A NEW METHOD OF CONSTRUCTING MAXIMAL PARTIAL SPREADS OF $PG(3, q)$, MAPPING $PG(3, q)$ OVER A NON-SINGULAR QUADRICAL $Q(4, q)$ OF $PG(4, q)$

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Abstract. We transfer the whole geometry of $PG(3, q)$ over a non-singular quadric $Q(4, q)$ of $PG(4, q)$ mapping suitably $PG(3, q)$ over $Q(4, q)$. More precisely the points of $PG(3, q)$ are the lines of $Q(4, q)$; the lines of $PG(3, q)$ are the tangent cones of $Q(4, q)$ and the reguli of the hyperbolic quadrics hyperplane section of $Q(4, q)$. A plane of $PG(3, q)$ is the set of lines of $Q(4, q)$ meeting a fixed line of $Q(4, q)$. We remark that this representation is valid also for a projective space $P^3$, $K$ over any field $K$ and we apply the above representation to construct maximal partial spreads $F$ in $PG(3, q)$. For $q$ even we get new cardinalities for $F$. For $q$ odd the cardinalities are partially known.

1. Introduction

Using a mapping of $PG(4, q)$ over the non-singular quadric $Q(4, q)$, we construct maximal partial spreads $F$ of $PG(3, q)$ with

$|F| = q^2 - q + 2$, $q$ odd, and $q > 3$;

$|F| = q^2 - q + 1$, $q$ even, $q \geq 8$, $n \in \mathbb{N}, n < \min \left\{ \frac{q-1}{4}, \frac{1+\sqrt{2q-1}}{2} \right\}$;

$|F| = q^2 - 2nq + 2n + 1$, $q$ odd, $q \geq 7$, $n < \min \left\{ \frac{q-1}{4}, \frac{1+\sqrt{8q-7}}{4} \right\}$.

The cardinalities $|F| = q^2 - q + 2$ and $|F| = q^2 - q + 1$ are known in the literature, but here the spreads are obtained in a different way. In particular A. Bruen [6], Theorem 17.6.9, obtained the cardinality $|F| = q^2 - q + 2$, $q$ odd and $q > 3$, but his construction is rather complicated.

2. The Mapping of $PG(3, q)$ Over $Q(4, q)$

Let $Q^+(5, q)$ be the Klein quadric of $PG(5, q)$. Let $Q(4, q)$ be the non-singular quadric hyperplane section of $Q^+(5, q)$, $Q(4, q) = Q^+(5, q) \cap S_4$, where $S_4$ is a hyperplane of $PG(5, q)$.

Let $L$ be the set of lines of $Q(4, q)$. We call linear complex of $PG(3, q)$ a set of lines whose Plücker coordinates $P_{i\bar{j}}$, $i, j = 0, 1, 2, 3$, $i < j$, satisfy a linear equation, that is a hyperplane section of the Klein quadric. A general linear complex $\mathcal{C}$ is represented by the linear equation

$$\sum_{i<j} a_{ij} P_{i\bar{j}} = 0,$$
where } det |a_{ij}| \neq 0, a_{ij} = -a_{ji}, a_{ii} = 0, a_{ij} \in GF(q). \text{ The points of } Q(4,q) \text{ represent therefore the lines of a general linear complex } C \text{ of } PG(3,q).

Let } \psi \text{ be the Klein mapping, that is the bijection which sends a line } r \text{ of } PG(3,q) \text{ to a point of } Q^+(5,q),

\begin{align*}
\psi : r & \rightarrow (P_{ij}) \in Q^+(5,q), \ i < j.
\end{align*}

The general linear complex } C \text{ determines the null polarity } f \text{ of } PG(3,q) \text{ which associates a point } X = (x_0, x_1, x_2, x_3) \text{ of } PG(3,q) \text{ with its polar plane } f(X) \text{ with equation}

\begin{align*}
\sum_{i=0}^{3} \left( \sum_{j=0}^{3} a_{ij} x_j \right) x_i = 0.
\end{align*}

Now let } x \text{ be a point of } PG(3,q) \text{ and } F_x \text{ the pencil of lines of } C \text{ through } x: \text{ the set } \psi(F_x) \text{ is a line } s \text{ of } Q(4,q). \text{ Let } \varphi \text{ be the following bijection:}

\begin{align*}
\varphi : x \in PG(3,q) & \rightarrow s \in \mathcal{L}.
\end{align*}

So } \varphi \text{ sends the points of } PG(3,q) \text{ to the lines of } Q(4,q).

Now let us represent the lines of } PG(3,q) \text{ over } Q(4,q). \text{ To do this, first we consider the lines of } C. \text{ Let } r \text{ be a line of } C \text{ and let } x_1, x_2, \ldots, x_{q+1} \text{ be the points of } r. \text{ The lines } \varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{q+1}) \text{ of } Q(4,q) \text{ are distinct and pass trough the point } \psi(r). \text{ Therefore } \varphi(r) \text{ is the tangent cone to } Q(4,q) \text{ at its vertex } \psi(r). \text{ It follows that } \varphi \text{ sends the lines of } C \text{ to the tangent cones of } Q(4,q). \text{ Therefore}

\begin{align*}
|C| = |Q(4,q)| = \theta_4 = q^3 + q^2 + q + 1.
\end{align*}

Now let } r \text{ be a line of } PG(3,q). \text{ The line } r', \text{ polar of } r \text{ under } f, \text{ is the axis of the pencil of polar planes of the points of } r. \text{ It follows that the polar of a line } r \in C \text{ coincides with } r.

Now, let us consider a line } r \text{ of } PG(3,q), \ r \notin C, \text{ whose polar is } r'. \text{ Obviously } r \cap r' = \emptyset. \text{ Let } x_1, x_2, \ldots, x_{q+1} \text{ be the points of } r \text{ and let } x'_1, x'_2, \ldots, x'_{q+1} \text{ be the points of } r'. \text{ The lines } \varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{q+1}) \text{ are mutually disjoint. Every line } \varphi(x'_j), \ j = 1, \ldots, q+1, \text{ meets all the lines } \varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{q+1}). \text{ The hyperplane } S_3 \text{ of } S_4 \text{ through } \varphi(x_1) \text{ and } \varphi(x_2), \text{ contains all the lines } \varphi(x'_1) \text{ and all the lines } \varphi(x_j), \ j = 1, \ldots, q+1. \text{ It follows that } S_3 \text{ meets } Q(4,q) \text{ at a hyperbolic quadric } Q(3,q).

We call regulus of } Q(4,q) \text{ a regulus of a hyperbolic quadric hyperplane section of } Q(4,q). \text{ Therefore the lines } \varphi(x_1), \varphi(x_2), \ldots, \varphi(x_{q+1}) \text{ and the lines } \varphi(x'_1), \varphi(x'_2), \ldots, \varphi(x'_{q+1}) \text{ are reguli of } Q(4,q). \text{ So } \varphi \text{ sends the lines of } PG(3,q) \text{ not belonging to } C \text{ to the reguli of } Q(4,q).

So the following Theorem holds.

**Theorem 1.** The Galois space } PG(3,q) \text{ is mapped over } Q(4,q) \text{ as follows:}

- The points of } PG(3,q) \text{ are the lines of } Q(4,q).
- The lines of } PG(3,q) \text{ are the tangent cones and the reguli of } Q(4,q).
- A plane } \pi \text{ of } PG(3,q) \text{ is the set of lines of } Q(4,q) \text{ meeting a fixed line of } Q(4,q).

More precisely the tangent cones of } Q(4,q) \text{ represent the lines of a general linear complex } C \text{ of } PG(3,q), \text{ and the other lines of } PG(3,q) \text{ are the reguli of } Q(4,q). \text{ A plane } \pi \text{ of } PG(3,q) \text{ is represented by the lines of } Q(4,q) \text{ meeting a fixed line of } Q(4,q), \text{ which is the pole of } \pi \text{ in } PG(3,q) \text{ with respect to the null polarity determined by } C.
In other words the correspondence ϕ between \( PG(3, q) \) and the set of lines of \( Q(4, q) \) uses the symplectic polarity \( η \) of \( PG(3, q) \) and the Klein correspondence between the lines of \( PG(3, q) \) and the set of points of the hyperbolic quadric \( Q^+(5, q) \).

Let \( η \) be a symplectic polarity. Then the totally isotropic lines of \( η \) are mapped by the Klein correspondence to the points of a particular parabolic quadric \( Q(4, q) \) of \( Q^+(5, q) \). These points can be identified with their tangent cones to \( Q(4, q) \). The lines \( L \) of \( PG(3, q) \) which are not totally isotropic with respect to \( η \) have a polar line \( L^η \) with respect to \( η \). It is a known fact that \( L \) and \( L^η \) can be made to correspond to a 3-dimensional hyperbolic quadric of \( Q(4, q) \).

3. The Non-singular Quadric \( Q(4, q) \) of \( PG(4, q) \)

Let \( Q(4, q) \) be a non-singular quadric of \( PG(4, q) \). If \( P \in Q(4, q) \), we denote by \( Γ_P \) the tangent cone at \( P \) to \( Q(4, q) \). \( Γ_P = S_1 \cap Q(4, q) \), where \( S_1 \) is the tangent hyperplane at \( P \) to \( Q(4, q) \). A hyperplane of \( PG(4, q) \) meets \( Q(4, q) \) either in an elliptic quadric \( E \), or in a cone \( Γ_P \), or in a hyperbolic quadric \( I \).

We get:

\[
|E| = q^2 + 1, \quad |Γ_P| = q^2 + q + 1, \quad |I| = q^2 + 2q + 1.
\]

It is easy to prove that the following Lemmas hold:

**Lemma 1.** The set \( I \cap Γ_P \), with \( P \notin I \) is a non-singular conic.

**Lemma 2.** Let \( I_1 \) and \( I_2 \) be two hyperbolic quadrics, hyperplane sections of \( Q(4, q) \). Then either \( I_1 \cap I_2 \) is a conic consisting of two distinct lines, or it is a non-singular conic.

**Lemma 3.** The set \( Γ_P \cap E \), \( P \notin E \) is a non-singular conic.

**Lemma 4.** The set \( E \cap I \) is a non-singular conic.

**Lemma 5.** The set \( Γ_P \cap Γ_Q \), \( Q \notin Γ_P \) is a non-singular conic.


**Theorem 2** (Tallini theorem). Let \( α \) be a plane of \( PG(4, q) \) meeting \( Q(4, q) \) at a non-singular conic. Let \( q \) be odd. If the line \( r \), polar of \( α \) with respect to \( Q(4, q) \), is external to \( Q(4, q) \), we have \((q + 1)/2\) hyperplanes through \( α \) meeting \( Q(4, q) \) at elliptic quadrics and \((q + 1)/2\) hyperplanes through \( α \) meeting \( Q(4, q) \) at hyperbolic quadrics. If \( r \) is a secant of \( Q(4, q) \), through \( α \) there are two tangent hyperplanes to \( Q(4, q) \), \((q - 1)/2\) hyperplanes meeting \( Q(4, q) \) at an elliptic quadric and \((q - 1)/2\) hyperplanes meeting \( Q(4, q) \) at a hyperbolic quadric. Let \( q \) be even. If the plane \( α \) does not contain the nucleus \( N \) of \( Q(4, q) \), the hyperplane joining \( α \) and \( N \) is tangent to \( Q(4, q) \). There are \( q/2 \) hyperplanes through \( α \) meeting \( Q(4, q) \) at elliptic quadrics and \( q/2 \) at hyperbolic quadrics. If \( N \in α \) every hyperplane through \( α \) is tangent to \( Q(4, q) \).

Let \( I \) resp. \( E \) be the set of all hyperbolic resp. elliptic quadrics which can be obtained as the sections \( S_2 \cap Q(4, q) \) of a hyperplane \( S_3 \) with the quadric \( Q(4, q) \).

Let:

\[
Γ := \{Γ_P = T_P \cap Q(4, q) | P \in Q(4, q)\},
\]

\( T_P \) tangent hyperplane in \( P \), be the set of all tangent cones. For \( I \in I \), let \( I_1 \) and \( I_2 \) be the two reguli of \( I \), i.e. for \( i \in \{1, 2\} I_i \) is a set of lines contained in \( I \), which partitions the points of \( I \) and let \( T_I := \{I_1, I_2 | I \in I\} \) be the set of all reguli.
For $\Gamma_P \in \Gamma$, let $\Gamma'_P$ be the set of all lines contained in $\Gamma_P$. Now let $\mathcal{C}$ be a non-singular conic plane section of $Q(4, q)$.

Then we set:

$$F_h(\mathcal{C}) := \{ I_i | i \in \{1, 2\} : \mathcal{C} \subseteq I \},$$

$$F_\gamma(\mathcal{C}) := \{ \Gamma'_P | \Gamma_P \in \Gamma : \mathcal{C} \subseteq \Gamma'_P \},$$

and

$$F(\mathcal{C}) = F_h(\mathcal{C}) \cup F_\gamma(\mathcal{C}).$$

From Tallini Theorem it follows that $|F(\mathcal{C})| = q + 1$.

### 4. Known Results About Partial Spreads in $PG(3, q)$

A line partial spread of $PG(3, q)$ is a set $\mathfrak{F}$ of lines mutually disjoint. The partial spread $\mathfrak{F}$ is called spread if $\mathfrak{F}$ is a covering of $PG(3, q)$.

Obviously

$$\mathfrak{F} \text{ spread } \iff |\mathfrak{F}| = q^2 + 1.$$  

The partial spread $\mathfrak{F}$ is maximal if it is not properly contained in another partial spread.

Maximal partial spreads have been investigated since a long time by several Authors, but a complete knowledge of them is far from being complete.

The known results about them are in [3], [4], [5], [9], [10] and are summarized in the following

**Theorem 3.** A maximal partial line spread of $PG(3, q)$ of cardinality $r$ exist in the following cases:

- $q \geq 7$, $q$ odd $\Rightarrow$ $q \equiv 1, 7, 9, 13, 15, 21 \mod 24$, $\frac{q^2 + 7}{2} \leq r \leq q^2 - q + 2$;
- $q \geq 7$, $q$ odd $\Rightarrow$ $q \equiv 3 \mod 4$, $\frac{5q^2 + 4q - 1}{8} \leq r \leq q^2 - q + 2$;
- $q \geq 7$, $q$ odd $\Rightarrow$ $q \equiv 1, 7, 9, 13, 15, 21 \mod 24$, $\frac{q^2 + 7}{2} \leq r \leq q^2 - q + 2$;
- $q \geq 4$ $\Rightarrow$ $r = q^2 - q + 2$;
- $q > 2$ $\Rightarrow$ $r = q^2 - q + 1$;
- $\gcd(q + 1, 3) = 1$ $\Rightarrow$ $r = \frac{q^2 + q + 2}{2}$;
- $\gcd(q + 1, 12) = 2$ $\Rightarrow$ $r = \frac{q^2 + 3}{2}$;
- $\gcd(q + 1, 24) = 4$ $\Rightarrow$ $r = \frac{q^2 + 5}{2}$;
- $q = 4$ $\Rightarrow$ $r = 11, 12, 13, 14$;
- $q = 5$ $\Rightarrow$ $15 \leq r \leq 21$;
- $q = 7$ $\Rightarrow$ $23 \leq r \leq 45$;

O. Heden [2], [3], [4], [5] found an example of a maximal partial spread $\mathfrak{F}$, with $|\mathfrak{F}| = 45$ $> q^2 - q + 2 = 44$;

- $q = 11$ $\Rightarrow$ $58 \leq r \leq 67$;
- $\forall q$, $r = q^2 + 1 - nq$, $0 < n \leq \frac{1}{2}q - 1$ [1].
5. New Examples of Maximal Partial Spreads of $PG(3,q)$

The map $\varphi$ of sect. 2 allows us to construct maximal partial spreads $\mathfrak{F}$ of $PG(3,q)$, as follows.

**Example 1.** Let $I$ be a hyperbolic quadric, hyperplane section of $Q(4,q)$ of $PG(4,q)$, $q$ odd, $q > 3$, $I = S_3 \cap Q(4,q)$, $S_3$ a hyperplane of $PG(4,q)$. Let $\pi$ be a plane of $S_3$ meeting $I$ at a non-singular conic $C$. Let $A$ and $B$ be two distinct points of $C$. Let $R_1$ and $R_2$ be the reguli of $I$ through $A$ and $B$ respectively. Let $s_A$ and $s_B$ be the lines of $R_2$ through $A$ and $B$ respectively. The lines $s_A$ and $s_B$ meet at a point $C$. Similarly the lines $s_A$ and $s_B$ meet at a point $D$. The points $A, B, C, D$ are distinct and $C, D \notin \pi$. Let $H = \pi \cap CD$. We get $H \notin C$.

The pair $(C, D)$ is called associated pair with the pair $(A, B)$, since $q$ is odd, there is a line $t$ of $\pi$ through $H$ external to $C$ (if $q$ is even, $H$ is the nucleus of $C$, and every line of $\pi$ through $H$ is tangent to $C$). The planes of $S_3$ through $t$ are $\pi$, the plane $\pi'$ through $DC$ and $q - 1$ planes $\pi_1, \pi_2, \ldots, \pi_{q-1}$. They meet $I$ at non singular conics.

Set $C_i = \pi_i \cap I$, $i = 1, \ldots, q - 1$. Let $\mathfrak{F}$ be the following set of tangent cones and reguli of $Q(4,q)$:

$$\mathfrak{F} = \{\Gamma_C, \Gamma_D, \Gamma_X, X \in C - \{A, B\}\} \bigcup \bigcup_{i=1}^{q-1} (F(C_i) - \{R_1, R_2\})$$

By means of Theorem 1 we identify $PG(3,q)$ with defined structure on $Q(4,q)$. So $\mathcal{F}$ can be considered a set of lines of $PG(3,q)$.

We prove that $\mathfrak{F}$ is a partial spread of $PG(3,q)$. First we remark that if $\Gamma_{V'}$ and $\Gamma_{V''}$ are two distinct tangent cones of $Q(4,q)$, $V' \neq V''$, $V' \in I$, $V'' \in I$, having a common line, then the line $V'V'' \subseteq I$. From this it follows that the cones $\Gamma_C, \Gamma_D, \Gamma_X, X \in C - \{A, B\}$, have no common line. Obviously two elements of $F(C_i) - \{R_1, R_2\}$, $i = 1, \ldots, q - 1$, have no common line too. Moreover if $\mathcal{G} \in F(C_i) - \{R_1, R_2\}$ and $\mathcal{G}' \in F(C_j) - \{R_1, R_2\}$, $i \neq j$, $i, j = 1, \ldots, q - 1$, $\mathcal{G}$ and $\mathcal{G}'$ have no common line. It follows that the cones and the reguli of

$$\bigcup_{i=1}^{q-1} (F(C_i) - \{R_1, R_2\})$$

have no common line. Obviously, a cone of the set

$$\{\Gamma_C, \Gamma_D, \Gamma_X, X \in C - \{A, B\}\}$$

and an element of

$$\bigcup_{i=1}^{q-1} (F(C_i) - \{R_1, R_2\})$$

have no common line. This proves that $\mathfrak{F}$ is a partial spread of $PG(3,q)$. Let us prove that $\mathfrak{F}$ is maximal. This means that every line of $PG(3,q)$ meets a line of $\mathfrak{F}$. This is equivalent to prove that every regulus and every tangent cone of $Q(4,q)$ has a line in common with either a regulus or a tangent cone of $\mathfrak{F}$.

First we prove that every tangent cone $\Gamma_Y$ of $Q(4,q)$, $Y \in Q(4,q)$, has a line
in common with either a cone, or a regulus of $\mathfrak{H}$. This is obvious if $Y \in \mathcal{E} = \mathcal{C} \cup r_A \cup r_B \cup s_A \cup s_B$. If $Y \in I - \mathcal{E}$, let $v$ be a line of $I$ through $Y$. If $v \in R_1$, then the line $v$ meets $\mathcal{C}$ at a point $M$, with $M \neq A, B$. Therefore $\Gamma_Y$ and $\Gamma_M$, $\Gamma_M \in \mathfrak{H}$, have the line $v$ in common. The same happens, if $v \in R_2$. Now let $Y \notin I$. The cone $\Gamma_Y$ meets $I$ at a non-singular conic $\mathcal{C}$, by Lemma 1. Let $\pi$ be the plane of $S_3$ through $\mathcal{C}$. If $\pi$ contains the line $t$ (the line of $\pi$ through $H$ external to $C$), then

$$\mathcal{C} \in \mathcal{U} = \{\mathcal{C}, \mathcal{C}' = \pi' \cap I, C_i, i = 1, \ldots, q - 1\}.$$

If $\mathcal{C} = \mathcal{C}'$, the cone $\Gamma_Y$ has a line in common with every cone $\Gamma_X$, $X \in \mathcal{C} - \{A, B\}$. If $\mathcal{C} = \mathcal{C}'$, the cone $\Gamma_Y$ has a line in common with both $\Gamma_C$ and $\Gamma_D$. If $\mathcal{C} = \mathcal{C}_i$, $1 \leq i \leq q - 1$, the cone $\Gamma_Y$ belongs to $\mathfrak{H}$. If the plane $\pi$ does not contain $t$, then $\mathcal{C} \notin \mathcal{U}$ and $\mathcal{C}$ has a point $Z$ in common with a conic $\mathcal{C}_i$, $1 \leq i \leq q - 1$, since $|\mathcal{C} \cap \mathcal{C}| \leq 2$, $|\mathcal{C} \cap \mathcal{C}'| \leq 2$, $q > 3$, and $\mathcal{U}$ is a partition of $I$. Then obviously there is an element of $F(C_i) - \{R_1, R_2\}$ containing the line of $\Gamma_Y$ through $Z$.

Secondly, let us prove that every regulus of $Q(4, q)$ has a line in common with either a cone, or a regulus of $Q(4, q)$ of $\mathfrak{H}$. Let $\mathcal{R}$ be a regulus of $Q(4, q)$ and let $I_{\mathcal{R}}$ be the hyperbolic quadric, hyperplane section of $Q(4, q)$, containing the regulus $\mathcal{R}$. By Lemma 2, we get $I_{\mathcal{R}} \cap I = \mathcal{C}'$, where $\mathcal{C}'$ is either a singular conic consisting of two distinct lines, or it is non-singular. If $\mathcal{C}'$ is singular, then a line of $\mathcal{C}'$ is a line $z$ of $\mathcal{R}$. Obviously $z$ belongs to one of the cones $\Gamma_C$, $\Gamma_D$, $\Gamma_X$, $X \in \mathcal{C} - \{A, B\}$. If $\mathcal{C}'$ is non-singular, we prove that $\mathcal{R}$ has a line in common with some element of $\mathfrak{H}$. The proof is the same as in the case of the cone $\Gamma_Y$, $Y \notin I$. Therefore $\mathfrak{H}$ is a maximal spread of $PG(3, q)$. By the previous construction we get:

$$|\mathfrak{H}| = q^2 - q + 2.$$

This example has the same cardinality of the example constructed by Bruen [6], Theorem 17.6.9. Our construction is simpler.

**Example 2.** Let $E$ be an elliptic quadric, hyperplane section of $Q(4, q)$, $E = S_3 \cap Q(4, q)$, $S_3$ a hyperplane of $PG(4, q)$. Let $X$ be a point of $Q(4, q) - E$ and let $\Gamma_X$ be the tangent cone to $Q(4, q)$ with vertex $X$. The set $\Gamma_X \cap E$ is a non-singular conic $\mathcal{C}$, by Lemma 3. Let $\ell$ be a line of $\Gamma_X$ and let $L = \ell \cap E$. Let $Y$ be a point of $E - \{L, X\}$ and let $\Gamma_Y$ be the tangent cone to $Q(4, q)$ with vertex $Y$. The cone $\Gamma_Y$ meets $E$ at a non-singular conic $\mathcal{C}'$. Obviously, we have $L \in \mathcal{C}'$, $\Gamma_X \cap \Gamma_Y = \ell$. It follows that $\mathcal{C} \neq \mathcal{C}'$, $\mathcal{C} \cap \mathcal{C}' = \{L\}$. Let us consider the set:

$$\mathfrak{H}' = |F(\mathcal{C}) - \Gamma_X| \cup \{\Gamma_Y\} \cup \{\Gamma_Z, Z \in E - \mathcal{C} \cup \mathcal{C}'\}.$$

Obviously the elements of $F(\mathcal{C}) - \Gamma_X$ have no line in common. It easy to prove that an element of $F(\mathcal{C}) - \Gamma_X$ has no line in common with $\Gamma_Y$. Therefore the set $\mathcal{A} = |F(\mathcal{C}) - \Gamma_X| \cup \{\Gamma_Y\}$ consists of elements pairwise without line in common. The cones of the set

$$\mathcal{B} = \{\Gamma_Z, Z \in E - \mathcal{C} \cup \mathcal{C}'\}$$

have no line in common. Moreover an element of $\mathcal{A}$ and an element of $\mathcal{B}$ have no line in common. It follows that $\mathfrak{H}'$ is a partial spread of $PG(3, q)$. As we proved in the Example 1, we prove that $\mathfrak{H}'$ is maximal. We get:

$$|\mathfrak{H}'| = q^2 - q + 1.$$
This example has the same cardinality of the example constructed by Beutelspacher [1], but our construction is different.

**Example 3.** Let \(\pi_q, q\) odd, be a projective plane. Let \(\mathcal{C}\) be a \((q + 1)\)-arc and let \(K\) be a set of \(\pi_q\) satisfying the following conditions:

a) \(K \cap \mathcal{C} = \emptyset\),

b) every external line to \(\mathcal{C}\) meets \(K\).

Let \(m\) be the number of pairs consisting of a point \(P \in K\) and a line external to \(\mathcal{C}\) through \(P\). We have \(m \geq (q^2 - q)/2\), since \((q^2 - q)/2\) is the number of the external lines to \(\mathcal{C}\). Moreover is easy to prove that \(m \leq |K| \cdot (q + 1)/2\). It follows that \((q^2 - q)/2 \leq |K| \cdot (q + 1)/2\), and therefore

\[|K| \geq q - 2 + \frac{2}{q + 1}.\]

Therefore \(|K| \geq q - 1\) and the following theorem holds:

**Theorem 4.** In \(\pi_q, q\) odd, let \(\mathcal{C}\) be a \((q + 1)\)-arc. If \(K\) is a set of \(\pi_q\) satisfying a) and b), we get \(|K| \geq q - 1\).

Let \(Q(4, q)\) be a non-singular quadric of \(PG(4, q)\) and let \(I\) be the hyperbolic quadric, hyperplane section of \(Q(4, q)\), \(I = S_H \cap Q(4, q)\), \(S_H\) a hyperplane of \(PG(4, q)\). Let \(\pi\) be a plane of \(S_H\) meeting \(I\) at a non-singular conic \(\mathcal{C}\). Let \(n\) be a positive integer, with \(2n \leq q + 1\). Let \(A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n\) be \(2n\) distinct points of \(\mathcal{C}\). We associate the pair \((A_i, B_i)\), \(1 \leq i \leq n\), with the pair \((C_i, D_i)\) as follows. Denote by \(r_{A_i}\) and \(s_{A_i}\), the two lines of \(I\) through \(A_i\) and by \(r_{B_i}\) and \(s_{B_i}\), the two lines of \(I\) through \(B_i\). The lines \(r_{A_i}\) and \(r_{B_i}\) belong to the same regulus and similarly \(s_{A_i}\) and \(s_{B_i}\). We set \(C_i = r_{A_i} \cap s_{B_i}\), \(D_i = r_{B_i} \cap s_{A_i}\). Let \(H_i = C_i \cup D_i \cup \pi\). Let \(\mathcal{B} = \{C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n\}\). Let \(\mathcal{D}\) be the set of lines joining two distinct points of \(\mathcal{B}\). Obviously we get \(|\mathcal{D}| = \binom{2n}{2}\). Every line of \(\mathcal{D}\) meets \(\pi\) at a point outside \(\mathcal{C}\). Assume \(q\) odd and \(n < \min\{(q - 1)/4, (1 + \sqrt{8q - 7})/4\}\). It follows that \(q \geq 7\). Let \(K\) be the set consisting of the points common to \(\pi\) and the lines of \(\mathcal{D}\). Obviously we have \(K \cap \mathcal{C} = \emptyset\), and \(|K| \leq |\mathcal{D}| = \binom{2n}{2}\). It follows \(|K| \leq \frac{q^2 - 2q}{2} < q - 1\). By Theorem 4, it follows that there is a line \(\ell\) of \(\pi\) external to \(\mathcal{C}\) such that \(\ell \cap K = \emptyset\). The planes \(\pi_{C_1}, \pi_{C_2}, \ldots, \pi_{C_n}, \pi_{D_1}, \pi_{D_2}, \ldots, \pi_{D_n}\), joining the line \(\ell\) with the points \(C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n\), are distinct, since \(\ell \cap K = \emptyset\). The planes of \(S_H\) through \(\ell\) are \(\pi, \pi_{C_1}, \pi_{C_2}, \ldots, \pi_{C_n}, \pi_{D_1}, \pi_{D_2}, \ldots, \pi_{D_n}\), and other \(q - 2n\) planes \(\pi_{\alpha}, \pi_{\beta}, \ldots, \pi_{\gamma - 2n}\), meeting \(I\) at a non-singular conic. Let \(C_i = \pi_i \cap \mathcal{I}, i = 1, \ldots, q - 2n\). Since \(n < \frac{q - 1}{2}\), it follows that a non-singular conic plane section of \(I\) with a plane different from \(\pi, \pi_{C_1}, \pi_{C_2}, \ldots, \pi_{C_n}, \pi_{D_1}, \pi_{D_2}, \ldots, \pi_{D_n}\), contains a point of \(\bigcup_{i=1}^{q-2n} C_i\). Let \(\mathcal{F}'\) be the following set:

\[
\mathcal{F}' = \left\{ \Gamma_{C_1}, \ldots, \Gamma_{C_n}, \Gamma_{D_1}, \ldots, \Gamma_{D_n}, \Gamma_X, X \in \mathcal{C} - \{A_1, \ldots, A_n, B_1, \ldots, B_n\} \right\} \cup \left\{ \bigcup_{i=1}^{q-2n} (F(C_i) - \{R_1, R_2\}) \right\}.
\]

As in the previous examples we prove that the set \(\mathcal{F}'\) is a maximal partial spread of \(PG(3, q)\), with \(|\mathcal{F}'| = q^2 - 2qn + 2n + 1\).

**Example 4.** In \(\pi_q, q\) even, let \(\mathcal{C}\) be a \((q + 1)\)-arc and let \(N\) be its nucleus. Let \(K'\) be a set of \(\pi_q\) satisfying the following conditions:
a') $K' \cap \mathcal{C} = \emptyset$,

b') $N \in K'$,

c') every external line to $\mathcal{C}'$ meets $K'$.

Let $m$ be the number of pairs consisting of a point $P \in K'$ and a line external to $\mathcal{C}'$ through $P$. We have $m \geq (q^2 - q)/2$, since this is the number of the external lines to $\mathcal{C}'$. Obviously, we get

$$m = (|K'| - 1) \frac{q}{2}.$$  

It follows that $|K'| \geq q$ and the following theorem holds.

**Theorem 5.** In $\pi_q$, $q$ even, let $\mathcal{C}'$ be a $(q+1)$-arc and let $N$ be its nucleus. If $K'$ is a set of $\pi_q$ satisfying $a')$, $b')$, $c')$, we have $|K'| \geq q$.

Following the notations of Example 3, assume $q$ even, $n < \min\{(q-1)/4, (1 + \sqrt{2q-1})/2\}$. It follows that $q \geq 8$. It is easy to prove that all the lines $C_i D_i$, $i = 1, \ldots, n$, contain the nucleus $N$ of $\mathcal{C}'$. Obviously a line $u \in D$, different from $C_i D_i$, $i = 1, \ldots, n$, does not contain $N$. Then the set $K'$ consisting of the points common to the lines of $D$ and $\pi$, $K' := \{d \cap \pi \mid d \in D\}$, is such that $N \in K'$ and $K' \cap \mathcal{C}' = \emptyset$. Since there are $n$ lines of $D$ through $N$, it follows that $|K'| \leq (\frac{q}{2^n}) - n + 1$. From this and from $n < (1 + \sqrt{2q-1})/2$ it follows

$$|K'| < q.$$  

By Theorem 5 we get that there is a line $\ell$ of $\pi$ external to $\mathcal{C}'$ such that $\ell \cap K' = \emptyset$. The planes $\pi_{C_1}, \pi_{C_2}, \ldots, \pi_{C_n}, \pi_{D_1}, \pi_{D_2}, \ldots, \pi_{D_n}$ spanned by $\ell$ and the points $C_1, \ldots, C_n, D_1, \ldots, D_n$ respectively, are distinct, since $\ell \cap K' = \emptyset$. We consider the following line set:

$$\mathfrak{S}''' = \{ \Gamma_{C_1}, \ldots, \Gamma_{C_n}, \Gamma_{D_1}, \ldots, \Gamma_{D_n}, \Gamma_X, X \in \mathcal{C}' - \{A_1, \ldots, A_n, B_1, \ldots, B_n\} \} \cup$$

$$\bigcup_{i=1}^{\frac{q-2}{2}} \{ F(C'_i) - \{ R_1, R_2 \} \}, C'_i = \pi_i \cap I, i = 1, \ldots, q - 2n.$$  

As in the Example 3, we prove that $\mathfrak{S}'''$ is a maximal partial spread of $PG(3, q)$, with

$$|\mathfrak{S}'''| = q^2 - 2nq + 2n + 1.$$  

This cardinality is quite new and therefore gives rise to new maximal partial spreads of $PG(3, q)$.

**References**


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