ON PL-ABSOLUTE TOTAL CURVATURE OF 2-KNOTS

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Abstract. The PL-absolute total curvature of an m-manifold in n-space is defined as half of the average number of critical points with respect to unit vectors in n-space. We give a lower bound of the PL-absolute total curvature of a 2-knot.

1. Introduction

A knot is an oriented circle $S^1$ embedded in $\mathbb{R}^3$. A surface-knot is an oriented closed connected surface embedded in $\mathbb{R}^4$. A surface-knot $F$ is trivial if it bounds a handlebody in $\mathbb{R}^4$. A genus-zero surface-knot $F$ is called a 2-knot.

J. Milnor [6] studied the total curvature of a closed curve $C$ embedded in $\mathbb{R}^n$. He proved that for a knot $K$, the total curvature of $K$ exceeds $4\pi$. Here $4\pi$ can be interpreted as the volume of $S^2$.

T. Homma [5] defined the PL-absolute total curvature of an m-manifold $M$ nicely embedded in $\mathbb{R}^n$ denoted by $\tau(M)$. He proved that if $\tau(M) = 1$, then $M$ is the boundary of a convex $(m+1)$-dimensional polygon $W$ embedded in $\mathbb{R}^n$. This implies that if $F$ is a non-trivial 2-knot, then $1 < \tau(F)$. Here 1 may be interpreted as one unit of the volume of $S^3$. This is a generalisation of Milnor’s result above.

Let $N(F)$ denote a tubular neighbourhood of a surface-knot $F$ in $\mathbb{R}^4$. The surface-knot group of $F$ is the fundamental group $\pi_1(\mathbb{R}^4 \setminus N(F))$ denoted by $\pi F$.

We obtain a lower bound of the PL-absolute total curvature of a 2-knot with $\pi F \not\cong \mathbb{Z}$.

**Theorem 1.1.** Let $F$ be a 2-knot with $\pi F \not\cong \mathbb{Z}$. Then

$$3 \leq \tau(F).$$

We will prove this theorem in Section 4.

This paper is organised as follows. In Section 2, we introduce PL-absolute total curvature. In Section 3 we will discuss the normal form of 2-knots. In Section 4 we will prove Theorem 1.1.

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2. **PL-absolute Total Curvature**

In this section we briefly describe the **PL-absolute total curvature of an m-manifold** $M$ embedded in $\mathbb{R}^n$. Such an m-manifold $M$ is critical at $p \in M$ with respect to $v \in S^{n-1}$, if the intersection of a star-neighbourhood $st(p)$ with the $n$-dimensional half space $H^+_n(p,v) = \{ q \in \mathbb{R}^n \mid v \cdot pq \geq 0 \}$ or $H^-_n(p,v) = \{ q \in \mathbb{R}^n \mid v \cdot pq \leq 0 \}$ is not an $m$-disc. If $M$ is critical at $p$ with respect to $v$, then $p$ will be called a critical point with respect to $v$. The angle $\hat{\angle}(p,M)$ at $p \in M$ is defined by

$$
\frac{1}{2|n-1|} \omega \{ v \in S^{n-1} \mid M \text{ is critical at } p \text{ with respect to } v \},
$$

(2)

where $\Gamma^{n-1}$ is the volume of $S^{n-1}$ and $\omega$ is the standard measure on $S^{n-1}$. The PL-absolute total curvature of $M \subset \mathbb{R}^n$ is defined by $\tau(M) := \sum_{p \in M} \hat{\angle}(p,M)$.

For a unit vector $v \in S^{n-1}$, $\mu(M,v)$ denotes the number of critical points of $M$ with respect to $v$. Then the following holds.

**Lemma 2.1** (T. Homma). Let $M$ be an oriented closed m-manifold nicely embedded in $\mathbb{R}^n$.

$$
\tau(M) = \frac{1}{2|n-1|} \int_{S^{n-1}} \mu(M,v) d\omega,
$$

(3)

where $d\omega$ is the volume element of $S^{n-1}$, $\mu(v,M)$ is the number of critical points with respect to a vector $v \in S^{n-1}$, and $\Gamma^{n-1}$ is the volume of $S^{n-1}$.

Set $\mu_{\min} = \min_{v \in S^{n-1}} \mu(M,v)$. Then the following is an immediate consequence of Lemma 2.1.

**Corollary 2.1.** Let $M$ be an m-manifold nicely embedded in $\mathbb{R}^n$.

$$
\frac{1}{2} \mu_{\min} \leq \tau(M).
$$

(4)

**Proof.** The inequality $\mu_{\min} \leq \mu(M,v)$ holds. This implies the desired inequality. \qed

3. **2-knots in Normal Forms**

We denote the set $\{(x_1,x_2,x_3,t) \mid t \in [a,b]\} \subset \mathbb{R}^4$ by $\mathbb{R}^3[a,b]$. We denote $\mathbb{R}^3[a,a]$ by $\mathbb{R}^3[a]$. Let $v \in \mathbb{S}^3$. Let $\mathbb{R}_v$ denote the line generated by $v$. Define $\pi_v : \mathbb{R}^4 \rightarrow \mathbb{R}_v$ by $\pi_v(x) = v \cdot x$, where $v \cdot x$ is the inner product. Let $F$ be a surface-knot. After perturbing $F$, $\pi_v|_F$ can be a Morse function on $F$.

The following is proved in [4].

**Lemma 3.1.** Let $v \in \mathbb{S}^3$. Let $F$ be a 2-knot with $m_1$ maximal points, $m_2$ minimal points and $s$ saddle points with respect to $v$. Then $F$ is equivalent to $F'$ such that

(1) $F'$ has the same number of maximal points and minimal points and saddle points as has $F$ with respect to $v$.

(2) All the minimal points are contained in $\mathbb{R}^3[-2]$.

(3) All the maximal points are contained in $\mathbb{R}^3[2]$. 


(4) All the saddle points are contained in $\mathbb{R}^3(-2, 2)$.

A proof can be found in [4].

We slightly modify $F$ by an isotopy deformation so that a critical point at a critical value $t (\pi_v|_F)^{-1}(t)$ is deformed into a disc if the index is 0 or 2 or deformed into a band if the index is 1. We still call the perturbed function a Morse function.

If a band corresponding to a saddle point reduces the number of components, then it is called a fusion (band). If it increases the number of components, then it is called a fission (band).

A normal form for a 2-knot $F$ is an embedding with a projection $\pi_v: \mathbb{R}^4 \to \mathbb{R}_v$ which restricts to a Morse function on $F$, and satisfies the following conditions.

1. $F \subset \mathbb{R}^3[-2, 2]$
2. The critical values of $\pi_v|_F$ are $\pm 1$ and $\pm 2$, and $(\pi_v|_F)^{-1}(0)$ is a knot; that is, connected.
3. All the minimal discs are in $\mathbb{R}^3[-2]$.
4. All the saddle bands with fusion bands are in $\mathbb{R}^3[-1]$.
5. All the saddle bands with fission bands are in $\mathbb{R}^3[1]$.
6. All the maximal discs are in $\mathbb{R}^3[2]$.

The following are proved in [4].

**Lemma 3.2** (Kawauchi-Shibuya-Suzuki). Any 2-knot is equivalent to a 2-knot in a normal form.

**Lemma 3.3** (Kawauchi-Shibuya-Suzuki). Let $F$ be a 2-knot. Suppose that $F$ is in a normal form. If $\pi_v|_F^{-1}(0)$ is trivial, then $\pi_F \cong \mathbb{Z}$.

**Example 3.1.** Let $F$ be Fox-Milnor’s 2-knot. Diagrams in Figure 1 are $\mathbb{R}^3[t] \cap F$ for $t = -1.5, -1, 0, 1, 1.5$, with $F$ in a normal form.

![Figure 1](image-url)
4. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $v \in S^3$. Let $F$ be a 2-knot. The 2-knot $F$ can be deformed in a normal form without changing the number of critical discs.

Let $m_1$ be the number of local maximal points with respect to $v$, and let $m_2$ be the number of local minimal points with respect to $v$. We claim that $m_1 > 1$ and $m_2 > 1$. Suppose that $m_i = 1$ for $i = 1, 2$. Then in the normal form the component that is the same as $\pi_v|^{-1}(0)$, vanishes at $R^3[2]$ or $R^3[-2]$. Since $\pi F \not\cong \mathbb{Z}$, by Lemma 3.3 $\pi_v|^{-1}(0)$ is not trivial. This means that $F$ is not a locally flat. This is a contradiction. Thus $m_i > 1$ for $i = 1, 2$. This implies that the number of fusion bands is at least one and the number of fission bands is at least one. We can perturb $F$ to $F'$ so that $F'$ has critical points at $t = \pm 1, \pm 2$ with respect to $\pi_v|F'$. Thus the number of critical points of $F'$ is the same as the number of critical discs of $F$ in the normal form. Therefore, for any $v \in S^3$,

$$2 + 1 + 1 + 2 = 6 \leq \mu(F, v).$$

By Corollary 2.1

$$3 \leq \tau(F).$$

By Corollary 2.1

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