UNIQUENESS OF MEROMORPHIC FUNCTIONS AND SHARING OF THREE VALUES WITH SOME WEIGHT

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Abstract. We prove some uniqueness theorems for meromorphic functions sharing three values with some weight which improve some known results.

1. Introduction, Definitions and Results

Let \( f \) and \( g \) be two nonconstant meromorphic functions defined in the open complex plane \( \mathbb{C} \). For \( b \in \mathbb{C} \cup \{\infty\} \) we say that \( f \) and \( g \) share the value \( b \) \( \text{CM} \) (counting multiplicities) if \( f \) and \( g \) have the same \( b \)–points with the same multiplicities. If the multiplicities are ignored, we say that \( f \) and \( g \) share the value \( b \) \( \text{IM} \) (ignoring multiplicities). We do not explain the standard notations and definitions of the value distribution theory because those are available in [2]. However, we now explain some notations and definitions which will be needed in the sequel.

Definition 1.1 ([3]). If \( s \) is a positive integer, we denote by \( N(r,a;f|\geq s) \) the counting function of those \( a \)–points of \( f \) whose multiplicities are greater than or equal to \( s \), where each \( a \)–point is counted only once.

Definition 1.2 ([3, 10]). If \( s \) is a positive integer, we denote by \( N_s(r,a;f) \) the counting function of \( a \)–points of \( f \), where an \( a \)–point with multiplicity \( m \) is counted \( m \) times if \( m \leq s \) and \( s \) times if \( m > s \). We put \( N_\infty(r,a;f) \equiv N(r,a;f) \).

Definition 1.3. Let \( f, g \) share a value \( a \) \( \text{IM} \). Let \( z \) be an \( a \)–point of \( f \) and \( g \) with multiplicities \( p_f(z) \) and \( p_g(z) \) respectively. If \( z \) is not an \( a \)–point of \( f \) and \( g \) then we suppose that \( p_f(z) = p_g(z) = 0 \). We put

\[
\nu_f(z) = \begin{cases} 
  p_f(z) & \text{if } p_f(z) = p_g(z) \neq 0 \\
  0 & \text{if } p_f(z) = p_g(z) = 0 \\
  0 & \text{if } p_f(z) \neq p_g(z);
\end{cases}
\]

and

\[
\varpi_f(z) = \begin{cases} 
  1 & \text{if } p_f(z) = p_g(z) \neq 0 \\
  0 & \text{if } p_f(z) = g_g(z) = 0 \\
  0 & \text{if } p_f(z) \neq p_g(z).
\end{cases}
\]

Clearly \( \nu_f(z) = \nu_g(z) \) and \( \varpi_f(z) = \varpi_g(z) \).

Now we put \( n_E(r,a;f,g) = \sum_{|z| \leq r} \nu_f(z) \) and \( \pi_E(r,a;f,g) = \sum_{|z| \leq r} \varpi_f(z) \).

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Also we denote by \(N_E(r, a; f, g)\) and \(\overline{N}_E(r, a; f, g)\) the integrated counting functions obtained from \(n_E(r, a; f, g)\) and \(\pi_E(r, a; f, g)\) respectively.

Finally we define \(N_s(r, a; f, g)\) and \(\overline{N}_s(r, a; f, g)\) as follows

\[
N_s(r, a; f, g) = N(r, a; f) - N_E(r, a; f, g)
\]

and

\[
\overline{N}_s(r, a; f, g) = \overline{N}(r, a; f) - \overline{N}_E(r, a; f, g).
\]

Clearly \(N_E(r, a; f, g) \equiv N_E(r, a; g, f)\), \(\overline{N}_E(r, a; f, g) \equiv \overline{N}_E(r, a; g, f)\) and \(\overline{N}_s(r, a; f, g) \equiv \overline{N}_s(r, a; g, f)\).

**Definition 1.4** ([3]). Let \(f, g\) share a value \(a\) IM. We denote by \(\overline{N}(r, a; f < g)\) \((\overline{N}(r, a; f > g))\) the counting function of those \(a\)-points of \(f\) whose multiplicities are less (greater) than the multiplicities of the corresponding \(a\)-points of \(g\), where each \(a\)-point is counted only once.

**Definition 1.5.** We put

\[
\delta_s(a; f) = 1 - \limsup_{r \to \infty} \frac{N_s(r, a; f)}{T(r, f)}
\]

where \(s\) is a positive integer.

Clearly \(\delta(a; f) \leq \delta_s(a; f) \leq \delta_{s-1}(a; f) \leq \cdots \leq \delta_1(a; f) = \Theta(a; f) \leq 1\).

H. Ueda [6] proved the following result.

**Theorem A** ([6]). Let \(f\) and \(g\) be two distinct nonconstant entire functions sharing \(0, 1\) CM and let \(a(\neq 0, 1)\) be a finite complex number. If \(a\) is lacunary for \(f\) then \(1 - a\) is lacunary for \(g\) and \((f - a)(g + a - 1) \equiv a(1 - a)\).

Improving Theorem A H.X. Yi [8] proved the following theorem.

**Theorem B** ([8]). Let \(f\) and \(g\) be two distinct nonconstant entire functions sharing \(0, 1\) CM and let \(a(\neq 0, 1)\) be a finite complex number. If \(\delta(a; f) > \frac{1}{2}\) then \(a\) and \(1 - a\) are Picard exceptional values of \(f\) and \(g\) respectively and \((f - a)(g + a - 1) \equiv a(1 - a)\).

S.Z. Ye [7] extended Theorem B to meromorphic functions and proved the following result.

**Theorem C** ([7]). Let \(f\) and \(g\) be two distinct nonconstant meromorphic functions such that \(f\) and \(g\) share \(0, 1, \infty\) CM. Let \(a(\neq 0, 1)\) be a finite complex number. If \(\delta(a; f) + \delta(\infty; f) > \frac{1}{2}\) then \(a\) and \(1 - a\) are Picard exceptional values of \(f\) and \(g\) respectively and also \(\infty\) is a Picard exceptional value of both \(f\) and \(g\) and \((f - a)(g + a - 1) \equiv a(1 - a)\).

The following two examples show that in the above theorems the sharing of 0 and 1 can not be relaxed from CM to IM.

**Example 1.6.** Let \(f = e^z - 1, g = (e^z - 1)^2\) and \(a = -1\). Then \(f, g\) share 0 IM and \(1, \infty\) CM. Also \(N(r, \infty; f) \equiv 0\) and \(N(r, a; f) \equiv 0\) but \((f - a)(g + a - 1) \neq a(1 - a)\).

**Example 1.7.** Let \(f = 2 - e^z, g = e^z(2 - e^z)\) and \(a = 2\). Then \(f, g\) share 1 IM and \(0, \infty\) CM. Also \(N(r, \infty; f) \equiv 0\) and \(N(r, a; f) \equiv 0\) but \((f - a)(g + a - 1) \neq a(1 - a)\).
Now one may ask the following question: **Is it possible in any way to relax the nature of sharing of values in the theorems stated above?**

In the paper we study this problem. To this end we now explain a gradation of sharing values which measures how close a shared value is to being shared IM or to being shared CM and is called weight of sharing.

**Definition 1.8** ([3, 4]). Let $k$ be a nonnegative integer or infinity. For $a \in C \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$–points of $f$ where an $a$–point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f, g$ share the value $a$ with weight $k$.

The definition implies that if $f, g$ share a value $a$ with weight $k$ then $z_0$ is an $a$–point of $f$ with multiplicity $m(\leq k)$ if and only if it is an $a$–point of $g$ with multiplicity $m(\leq k)$ and $z_0$ is an $a$–point of $f$ with multiplicity $m(> k)$ if and only if it is an $a$–point of $g$ with multiplicity $m(> k)$ where $m$ is not necessarily equal to $n$.

We write $f, g$ share $(a, k)$ to mean that $f, g$ share the value $a$ with weight $k$. Clearly if $f, g$ share $(a, k)$ then $f, g$ share $(a, p)$ for all integers $p, 0 \leq p < k$. Also we note that $f, g$ share a value $a$ IM or CM if and only if $f, g$ share $(a, 0)$ or $(a, \infty)$ respectively.

Now we state the main results of the paper.

**Theorem 1.9.** Let $f$ and $g$ be two distinct meromorphic functions sharing $(0, 1)$, $(1, \infty)$, $(\infty, 11)$. If $a(\neq 0, 1, \infty)$ is a complex number such that $3\delta_2(a; f) + 3\delta(\infty; f) > 4$ then $a$ and $1 - a$ are Picard exceptional values of $f$ and $g$ respectively and also $\infty$ is a Picard exceptional value of both $f$ and $g$ and $(f - a)(g + a - 1) \equiv a(1 - a)$.

**Theorem 1.10.** Let $f$ and $g$ be two distinct meromorphic functions sharing $(0, 1)$, $(1, \infty)$, $(\infty, 0)$. If $a(\neq 0, 1, \infty)$ is a complex number such that $3\delta_2(a; f) + 14\delta(\infty; f) > 15$ then $a$ and $1 - a$ are Picard exceptional values of $f$ and $g$ respectively and also $\infty$ is a Picard exceptional value of both $f$ and $g$ and $(f - a)(g + a - 1) \equiv a(1 - a)$.

**Corollary 1.11.** Let $f$ and $g$ be two distinct meromorphic functions sharing $(0, \infty)$, $(1, 1)$, $(\infty, 11)$. If $a(\neq 0, 1, \infty)$ is a complex number such that $3\delta_2(a; f) + 3\delta(\infty; f) > 4$ then $a$ and $1 - a$ are Picard exceptional values of $f$ and $g$ respectively and also $\infty$ is a Picard exceptional value of both $f$ and $g$ and $(f - a)(g + a - 1) \equiv a(1 - a)$.

**Corollary 1.12.** Let $f$ and $g$ be two distinct meromorphic functions sharing $(0, \infty)$, $(1, 1)$, $(\infty, 0)$. If $a(\neq 0, 1, \infty)$ is a complex number such that $3\delta_2(a; f) + 14\delta(\infty; f) > 15$ then $a$ and $1 - a$ are Picard exceptional values of $f$ and $g$ respectively and also $\infty$ is a Picard exceptional value of both $f$ and $g$ and $(f - a)(g + a - 1) \equiv a(1 - a)$.

**Example** 1.6 shows that in **Theorem 1.9** and **Theorem 1.10** sharing $(0, 1)$ cannot be relaxed to sharing $(0, 0)$. Also **Example 1.7** shows that in **Corollary 1.11** and **Corollary 1.12** sharing $(1, 1)$ cannot be relaxed to sharing $(1, 0)$.

Throughout the paper we denote by $f, g$ two nonconstant meromorphic functions defined in $C$. 
2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([3]). Let \( f, g \) share \((0, 1), (1, \infty), (\infty, 0)\) and \( f \neq g \). Then

\[
\mathcal{N}(r, 0; f) \geq 2 = \mathcal{N}(r, 0; g) \leq \mathcal{N}(r, \infty; f, g) + S(r, f).
\]

Lemma 2.2 (cf. [1, 3]). Let \( f, g \) share \((0, 0), (1, 0), (\infty, 0)\). Then

(i) \( T(r, g) \leq 3T(r, f) + S(r, g) \),

(ii) \( T(r, f) \leq 3T(r, g) + S(r, f) \).

Lemma 2.3. Let \( f, g \) share \((0, 0), (1, \infty), (\infty, 0)\) and \( f \neq g \). If \( \alpha = (f-1)/(g-1) \) and \( h = \frac{g}{f} \) then

(i) \( \mathcal{N}(r, 0; \alpha) = \mathcal{N}(r, \infty; f < g) \),

(ii) \( \mathcal{N}(r, \infty; \alpha) = \mathcal{N}(r, \infty; f > g) \),

(iii) \( \mathcal{N}(r, 0; h) = \mathcal{N}(r, 0; f < g) + \mathcal{N}(r, \infty; f > g) \),

(iv) \( \mathcal{N}(r, \infty; h) = \mathcal{N}(r, 0; f > g) + \mathcal{N}(r, \infty; f < g) \).

Proof. Let \( z_0 \) be a pole of \( f \) and \( g \) with multiplicities \( m \) and \( n \) respectively. Since \( h = \frac{g}{f} \), it follows that \( z_0 \) will be a pole of \( h \) if \( m < n \).

Again let \( z_1 \) be a zero of \( f \) and \( g \) with multiplicities \( p \) and \( q \) respectively. It is easy to verify that \( z_1 \) will be a pole of \( h \) if \( p > q \).

Therefore

\[
\mathcal{N}(r, \infty; h) = \mathcal{N}(r, 0; f > g) + \mathcal{N}(r, \infty; f < g).
\]

By similar arguments we can prove (i), (ii) and (iii). This proves the lemma. \( \square \)

Lemma 2.4. Let \( af + bg \equiv c \), where \( a, b, c \) be nonzero constants. Then

\[
T(r, f) < \mathcal{N}(r, 0; f) + \mathcal{N}(r, 0; g) + \mathcal{N}(r, \infty; f) + S(r, f).
\]

Proof. The lemma follows as a consequence of the second fundamental theorem. \( \square \)

Lemma 2.5. Let \( f, g \) share \((0, 1), (1, \infty), (\infty, 0)\) and \( \alpha, h \) be defined as in Lemma 2.3. If \( \alpha h + b \alpha \equiv c \) for nonzero constants \( a, b, c \) then

\[
T(r, f) \leq 6\mathcal{N}(r, \infty; f < g) + S(r, f).
\]

Proof. If one of \( \alpha \) and \( \alpha h \) is constant then from the given condition we see that the other is also constant. Since \( f = (1 - \alpha)/(1 - \alpha h) \), it follows that \( f \) becomes a constant, which is a contradiction. So \( \alpha \) and \( \alpha h \) are nonconstant.

Let \( z_0 \) be a pole of \( f \) and \( g \) with multiplicities \( p \) and \( q \) respectively. If \( p > q \) then \( z_0 \) is a zero of \( 1/\alpha \) and \( h \). So from \( \alpha h + b \equiv c/\alpha \), it follows that \( b = 0 \), which is a contradiction. So \( N(r, \infty; f > g) \equiv 0 \).

Again let \( z_1 \) be a zero of \( f \) and \( g \) with multiplicities \( m \) and \( n \) respectively. If \( m > n \) then \( z_0 \) is a pole of \( h \) but \( z_0 \) is a regular point of \( \alpha \) with \( \alpha(z_0) = 1 \). Since \( \alpha h \equiv (c/\alpha) - b \), this implies a contradiction. So \( N(r, 0; f > g) \equiv 0 \).

By Lemmas 2.2, 2.3 and 2.4 we get in view of the first fundamental theorem

\[
T(r, \alpha) \leq \mathcal{N}(r, 0; \alpha) + \mathcal{N}(r, \infty; \alpha) + \mathcal{N}(r, 0; h) + S(r, \alpha)
\leq \mathcal{N}(r, 0; f < g) + \mathcal{N}(r, \infty; f < g) + S(r, f).
\]  \( (2.1) \)
Again since \( ah + b \equiv c/\alpha \), it follows from Lemma 2.3 that
\[
\overline{N}(r, \frac{-b}{a}; h) = \overline{N}(r, \infty; \alpha) = \overline{N}(r, \infty; f > g) \equiv 0,
\]
and
\[
\overline{N}(r, 0; h) = \overline{N}(r, 0; f < g),
\]
By the second fundamental theorem we get
\[
T(r, h) \leq \overline{N}(r, \infty; h) + \overline{N}(r, 0; h) + \overline{N}(r, -b/a; h) + S(r, h)
\leq \overline{N}(r, 0; f < g) + \overline{N}(r, \infty; f < g) + S(r, h).
\]
(2.2)
From the definitions of \( \alpha \) and \( h \) it follows in view of Lemma 2.2 and the first fundamental theorem that \( S(r, \alpha) = S(r, f) \) and \( S(r, h) = S(r, f) \).

Since \( f, g \) share \((0,1), (1, \infty), (\infty,0)\), it follows from Lemma 2.1 that
\[
\overline{N}(r, 0; f < g) \leq \overline{N}(r, 0; f \geq 2)
\leq \overline{N}(r, \infty; f, g) + S(r, f)
= \overline{N}(r, \infty; f < g) + S(r, f).
\]
(2.3)
Again since \( f = (1 - \alpha)/(1 - \alpha h) \), it follows from (2.1), (2.2), (2.3) and the first fundamental theorem that
\[
T(r, f) \leq 2T(r, \alpha) + T(r, h) + O(1)
\leq 3\overline{N}(r, 0; f < g) + 3\overline{N}(r, \infty; f < g) + S(r, f)
\leq 6\overline{N}(r, \infty; f < g) + S(r, f).
\]
This proves the lemma. \( \square \)

**Lemma 2.6 ([5]).** Let \( f_1, f_2, f_3 \) be nonconstant meromorphic functions such that \( f_1 + f_2 + f_3 \equiv 1 \). If \( f_1, f_2, f_3 \) are linearly independent then
\[
T(r, f_i) \leq \sum_{i=1}^{3} N_2(r, 0; f_i) + \sum_{i=1}^{3} \overline{N}(r, \infty; f_i) + S(r),
\]
where \( S(r) = \sum_{i=1}^{3} S(r, f_i) \).

**Lemma 2.7 ([9]).** Let \( f_1, f_2, f_3 \) be nonconstant meromorphic functions such that \( f_1 + f_2 + f_3 \equiv 1 \) and let \( g_1 = -f_1/f_3, g_2 = 1/f_3, g_3 = -f_2/f_3 \). If \( f_1, f_2, f_3 \) are linearly independent then \( g_1, g_2, g_3 \) are linearly independent.

**Lemma 2.8.** Let \( f, g \) share \((0,1), (1, \infty), (\infty,0)\) and \( f \neq g \). Let
\[
f_1 = \frac{(f - a)(1 - \alpha h)}{1 - a}, \quad f_2 = -\frac{a\alpha h}{1 - a}, \quad \text{and} \quad f_3 = \frac{\alpha}{1 - a},
\]
where \( a(\neq 0,1,\infty) \) be a complex number and \( \alpha, h \) are defined as in Lemma 2.3. If \( f_1, f_2, f_3 \) are linearly independent then
\[
(i) \quad \overline{N}(r, 0; f) \leq N_2(r, \infty; f) + 5\overline{N}_*(r, \infty; f, g) + N_*(r, \infty; f, g) + S(r, f),
\]
\[
(ii) \quad \overline{N}(r, 1; f) \leq N_2(r, \infty; f) + 6\overline{N}_*(r, \infty; f, g) + N_*(r, \infty; f, g) + S(r, f).
\]
Proof. Since \((1 - a)f_1 = (1 - \alpha) - a(1 - ah)\), it follows that

\[
\overline{N}(r, \infty; f_1) \leq \overline{N}(r, \infty; f > g) + \overline{N}(r, 0; f > g)
\]

because if \(z_o\) is a pole of \(f\) and \(g\) with multiplicities \(m\) and \(n\) respectively then at \(z_o, \alpha h = \frac{g(f - 1)}{f(1 - \alpha h)}\) has no pole and at \(z_o, \alpha\) has a pole if \(m > n\).

Also \(\overline{N}(r, \infty; f_2) \leq \overline{N}(r, 0; f > g)\) and \(\overline{N}(r, \infty; f_3) \leq \overline{N}(r, \infty; f > g)\). First we suppose that \(\alpha\) is nonconstant. Since by Lemma 2.2 \(\sum_{i=1}^{3} S(r, f_1) = S(r, f)\), we get by Lemma 2.6

\[
T(r, \alpha) \leq N_2(r, 0; f_1) + N_2(r, 0; f_2) + N_2(r, 0; f_3)
\]

\[
+ \overline{N}(r, \infty; f_1) + \overline{N}(r, \infty; f_2) + \overline{N}(r, \infty; f_3) + S(r, f)
\]

\[
\leq N_2(r, 0; f_1) + 2\overline{N}(r, 0; f < g) + 2\overline{N}(r, \infty; f < g)
\]

\[
+ \overline{N}(r, \infty; f > g) + \overline{N}(r, 0; f > g) + \overline{N}(r, 0; f > g) + S(r, f)
\]

\[
= N_2(r, 0; f_1) + 2\overline{N}_*(r, 0; f, g) + 2\overline{N}_*(r, \infty; f, g) + S(r, f).
\]

We see that \((1 - a)f_1 \equiv (f - a)(1 - \alpha h) \equiv (1 - \alpha) - a(1 - ah)\) and \(f \equiv (1 - a)/\alpha\). So \(z_o\) is a possible zero of \(f_1\) if either \(z_o\) is a zero of \(f - a\) or \(z_o\) is a common zero of \(1 - \alpha\) and \(1 - ah\). Therefore,

\[
N_2(r, 0; f_1) \leq N_2(r, a; f) + N(r, 0; 1 - ah) - N(r, \infty; f | \alpha \neq \infty),
\]

where \(N(r, \infty; f | \alpha \neq \infty)\) denotes the counting function of those poles of \(f\), counted with proper multiplicities, which are not the poles of \(\alpha\).

Since \(f, g\) share \((0, 1), (1, \infty), (\infty, 0)\), by Lemma 2.1 we obtain

\[
\overline{N}_*(r, 0; f, g) \leq \overline{N}(r, 0; f \geq 2) \leq \overline{N}_*(r, \infty; f, g) + S(r, f).
\]

Therefore,

\[
T(r, \alpha) \leq N_2(r, a; f) + 4\overline{N}_*(r, \infty; f, g) + N(r, 0; 1 - ah)
\]

\[
- N(r, \infty; f | \alpha \neq \infty) + S(r, f).
\]

Now we note that

\[
N(r, \infty; f) - N(r, \infty; f | \alpha \neq \infty) = N(r, \infty; f | \alpha = \infty)
\]

\[
= N(r, \infty; f > g) \leq \overline{N}_*(r, \infty; f, g),
\]

where \(N(r, \infty; f | \alpha = \infty)\) denotes the counting function of those poles of \(f\), counted with proper multiplicities, which are also poles of \(\alpha\).
Since $f \equiv (1 - \alpha)/(1 - \alpha h)$, it follows that
\[
\overline{N}(r, 0; f) \leq N(r, 0; 1 - \alpha) + N(r, 0; 1 - \alpha h) + N(r, \infty; f) + \overline{N}(r, \infty; \alpha h)
\]
\[
\leq T(r, \alpha) - N(r, 0; 1 - \alpha h) + N(r, \infty; f) + \overline{N}(r, 0; f > g) + O(1)
\]
\[
\leq N_2(r, \alpha; f) + 4\overline{N}_*(r, \infty; f, g) + \overline{N}(r, 0; f \geq 2)
\]
\[
+ \overline{N}_*(r, \infty; f, g) + S(r, f).
\]
Hence by Lemma 2.1 we get
\[
\overline{N}(r, 0; f) \leq N_2(r, \alpha; f) + 5\overline{N}_*(r, \infty; f, g) + N_*(r, \infty; f, g) + S(r, f).
\]
If $\alpha$ is a constant, it follows that $\overline{N}(r, 0; f) \equiv 0$ because $f - 1 \equiv \alpha(g - 1), f \not\equiv g$ and $f, g$ share $(0, 1)$.
If $h$ is a constant then $\overline{N}(r, 1; f) \equiv 0$ because $g \equiv hf, f \not\equiv g$ and $f, g$ share $(1, \infty)$. So we suppose that $h$ is nonconstant. Let
\[
g_1 = \frac{-f_1}{f_3} = \frac{-(f - a)(1 - \alpha h)}{\alpha}, \quad g_2 = \frac{1}{f_3} = \frac{1 - a}{\alpha} \quad \text{and} \quad g_3 = \frac{-f_2}{f_3} = ah.
\]
Then $g_1 + g_2 + g_3 \equiv 1$ and by Lemma 2.7 $g_1, g_2, g_3$ are linearly independent. Since by Lemma 2.2 $\sum_{i=1}^3 N(r, g_i) = S(r, f)$, applying Lemma 2.6 to $g_1, g_2, g_3$ we get
\[
T(r, h) \leq \sum_{i=1}^3 N_2(r, 0; g_i) + \sum_{i=1}^3 \overline{N}(r, \infty; g_i) + S(r, f)
\]
\[
\leq N_2(r, 0; g_1) + 2\overline{N}(r, \infty; f > g) + 2\overline{N}(r, 0; f < g)
\]
\[
+ 2\overline{N}(r, \infty; f > g) + \overline{N}(r, \infty; g_1) + \overline{N}(r, \infty; f < g)
\]
\[
+ \overline{N}(r, 0; f > g) + \overline{N}(r, \infty; f < g) + S(r, f).
\]
(2.4)
Since $g_1 = (1 - a/f)(1 - (g - 1)/(f - 1))$ and $f, g$ share $(0, 1), (1, \infty), (\infty, 0)$, it follows that possible poles of $g_1$ occur at the zeros and poles of $f$ and $g$.
Let $z_0$ be a zero of $f$ and $g$ with multiplicities $m$ and $n$ respectively. Then in some neighbourhood of $z_0$, we get
\[
g_1(z) = \frac{(z - z_0)^m \phi - a}{(z - z_0)^m \phi - 1}\frac{(z - z_0)^n \psi - \phi}{(z - z_0)^n \psi - \phi - 1},
\]
where $\phi, \psi$ are analytic at $z_0$ and $\phi(z_0) \neq 0, \psi(z_0) \neq 0$. This shows that $z_0$ is a pole of $g_1$ if $m > n$.
Again let $z_1$ be a pole of $f$ and $g$ with multiplicities $p$ and $q$ respectively. Then in some neighbourhood of $z_1$ we get
\[
g_1(z) = \frac{\lambda(z - z_1)^q - p - \mu}{(z - z_1)^q - \lambda - (z - z_1)^p},
\]
where $\lambda, \mu$ are analytic at $z_1$ and $\lambda(z_1) \neq 0, \mu(z_1) \neq 0$. This shows that $z_1$ is a pole of $g_1$ if $q > p$.
Therefore
\[
\overline{N}(r, \infty; g_1) \leq \overline{N}(r, 0; f > g) + \overline{N}(r, \infty; f < g).
\]
Now from (2.4) we get
\[
T(r, h) \leq N_2(r, 0; g_1) + 3\overline{N}_*(r, \infty; f, g) + \overline{N}(r, \infty; f > g)
\]
\[
+ 2\overline{N}_*(r, 0; f, g) + S(r, f).
\]
(2.5)
Since \( f, g \) share \((0, 1), (1, \infty), (\infty, 0)\), it follows from Lemma 2.1 that
\[
\mathcal{N}_e(r, 0; f, g) \leq \mathcal{N}(r, 0; f \geq 2) \leq \mathcal{N}_e(r, \infty; f, g) + S(r, f).
\]
So from (2.5) we get
\[
T(r, h) \leq N_2(r, 0; g_1) + 5\mathcal{N}_e(r, \infty; f, g) + \mathcal{N}(r, \infty; f > g) + S(r, f).
\]
We see that
\[
g_1 = \frac{-(f-a)(1-ah)}{\alpha} = \frac{\alpha(1-ah) - (1-\alpha)}{\alpha} \quad \text{and} \quad f = 1 - \frac{\alpha}{1-ah}.
\]
So \( z_o \) is a possible zero of \( g_1 \) if (i) \( z_o \) is a zero of \( f - a \), (ii) \( z_o \) is a common zero of \( 1 - \alpha \) and \( 1 - ah \), (iii) \( z_o \) is a pole of \( \alpha \).
If \( z_o \) is a pole of \( \alpha = (f-1)/(g-1) \) then \( z_o \) is a pole of \( f \) and \( g \) with multiplicities \( m \) and \( n(\leq m) \) respectively because \( f, g \) share \((1, \infty), (\infty, 0)\). Again since \( g_1 = (1-a/f)(1-(g-1)/(f-1)) \), it follows that \( g_1(z_o) = 1 \). So a pole of \( \alpha \) is not a zero of \( g_1 \). Therefore
\[
N_2(r, 0; g_1) \leq N_2(r, \alpha; f) + N(r, 0; 1-ah) - N(r, \infty; f \mid \alpha \neq \infty).
\]
So
\[
T(r, h) \leq N_2(r, \alpha; f) + N(r, 0; 1-ah) - N(r, \infty; f \mid \alpha \neq \infty)
+ 5\mathcal{N}_e(r, \infty; f, g) + \mathcal{N}(r, \infty; f > g) + S(r, f).
\]
Now we note in view of Lemma 2.3(i) that
\[
\overline{\mathcal{N}}(r, 0; \alpha) + N(r, \infty; f) - N(r, \infty; f \mid \alpha \neq \infty)
= \overline{\mathcal{N}}(r, \infty; f < g) + N(r, \infty; f \mid \alpha = \infty)
= \overline{\mathcal{N}}(r, \infty; f < g) + N(r, \infty; f > g)
\leq \overline{\mathcal{N}}(r, \infty; f < g) + N_*(r, \infty; f, g).
\]
Since
\[
f - 1 \equiv \frac{(1-h)\alpha}{1-ah},
\]
we get
\[
\overline{\mathcal{N}}(r, 1; f) \leq N(r, 1; h) + \overline{\mathcal{N}}(r, 0; \alpha) - N(r, 0; 1-ah) + N(r, \infty; f)
\leq T(r, h) + \overline{\mathcal{N}}(r, 0; \alpha) - N(r, 0; 1-ah) + N(r, \infty; f) + O(1)
\leq N_2(r, \alpha; f) + 5\mathcal{N}_e(r, \infty; f, g) + \mathcal{N}(r, \infty; f < g)
+ \overline{\mathcal{N}}(r, \infty; f > g) + N_*(r, \infty; f, g) + S(r, f)
= N_2(r, \alpha; f) + 6\mathcal{N}_e(r, \infty; f, g) + N_*(r, \infty; f, g) + S(r, f).
\]
This proves the lemma. \( \square \)
3. Proofs of the Main Results

We shall prove Theorem 1.9 and Corollary 1.11 only because Theorem 1.10 and Corollary 1.12 can be proved similarly noting that if \( f, g \) share \((\infty, 0)\) then \( \overline{N}_s(r, \infty; f, g) \leq \overline{N}(r, \infty; f) \) and \( N_s(r, \infty; f, g) \leq N(r, \infty; f) \).

**Proof of Theorem 1.12.** Let \( f_1, f_2, f_3 \) be defined as in Lemma 2.8. If possible, suppose that \( f_1, f_2, f_3 \) are linearly independent. Then by the second fundamental theorem and Lemma 2.8 we get

\[
2T(r, f) \leq \overline{N}(r, 0; f) + \overline{N}(r, 1; f) + \overline{N}(r, a; f) + \overline{N}(r, \infty; f) + S(r, f)
\]

\[
\leq 3N_2(r, a; f) + 11\overline{N}_s(r, \infty; f, g) + \overline{N}(r, \infty; f) + 2N_s(r, \infty; f, g) + S(r, f).
\]

(3.1)

Since \( f, g \) share \((\infty, 11)\), it follows that

\[
11\overline{N}_s(r, \infty; f, g) + \overline{N}(r, \infty; f) + 2N_s(r, \infty; f, g)
\]

\[
\leq 11\overline{N}(r, \infty; f | \geq 12) + \overline{N}(r, \infty; f) + 2N_s(r, \infty; f, g) \leq 3N(r, \infty; f).
\]

So we get from (3.1)

\[
2T(r, f) \leq 3N_2(r, a; f) + 3N(r, \infty; f) + S(r, f)
\]

which implies

\[
3\delta_2(a; f) + 3\delta(\infty; f) \leq 4.
\]

This contradicts the given condition. So there exist constants \( c_1, c_2, c_3 \), not all zero, such that

\[
c_1f_1 + c_2f_2 + c_3f_3 \equiv 0.
\]

(3.2)

If possible, let \( c_1 = 0 \). Then from (3.2) and the definitions of \( f_2, f_3 \) it follows that \( h \) is a constant. Since \( f \neq g \), we see that \( h \neq 1 \) and so \( 1 \) becomes a Picard exceptional value of \( f \) because \( f, g \) share \((1, \infty)\) and \( g \equiv hf \).

Also we note that \( \alpha = (f - 1)/(g - 1) = (f - 1)(hf - 1) \) has no pole because \( h \) is a constant and \( f, g \) share \((1, \infty)\). Since

\[
f \equiv \frac{1}{h} + \frac{h - 1}{h(1 - ah)}
\]

and \( \alpha \) has no pole, it follows that \( 1/h \) is also a Picard exceptional value of \( f \).

Therefore, \( \delta(\infty; f) = 0 \), which contradicts the given condition. So \( c_1 \neq 0 \).

Since \( f_1 + f_2 + f_3 \equiv 1 \), we get from (3.2)

\[
cf_2 + df_3 \equiv 1,
\]

(3.3)

where \( |c| + |d| \neq 0 \).

Now we consider the following cases.

**Case I.** Let \( c \neq 0 \) and \( d \neq 0 \). Then from (3.3) we get

\[
\frac{-aca}{1-a} + \frac{da}{1-a} \equiv 1.
\]
Since $f$, $g$ share $(0,1), (1,\infty), (\infty,11)$, by Lemma 2.5 we obtain
\[
T(r, f) \leq 6N(r, \infty; f < g) + S(r, f)
\]
\[
\leq 6N(r, \infty; f \geq 12) + S(r, f)
\]
\[
\leq N(r, \infty; f) + S(r, f).
\]
This shows that $\delta(\infty; f) = 0$, which contradicts the given condition.

**Case II.** Let $c = 0$ and $d \neq 0$. Then from (3.3) we see that $\alpha$ is a constant. Since $\alpha = (f - 1)/(g - 1)$ and $f \neq g$, it follows that $\alpha \neq 1$. So $N(r, 0; f) \equiv 0$ because $f$, $g$ share $(0,1)$.

Again since $f = (1 - a)/(1 - ah)$, we get from the second fundamental theorem that
\[
T(r, f) \leq N(r, 0; f) + N(r, 1 - \alpha; f) + N(r, \infty; f) + S(r, f)
\]
\[
= N(r, 0; h) + N(r, \infty; f) + S(r, f).
\]
Since $f$, $g$ share $(0,1), (1,\infty), (\infty,11)$ in view of Lemma 1 and Lemma 3 we get from above
\[
T(r, f) \leq N(r, 0; f < g) + N(r, \infty; f > g) + N(r, \infty; f) + S(r, f)
\]
\[
\leq N(r, 0; f \geq 2) + N_s(r, \infty; f, g) + N(r, \infty; f) + S(r, f)
\]
\[
\leq 2N_s(r, \infty; f, g) + N(r, \infty; f) + S(r, f)
\]
\[
\leq 2N(r, \infty; f \geq 12) + N(r, \infty; f) + S(r, f)
\]
\[
\leq N(r, \infty; f) + S(r, f).
\]
This implies $\delta(\infty; f) = 0$ which contradicts the given condition.

**Case III.** Let $c \neq 0$ and $d = 0$. Then from (3.3) we see that $ah = p$, say, a constant. Since $f \neq g$ and $ah = \frac{a(f - 1)}{f(g - 1)}$, it follows that $p \neq 1$. So we get
\[
f - a = \frac{(1 + ap - a) - \alpha}{1 - p}.
\]
(3.4)
If possible, let $1 + ap - a \neq 0$. Then by the second fundamental theorem and Lemma 2.3 we get
\[
T(r, \alpha) \leq N(r, \infty; \alpha) + N(r, 0; \alpha) + N(r, 1 + ap - a; \alpha) + S(r, f)
\]
\[
\leq N_s(r, \infty; f, g) + N(r, a; f) + S(r, f).
\]
Since $f = (1 - a)/(1 - p)$, it follows that $T(r, f) = T(r, \alpha) + O(1)$. So, noting that $f$, $g$ share $(\infty,11)$, we obtain from above
\[
T(r, f) \leq N(r, \infty; f \geq 12) + N(r, a; f) + S(r, f)
\]
\[
\leq N_2(r, a; f) + N(r, \infty; f) + S(r, f).
\]
This implies $\delta_2(a; f) + \delta(\infty; f) \leq 1$, which contradicts the given condition.

Therefore $1 + ap - a = 0$ i.e. $p = (a - 1)/a$. So from (3.4) we get
\[
f - a \equiv -\alpha.
\]
(3.5)
Since $g = hf$, we get from (3.5)

$$g + a - 1 \equiv \frac{a - 1}{\alpha}.$$  \hspace{1cm} (3.6)

From (3.5) and (3.6) we obtain

$$(f - a)(g + a - 1) \equiv a(1 - a).$$

This proves the theorem.

**Proof of Corollary 1.14.** Let $F = 1 - f$ and $G = 1 - g$. Then $F$, $G$ are distinct and share $(0, 1), (1, \infty), (\infty, 1)$. Also $\delta_2(1 - a; F) = \delta_2(a; f)$ and $\delta(\infty; F) = \delta(\infty; f)$. So by Theorem 1.9 we get

$$(F - 1 + a)(G - a) \equiv a(1 - a)$$

i.e.,

$$(f - a)(g + a - 1) \equiv a(1 - a).$$

This proves the corollary.

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