g_1+g_2+g_3 \equiv 1$ and by *Lemma* 2.7 g_1, g_2, g_3 are linearly independent. Since by *Lemma* 2.2 $\sum_{i=1}^{3} S(r,g_i) = S(r,f)$, applying *Lemma* 2.6 to g_1, g_2, g_3 we get

$$\begin{aligned}T(r,h) &\le \sum_{i=1}^{3} N_2(r,0;g_i) + \sum_{i=1}^{3} \overline{N}(r,\infty;g_i) + S(r,f)\\ &\le N_2(r,0;g_1) + 2\overline{N}(r,\infty;f>g) + 2\overline{N}(r,0;f<g)\\ &\quad + 2\overline{N}(r,\infty;f>g) + \overline{N}(r,\infty;g_1) + \overline{N}(r,\infty;f<g)\\ &\quad + \overline{N}(r,0;f>g) + \overline{N}(r,\infty;f<g) + S(r,f). \qquad (2.4)\end{aligned}$$

Since $g_1 = (1-a/f)\{1-(g-1)/(f-1)\}$ and f, g share $(0,1)$, $(1,\infty)$, $(\infty,0)$, it follows that possible poles of g_1 occur at the zeros and poles of f and g.

Let z_o be a zero of f and g with multiplicities m and n respectively. Then in some neighbourhood of z_o we get

$$g_1(z) = \frac{\{(z-z_o)^m\phi - a\}\{(z-z_o)^m\phi - (z-z_o)^n\psi\}}{(z-z_o)^m\phi\{(z-z_o)^m\phi - 1\}},$$

where ϕ, ψ are analytic at z_o and $\phi(z_o) \ne 0$, $\psi(z_o) \ne 0$. This shows that z_o is a pole of g_1 if $m > n$.

Again let z_1 be a pole of f and g with multiplicities p and q respectively. Then in some neighbourhood of z_1 we get

$$g_1(z) = \frac{\{\lambda(z-z_1)^{q-p} - \mu\}\{\lambda - a(z-z_1)^p\}}{\lambda(z-z_1)^{q-p}\{\lambda - (z-z_1)^p\}},$$

where λ, μ are analytic at z_1 and $\lambda(z_1) \ne 0$, $\mu(z_1) \ne 0$. This shows that z_1 is a pole of g_1 if $q > p$. Therefore

$$\overline{N}(r,\infty;g_1) \le \overline{N}(r,0;f>g) + \overline{N}(r,\infty;f<g).$$

Now from (2.4) we get

$$\begin{aligned}T(r,h) &\le N_2(r,0;g_1) + 3\overline{N}_*(r,\infty;f,g) + \overline{N}(r,\infty;f>g)\\ &\quad + 2\overline{N}_*(r,0;f,g) + S(r,f). \qquad (2.5)\end{aligned}$$

Since f, g share $(0,1)$, $(1,\infty)$, $(\infty,0)$, it follows from *Lemma* 2.1 that

$$\overline{N}_*(r,0;f,g) \le \overline{N}(r,0;f\mid\ge 2) \le \overline{N}_*(r,\infty;f,g)+S(r,f).$$

So from (2.5) we get

$$T(r,h) \le N_2(r,0;g_1)+5\overline{N}_*(r,\infty;f,g)+\overline{N}(r,\infty;f>g)+S(r,f).$$

We see that

$$g_1=\frac{-(f-a)(1-\alpha h)}{\alpha}=\frac{a(1-\alpha h)-(1-\alpha)}{\alpha} \quad \text{and} \quad f=\frac{1-\alpha}{1-\alpha h}.$$

So z_o is a possible zero of g_1 if (i) z_o is a zero of $f-a$, (ii) z_o is a common zero of $1-\alpha$ and $1-\alpha h$, (iii) z_o is a pole of α.

If z_o is a pole of $\alpha=(f-1)/(g-1)$ then z_o is a pole of f and g with multiplicities m and $n(<m)$ respectively because f, g share $(1,\infty)$, $(\infty,0)$. Again since $g_1=(1-a/f)\{1-(g-1)/(f-1)\}$, it follows that $g_1(z_o)=1$. So a pole of α is not a zero of g_1. Therefore

$$N_2(r,0;g_1)\le N_2(r,a;f)+N(r,0;1-\alpha h)-N(r,\infty;f\mid\alpha\ne\infty).$$

So

$$\begin{aligned} T(r,h) \le\ & N_2(r,a;f)+N(r,0;1-\alpha h)-N(r,\infty;f\mid\alpha\ne\infty)\\ &+5\overline{N}_*(r,\infty;f,g)+\overline{N}(r,\infty;f>g)+S(r,f). \end{aligned}$$

Now we note in view of *Lemma* 2.3(i) that

$$\begin{aligned} &\overline{N}(r,0;\alpha)+N(r,\infty;f)-N(r,\infty;f\mid\alpha\ne\infty)\\ &\qquad = \overline{N}(r,\infty;f<g)+N(r,\infty;f\mid\alpha=\infty)\\ &\qquad = \overline{N}(r,\infty;f<g)+N(r,\infty;f>g)\\ &\qquad \le \overline{N}(r,\infty;f<g)+N_*(r,\infty;f,g). \end{aligned}$$

Since

$$f-1 \equiv \frac{(1-h)\alpha}{1-\alpha h},$$

we get

$$\begin{aligned} \overline{N}(r,1;f) &\le N(r,1;h)+\overline{N}(r,0;\alpha)-N(r,0;1-\alpha h)+N(r,\infty;f)\\ &\le T(r,h)+\overline{N}(r,0;\alpha)-N(r,0;1-\alpha h)+N(r,\infty;f)+O(1)\\ &\le N_2(r,a;f)+5\overline{N}_*(r,\infty;f,g)+\overline{N}(r,\infty;f<g)\\ &\quad +\overline{N}(r,\infty;f>g)+N_*(r,\infty;f,g)+S(r,f)\\ &= N_2(r,a;f)+6\overline{N}_*(r,\infty;f,g)+N_*(r,\infty;f,g)+S(r,f). \end{aligned}$$

This proves the lemma. □

3. Proofs of the Main Results

We shall prove *Theorem* 1.9 and *Corollary* 1.11 only because *Theorem* 1.10 and *Corollary* 1.12 can be proved similarly noting that if f, g share $(\infty, 0)$ then $\overline{N}_*(r,\infty;f,g) \le \overline{N}(r,\infty;f)$ and $N_*(r,\infty;f,g) \le N(r,\infty;f)$.

Proof of Theorem 1.12. Let f_1, f_2, f_3 be defined as in *Lemma* 2.8. If possible, suppose that f_1, f_2, f_3 are linearly independent. Then by the second fundamental theorem and *Lemma* 2.8 we get

$$\begin{aligned} 2T(r,f) &\le \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,a;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\le 3N_2(r,a;f) + 11\overline{N}_*(r,\infty;f,g) + \overline{N}(r,\infty;f) \\ &\quad + 2N_*(r,\infty;f,g) + S(r,f). \end{aligned} \tag{3.1}$$

Since f, g share $(\infty, 11)$, it follows that

$$\begin{aligned} &11\overline{N}_*(r,\infty;f,g) + \overline{N}(r,\infty;f) + 2N_*(r,\infty;f,g) \\ &\qquad \le 11\overline{N}(r,\infty;f \mid \ge 12) + \overline{N}(r,\infty;f) + 2N_*(r,\infty;f,g) \le 3N(r,\infty;f). \end{aligned}$$

So we get from (3.1)

$$2T(r,f) \;\le\; 3N_2(r,a;f) + 3N(r,\infty;f) + S(r,f)$$

which implies

$$3\delta_2(a;f) + 3\delta(\infty;f) \;\le\; 4.$$

This contradicts the given condition. So there exist constants c_1, c_2, c_3, not all zero, such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 \;\equiv\; 0. \tag{3.2}$$

If possible, let $c_1 = 0$. Then from (3.2) and the definitions of f_2, f_3 it follows that h is a constant. Since $f \not\equiv g$, we see that $h \ne 1$ and so 1 becomes a Picard exceptional value of f because f, g share $(1,\infty)$ and $g \equiv hf$.

Also we note that $\alpha = (f-1)/(g-1) = (f-1)(hf-1)$ has no pole because h is a constant and f, g share $(1,\infty)$. Since

$$f \;\equiv\; \frac{1}{h} + \frac{h-1}{h(1-\alpha h)}$$

and α has no pole, it follows that $1/h$ is also a Picard exceptional value of f. Therefore, $\delta(\infty;f) = 0$, which contradicts the given condition. So $c_1 \ne 0$.

Since $f_1 + f_2 + f_3 \equiv 1$, we get from (3.2)

$$cf_2 + df_3 \;\equiv\; 1, \tag{3.3}$$

where $|c| + |d| \ne 0$.

Now we consider the following cases.

Case I. Let $c \ne 0$ and $d \ne 0$. Then from (3.3) we get

$$\frac{-ac\alpha h}{1-a} + \frac{d\alpha}{1-a} \;\equiv\; 1.$$

Since f, g share $(0,1)$, $(1,\infty)$, $(\infty,11)$, by *Lemma* 2.5 we obtain

$$\begin{aligned} T(r,f) &\leq 6\overline{N}(r,\infty;f<g)+S(r,f) \\ &\leq 6\overline{N}(r,\infty;f\mid\geq 12)+S(r,f) \\ &\leq N(r,\infty;f)+S(r,f). \end{aligned}$$

This shows that $\delta(\infty;f)=0$, which contradicts the given condition.

Case II. Let $c=0$ and $d\neq 0$. Then from (3.3) we see that α is a constant. Since $\alpha=(f-1)/(g-1)$ and $f\not\equiv g$, it follows that $\alpha\neq 1$. So $N(r,0;f)\equiv 0$ because f, g share $(0,1)$.

Again since $f=(1-\alpha)/(1-\alpha h)$, we get from the second fundamental theorem that

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,1-\alpha;f)+\overline{N}(r,\infty;f)+S(r,f) \\ &= \overline{N}(r,0;h)+\overline{N}(r,\infty;f)+S(r,f). \end{aligned}$$

Since f, g share $(0,1)$, $(1,\infty)$, $(\infty,11)$ in view of *Lemma 1* and *Lemma 3* we get from above

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f<g)+\overline{N}(r,\infty;f>g)+\overline{N}(r,\infty;f)+S(r,f) \\ &\leq \overline{N}(r,0;f\mid\geq 2)+\overline{N}_*(r,\infty;f,g)+\overline{N}(r,\infty;f)+S(r,f) \\ &\leq 2\overline{N}_*(r,\infty;f,g)+\overline{N}(r,\infty;f)+S(r,f) \\ &\leq 2\overline{N}(r,\infty;f\mid\geq 12)+\overline{N}(r,\infty;f)+S(r,f) \\ &\leq N(r,\infty;f)+S(r,f). \end{aligned}$$

This implies $\delta(\infty;f)=0$ which contradicts the given condition.

Case III. Let $c\neq 0$ and $d=0$. Then from (3.3) we see that $\alpha h=p$,say, a constant. Since $f\not\equiv g$ and $\alpha h=\frac{g(f-1)}{f(g-1)}$, it follows that $p\neq 1$. So we get

$$f-a \equiv \frac{(1+ap-a)-\alpha}{1-p}. \tag{3.4}$$

If possible, let $1+ap-a\neq 0$. Then by the second fundamental theorem and *Lemma* 2.3 we get

$$\begin{aligned} T(r,\alpha) &\leq \overline{N}(r,\infty;\alpha)+\overline{N}(r,0;\alpha)+\overline{N}(r,1+ap-a;\alpha)+S(r,f) \\ &\leq \overline{N}_*(r,\infty;f,g)+\overline{N}(r,a;f)+S(r,f). \end{aligned}$$

Since $f=(1-\alpha)/(1-p)$, it follows that $T(r,f)=T(r,\alpha)+O(1)$. So, noting that f, g share $(\infty,11)$, we obtain from above

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,\infty;f\mid\geq 12)+\overline{N}(r,a;f)+S(r,f) \\ &\leq N_2(r,a;f)+N(r,\infty;f)+S(r,f). \end{aligned}$$

This implies $\delta_2(a;f)+\delta(\infty;f)\leq 1$, which contradicts the given condition.

Therefore $1+ap-a=0$ i.e. $p=(a-1)/a$. So from (3.4) we get

$$f-a \equiv -a\alpha. \tag{3.5}$$

Since $g = hf$, we get from (3.5)

$$g + a - 1 \equiv \frac{a-1}{\alpha}. \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$(f - a)(g + a - 1) \equiv a(1 - a).$$

This proves the theorem.

Proof of Corollary 1.14. Let $F = 1-f$ and $G = 1-g$. Then F, G are distinct and share $(0, 1)$, $(1, \infty)$, $(\infty, 11)$. Also $\delta_2(1 - a; F) = \delta_2(a; f)$ and $\delta(\infty; F) = \delta(\infty; f)$. So by *Theorem* 1.9 we get

$$(F - 1 + a)(G - a) \equiv a(1 - a)$$

i.e.,

$$(f - a)(g + a - 1) \equiv a(1 - a).$$

This proves the corollary.

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