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A SELECTION THEOREM ABOUT A KIND OF WEAKLY LOWER SEMICONTINUOUS SET–VALUED MAPPINGS

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Abstract. The main purpose of this paper is to give a new selection theorem which unifies and generalizes some known results.

1. Introduction

The theory of continuous selections of set–valued mappings is a classical area of mathematics and has various applications in general topology, fixed point theory, game theory, mathematical economics, and many other modern mathematical branches. The following existence theorem of continuous selections is one of the fundamental results in selection theory.

Theorem 1.1 (Michael's selection theorem [**5**])**.** *If X is a paracompact space and Y is a Banach space, then any lower semi–continuous set–valued mapping with closed convex values from X into Y has a continuous selection.*

In recent years, many mathematicians have attempted to obtain continuous selections for set–valued mappings under minimal hypotheses. From [**5**], we know that the paracompactness of the domain is necessary. Gutev [**2**] showed that Theorem 1.1 is true when lower semi–continuity is replaced by quasi lower semi–continuity which is weaker than lower semi–continuity. In 1992, Preslawski and Rybinski [**7**] showed that Theorem 1.1 remains valid when lower semi–continuity is replaced by ball–uniformly lower semi–continuity. On the other hand, Curtis [**1**] introduced a kind of convex structure in a metric space (Y, d) to replace the convex structure in Banach spaces, and gave a selection theorem which improves Michael's result. This kind of convex structure is useful in hyperspace contractibility [**1**]. Hou [**3**] gave a selection theorem in H–spaces about w.l.s.c set–valued mappings. A nature question is whether this selection theorem is true or not in the class of metric spaces with convex structures in [**1**]. This paper is devoted to answer this question. Our theorem unifies and improves related results in [**1**], [**2**], [**4**], [**5**] and [**7**].

First, we introduce some definitions and notation. Let X be a topological space, and let (Y, d) be a metric space. We denote the family of all nonempty subsets of Y by 2^Y. For any $A \in 2^Y$ and $\varepsilon > 0$, let $S_\varepsilon(A) = \{y \in Y : d(y, A) < \varepsilon\}$. Given a set–valued mapping $F : X \to 2^Y$, for $r > 0$ and $x \in X$ define

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$$F_r(x) = \{y \in Y : \exists U \in \mathcal{U}(x), \quad \forall x' \in U, \quad d(y, F(x')) < r\}$$

and

$$F_0(x) = \bigcap \{F_r(x) : r > 0\},$$

where $\mathcal{U}(x)$ is a local base of x.

Definition 1.2. A set–valued mapping $F : X \to 2^Y$ is called

(1) *lower semicontinuous* (l.s.c.) if for $\forall \varepsilon > 0$, $\forall x \in X$ and $y \in F(x)$, there exists a neighborhood U of x such that $y \in \bigcap \{S_\varepsilon(F(x')) : x' \in U\}$.

(2) *quasi lower semicontinuous* (q.l.s.c.) [**2**] if for each $x \in X$ and every neighborhood V of x, and every $\varepsilon > 0$ there exists a point $x' \in V$ such that for every $y \in F(x')$ there exists a neighborhood U_y of x for which $y \in \bigcap \{S_\varepsilon(F(z)) : z \in U_y\}$.

(3) *locally uniformly weakly lower semicontinuous* (w.l.s.c.) [**3**], if for any $x \in X$, there exists an open neighborhood $U(x)$ of x with the property that for any $\varepsilon > 0$ and $y \in Y$ there exists $\delta > 0$ such that for any $z \in U(x)$ there exists $r > 0$ such that
$$\varnothing \neq S_r(y) \cap F_\delta(z) \subseteq S_\varepsilon(F_\mu(z))$$
for all $\mu > 0$.

Proposition 1.3. *Let X be a topological space, (Y, d) a metric space. If a set–valued mapping $F : X \to 2^Y$ is* q.l.s.c*, then it is* w.l.s.c. .

Proof. Fix $x_0 \in X$ and let $U(x_0) = X$. For any $y \in Y$, we take $\delta = \varepsilon$. For any $x \in U(x_0)$, $F_\delta(x) \neq \varnothing$ from the property of F. Take $x \in U(x_0)$, then there exists $r > 0$ such that $S_r(y) \cap F_\delta(x) \neq \varnothing$. If $y' \in S_r(y) \cap F_\delta(x)$, then there exists a neighborhood $U(x)$ of x such that $d(y', F(z)) < \delta$ for each $z \in U(x)$. For $\mu > 0$, since F is q.l.s.c., there exists $x' \in U(x)$ such that $F(x') \subseteq F_\mu(x)$. Thus $d(y', F(x')) < \delta = \varepsilon$, and hence $d(y', F_\mu(x)) < \delta$ and $y' \in S_\varepsilon(F_\mu(x))$. We conclude that F is w.l.s.c. . ∎

It is easy to see that l.s.c. ⇒ q.l.s.c. ⇒ w.l.s.c. . In general, the converses are not true (Refer to [**4**] for the relevant examples). A mapping $f : X \to Y$ is *δ–continuous* if there exists an open cover $\mathcal{U}$ of X such that **diam(** $\boldsymbol{f(U)}$**)**$< \delta$ for each $U \in \mathcal{U}$. Every continuous maping is *δ–continuous*, but the converse is not true. Let (Y, d) be a metric space, and for each positive integer n, let $P_n = \{(t_i) \in [0, 1]^n : \sum_1^n t_i = 1\}$.

Definition 1.4 ([**1**])**.** A convex structure on (Y, d) is a sequence of subsets $M_n \subset Y^n$ and continuous mappings $k_n : M_n \times P_n \to Y$ satisfying the following conditions:

(1) $k_n(y, \dots, y; t_1, \dots, t_n) = y$ for all $y \in Y$ and all $(t_i) \in P_n$;

(2) $k_n(y_1, \dots, y_n; t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)$
$= k_{n-1}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n; t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$
for all $y_i \in Y$ and $(t_i) \in P_n$ with $t_i = 0$;

(3) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in N$ and $(t_i) \in P_n$,

$$d\left(k_n\big((y_i);(t_i)\big), k_n\big((y_{i'});(t_i)\big)\right) < \varepsilon$$

if $d(y_i, y_{i'}) < \delta$ for each i.

A subset $C \subseteq Y$ is said to be *convex* with respect to this convex structure if for each $n, C^n \subseteq M_n$ and $k_n(C^n \times P_n) \subseteq C$. The equi–uniform continuity conditions (3) is crucial; this is a convex structure analogy to local convexity in a linear space. Throughout this paper, we use N to denote the set of positive integers.

2. Main Results

Theorem 2.1. *Let X be a paracompact topological space, (Y, d) a metric space with a convex structure and $F : X \to 2^Y$ a* w.l.s.c. *set–valued mapping with each $F(x)$ a complete and convex subset of Y. Then F admits a continuous selection.*

Proof. By the definition of w.l.s.c., for $\forall\varepsilon > 0$ and $x \in X$, $F_\varepsilon(x) \neq \varnothing$. We shall show that $F_0 : X \to 2^Y$ is l.s.c., and for any $x_0 \in X$ there exists an open neighborhood $U(x_0)$ with the property that for any $\varepsilon > 0$ and $y \in Y$, there exists $\delta > 0$ such that for all $x \in U(x_0)$,

$$d(y, F_0(x)) \;<\; d\big(y, F_\delta(x)\big) + \varepsilon$$

Fix $x_0 \in X$. Since F is w.l.s.c., there exists an open neighborhood $U(x_0)$ with the property that for any $\varepsilon > 0$ and $y \in Y$, there exists $\delta > 0$ such that for any $x \in U(x_0)$,

$$\varnothing \neq S_{r_x}(y) \cap F_\delta(x) \subseteq S_{\varepsilon/2^2}\big(F_\mu(x)\big)$$

for all $\mu > 0$ and some $r_x > 0$.

Fix $x \in U(x_0)$, let $r_1 = d\big(y, F_\delta(x)\big) + \varepsilon/2$. It is not difficult to see that $S_{r_1}(y) \cap F_\delta(x) \cap S_{r_x}(y) \neq \varnothing$. Pick $y_1 \in S_{r_1}(y) \cap F_\delta(x) \cap S_{r_x}(y)$, then $F_\mu(x) \cap S_{\varepsilon/2^2}(y_1) \neq \varnothing$ for any $\mu > 0$. We want to obtain a sequence $< y_n >$ satisfying that

(1) $y_{n+1} \in S_{\varepsilon/2^{n+1}}(y_n)$;

(2) $y_n \in S_{\varepsilon/2^{n+1}}\big(F_\mu(x)\big)$ for each $\mu > 0$.

Suppose that for $n = 1, 2, \ldots, k$ we have defined y_n satisfying (1) and (2). For y_k choose $\delta_k > 0$ and $r_k > 0$ such that $\varnothing \neq S_{r_k}(y_k) \cap F_{\delta_k}(x) \subseteq S_{\varepsilon/2^{k+2}}\big(F_\mu(x)\big)$ for all $\mu > 0$. Since $S_{\varepsilon/2^{k+1}}(y_k) \cap F_{\delta_k}(x) \neq \varnothing$, we can pick $y_{k+1} \in S_{\varepsilon/2^{k+1}}(y_k) \cap S_{r_k}(y_k) \cap F_{\delta_k}(x)$. Then $S_{\varepsilon/2^{k+2}}(y_{k+1}) \cap F_\mu(x) \neq \varnothing$ for all $\mu > 0$. Thus (1) and (2) are satisfied for $n = k + 1$. So $\langle y_n \rangle$ is a Cauchy sequence. Next, we shall show that

(1) $\langle y_n \rangle$ converges to some point y_0, and $y_0 \in S_{\varepsilon/2^n}(y_n)$ for every $n \in N$.

By (2), for each $n \in N$, we can take $y_{n'} \in F_{\varepsilon/2^{n+1}}(x)$ and $y_{n''} \in F(x)$ such that $d(y_{n'}, y_n) < \varepsilon/2^{n+1}$ and $d(y_{n'}, y_{n''}) < \varepsilon/2^{n+1}$. So $d(y_n, y_{n''}) < \varepsilon/2^n$ and $\langle y_{n''} \rangle$ is a Cauchy sequence of $F(x)$. Suppose $\langle y_{n''} \rangle$ converges to y_0. Then $\langle y_n \rangle$ converges to y_0.

(2) $F_0(x) \neq \varnothing$ for every $x \in X$.

Let $\mu > 0$. For each $k \in N$, pick $z_k \in S_{\varepsilon/2^{k+1}}(y_k) \cap F_{\mu/2}(x)$. Then $z_k \to y_0$. If $y_0 \overline{\in} F_\mu(x)$, then for each $U \in \mathcal{U}(x)$, there is $x' \in U$ such that $d\big(y_0, F(x')\big) \geq \mu$. There is $k \in N$ such that $S_{\mu/2}(z_k) \subseteq S_\mu(y_0)$, thus $d\big(z_k, F(x')\big) \geq \mu/2$. Hence $z_k \overline{\in} \bigcup_{U \in \mathcal{U}(x)} \bigcap_{x' \in U} S_{\mu/2}\big(F(x')\big) = F_{\mu/2}(x)$. A contradiction. We have shown $y_0 \in F_\mu(x)$ for every $\mu > 0$, which implies $y_0 \in F_0(x)$.

(3) $F_0(x)$ is a convex and complete subset of $F(x)$ for each $x \in X$.
Obviously, $F_0(x)$ is a closed set of Y, $F_0(x) = \overline{F_0(x)} \subseteq F(x)$. We only need to show that $F_0(x)$ is convex. Let $y_1, \ldots, y_n \in F_0(x)$, we have to prove that $k_n(y_1, \ldots, y_n; t_1, \ldots, t_n) \in F_0(x)$ for each $(t_i) \in P_n$. For $\varepsilon > 0$, let $\delta(\varepsilon) = r$. For this $r > 0$, $y_i \in \bigcap_{x' \in U_i} S_r\big(F(x')\big)$, where U_i is a neighborhood of x. Let $U = \bigcap_{i=1}^{i=n} U_i$. Then $y_i \in \bigcap_{x' \in U} S_r\big(F(x')\big)$ for each $i \leq n$. For each $x' \in U$, $\exists z_i \in F(x')$ such that $d(y_i, z_i) < \delta(\varepsilon)$. So

$$d\big(k_n(y_1, \ldots, y_n; t_1, \ldots, t_n), k_n(z_1, \ldots, z_n; t_1, \ldots, t_n)\big) < \varepsilon.$$

It follows that for any $x' \in U$,

$$k_n(y_1, \ldots, y_n; t_1, \ldots, t_n) \in S_\varepsilon\big(k_n(z_1, \ldots, z_n; t_1, \ldots, t_n)\big) \subseteq S_\varepsilon\big(F(x')\big).$$

Hence $k_n(y_1, \ldots, y_n; t_1, \ldots, t_n) \in F_0(x)$, and $F_0(x)$ is convex.

(4) F_0 is l.s.c., and whenever $x \in U(x_0)$, $d\big(y, F_\delta(x)\big) \leq d\big(y, F_0(x)\big) < d\big(y, F_\delta(x)\big) + \varepsilon$.
First of all, it is easy to see $d\big(y, F_\delta(x)\big) \leq d\big(y, F_0(x)\big) < d\big(y, F_\delta(x)\big) + \varepsilon$. Now, suppose $F_0(x) \cap V \neq \varnothing$ for some open subset of Y. Pick $y \in F_0(x) \cap V$, then there exists $\varepsilon > 0$ such that $S_\varepsilon(y) \subseteq V$, so we can find an open neighborhood $W(x)$ of x and $\delta > 0$ such that $d\big(y, F_0(x')\big) < d\big(y, F_\delta(x')\big) + \varepsilon$ for every $x' \in W(x)$. Since $y \in F_0(x)$, by the definition of F_0, there exists an open neighborhood $G(x)$ of x with $G(x) \subseteq W(x)$ such that $d\big(y, F(z)\big) < \delta$ whenever $z \in G(x)$. Hence, $y \in F_\delta(x')$ whenever $x' \in G(x)$, which implies that F_0 is l.s.c..

(5) F_0 admits a continuous selection.
We need to construct a sequence $\langle f_n \rangle$ of selections of F_0 such that

(i) each f_n is 2^{-n}–continuous;

(ii) $d(f_n, f_{n+1}) < 2^{-n}$ for all $n \in N$.

For $\varepsilon > 0$, let $\delta = \delta(\varepsilon)$ be the same as that in condition (3) of Definition 1.4, and assume that $\delta(\varepsilon) \leq \varepsilon$. First, we construct f_1. Pick an open cover $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ of Y such that $\operatorname{diam}(V_\alpha) < \delta\big(\delta(1/2)/5\big)$. Since F_0 is l.s.c., there exists a locally finite open cover $\{U_\alpha : \alpha \in \Lambda\}$ such that $U_\alpha \subseteq F_0^{-1}(V_\alpha)$. Let $\{p_\alpha : \alpha \in A\}$ be a partition of unity on X such that $X_\alpha = \overline{\{x : p_\alpha(x) > 0\}} \subseteq U_\alpha$. Well–order the index set A for $\{p_\alpha : \alpha \in A\}$, and for each $x \in X$ set $A(x) = \{\alpha \in A : x \in X_\alpha\} = \{\alpha_1, \ldots, \alpha_n\}$ with the ordering inherited from A. Take $y_{\alpha_i}(x) \in F_0(x) \cap V_{\alpha_i}$. We define that $f_1 : X \to Y$ such that

$$f_1(x) = k_n\big(y_{\alpha_1}(x), \ldots, y_{\alpha_n}(x); p_{\alpha_1}(x), \ldots, p_{\alpha_n}(x)\big).$$

Then for any $x \in X$, $f_1(x) \in F_0(x)$. Next, we show that f_1 is $\delta(1/2)$–continuous. For each $x \in X$, pick an open neighborhood $N(x)$ of x which only meets X_α satisfying that $x \in X_\alpha$. then when $x' \in N(x)$, $A(x') \subseteq A(x)$. By the continuity of k_n, we can assume that

$$d\left(k_n\big(y_{\alpha_1}(x), \ldots, y_{\alpha_n}(x); p_{\alpha_1}(x), \ldots, p_{\alpha_n}(x)\big), k_n\big(y_{\alpha_1}(x), \ldots, y_{\alpha_n}(x); p_{\alpha_1}(x'), \ldots, p_{\alpha_n}(x')\big)\right) < \delta(1/2)/5$$

for any $x' \in N(x)$. Let $A(x') = \{\alpha_{j_1}, \ldots, \alpha_{j_m}\}$, where $1 \leq j_1 < \ldots < j_m \leq n$. We may assume that $j_i = i$. Then since $\{y_{\alpha_i}(x), y_{\alpha_i}(x')\} \subseteq V_{\alpha_i}$ and

$\operatorname{diam}(V_{\alpha_i}) < \delta\big(\delta(1/2)/5\big)$ for each $i \leq m$, we have

$$d\left(k_m\big(y_{\alpha_1}(x), \dots, y_{\alpha_m}(x); p_{\alpha_1}(x'), \dots, p_{\alpha_m}(x')\big), k_m\big(y_{\alpha_1}(x'), \dots, y_{\alpha_m}(x'); p_{\alpha_1}(x'), \dots, p_{\alpha_m}(x')\big)\right) < \delta(1/2)/5.$$

Thus

$$\begin{aligned} d\big(f_1(x), f_1(x')\big) &< d\left(k_n\big(y_{\alpha_1}(x), \dots, y_{\alpha_n}(x); p_{\alpha_1}(x), \dots, p_{\alpha_n}(x)\big), k_n\big(y_{\alpha_1}(x), \dots, y_{\alpha_n}(x); p_{\alpha_1}(x'), \dots, p_{\alpha_n}(x')\big)\right) \\ &+ d\left(k_m\big(y_{\alpha_1}(x), \dots, y_{\alpha_m}(x); p_{\alpha_1}(x'), \dots, p_{\alpha_m}(x')\big), k_m\big(y_{\alpha_1}(x'), \dots, y_{\alpha_m}(x'); p_{\alpha_1}(x'), \dots, p_{\alpha_m}(x')\big)\right) \\ &< \delta(1/2)/5 + \delta(1/2)/5 < \delta(1/2)/2, \end{aligned}$$

and $\operatorname{diam}\left(f_1\big(N(x)\big)\right) < \delta(1/2)$. We have proved that f_1 is $\delta(1/2)$–continuous.

Suppose $f_i (i \leq n)$ is $\delta(1/2^i)$–continuous and $d\big(f_i(x), f_{i+1}(x)\big) < 1/2^i (i \leq n-1)$. By Lemma 2.3 in [**1**], there is a $\delta(1/2^{n+1})$–continuous selection f_{n+1} of F_0 such that $d\big(f_n(x), f_{n+1}(x)\big) < 1/2^n$. So we can inductively construct a sequence $\langle f_n \rangle$ of $\delta(2^{-n})$–continuous selections of F_0 satisfying (i) and (ii).

Let $f = \lim f_n$. Then f is a selection of F_0. To complete the proof, we shall show that f is continuous. Let $x \in X$ and V be an open neighborhood of $f(x)$. Then there exists $\varepsilon > 0$ such that $S_\varepsilon\big(f(x)\big) \subseteq V$. Take $n_0 \in N$ such that $1/2^{n_0-1} < \varepsilon/4$, then $d\big(f_{n_0}(x), f(x)\big) \leq 1/2^{n_0-1} < \epsilon/4$. Since f_{n_0} is $1/2^{n_0}$–continuous, there exists an open neighborhood G of x satisfying $\operatorname{diam}\big(f_{n_0}(G)\big) < 1/2^{n_0}$. For each $z \in G$,

$$d\big(f(x), f(z)\big) \leq d\big(f(x), f_{n_0}(x)\big) + d\big(f_{n_0}(x), f_{n_0}(z)\big) + d\big(f_{n_0}(z), f(z)\big) < \varepsilon.$$

Hence $f(G) \subseteq V$ and f is continuous. We have shown that f is a continous selection of F. □

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