NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 32 (2003), 195–201

A RESULT ABOUT THE NUMBER OF EXTENSIONS OF A POSET

Lin You and Tianming Wang

(Received February 2002)

Abstract. If P is a finite poset and $e(P)$ denotes the number of all linear extensions of P to a totally ordered finite set, then $e(P)$ is equal to the number of all maximal chains of $J(P)$ ([**1**]). In this paper, we show that $e(P)$ is also equal to the sum of the numbers of linear extensions of P's some special subsets to a totally ordered finite set.

1. Introduction and Definitions

Let n be a positive integer and $[n] = \{1, 2, \ldots, n\}$. Then the set $[n]$ with its usual order forms a n–element poset with the special property that any two elements are comparable. This poset is denoted by $\boldsymbol{n}_{\leq}$.

Definition 1.1. Let P be a poset (partially ordered set). If $|P| = n$, then an order–preserving bijective map $\sigma : P \longrightarrow \boldsymbol{n}_{\leq}$ is called a linear extension of P to a totally ordered set . We use $E(P)$ to denote the set of all linear extensions of P to a totally ordered set. While the number of all linear extension of P to a totally ordered set is denoted by $e(P)$.

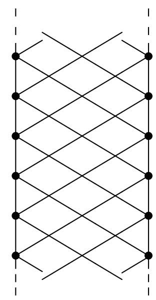
Definition 1.2. Let P be a poset.

1. If x, $y \in P$, then we say y covers x or y is a cover of x if no element $z \in P$ satisfies $x < z < y$.
2. A chain $C : \cdots < x_i < x_{i+1} < \cdots$ of P is called a saturated chain if x_{i+1} covers x_i for every i.
3. A saturated chain C of P is called a maximal chain if there does not exist any element x of P such that $C \bigcup \{x\}$ becomes a saturated chain.
4. An antichain(or Sperner family) of P is a subset A of P such that any two distinct elements of A are incomparable.
5. A cutset of P is a subset which intersects every maximal chain of P; Furthermore, a cutset A is called an antichain cutset if A is also an antichain.

Generally, a poset does not always have an antichain cutset. For example, the poset defined as the following figure has no antichain cutsets ([**5**]):

1991 *Mathematics Subject Classification* 05A15, 06A07.

Key words and phrases: Extension, Antichain, Maximal Chain.
This Project is supported by the Hainan Science Foundations (No. 10002).



But in a finite poset, there always exist antichain cutsets. For example, let P be a finite poset and A be the subset of all minimal elements of P. Then A is an antichain cutset. While the subset of all maximal elements of P is also an antichain cutset.

An order ideal of P is a subset I of P such that if $x \in I$ and $y \leq x$, then $y \in I$. The set of all order ideals of P forms a poset, denoted $J(P)$, when it is ordered by inclusion. In [**1**], Richard Stanley proved that the number $e(P)$ is equal to the number of all maximal chains of $J(P)$ by giving a bijective map if P is a finite poset. In the next, we prove that $e(P)$ is also equal to the sum of the numbers of linear extensions of P's some special subsets with the cardinality $|P| - 1$ to a totally ordered finite set.

2. Main Theorem

Theorem 2.1. *Let P be a finite poset and A be an antichain cutset of P . Then we have*

$$e(P) \;=\; \sum_{x \in A} e(P - x),$$

where $P - x$ denotes the subset of P excluding the element x of P.

Proof. Define a map ϕ from $E(P)$ to $\cup_{x \in A} E(P - x)$ as follows:

Take $\sigma \in E(P)$.

1. If there exists $x_0 \in A$ such that $\sigma(x_0) = n$, then let $\phi(\sigma) = \tau : \tau(y) = \sigma(y)$ for every $y \in P - x_0$. Then obviously $\tau \in E(P - x_0)$.
2. If $\sigma(y_0) = n$, while $y_0 \in P - A$, let

$$\begin{aligned}
\sigma(y_1) &= \max\{\sigma(y) : y \in P \text{ and } y_0 \text{ is a cover of } y\}\\
\sigma(y_2) &= \max\{\sigma(y) : y \in P \text{ and } y_1 \text{ is a cover of } y\}\\
&\cdots\ \cdots\\
\sigma(y_k) &= \max\{\sigma(y) : y \in P \text{ and } y_{k-1} \text{ is a cover of } y\}\\
&\cdots\ \cdots
\end{aligned}$$

Hence we have a maximal chain of P:

$$\cdots < y_k < y_{k-1} < \cdots < y_2 < y_1 < y_0.$$

By the definition of A we know that there exists an element $x_0 \in A$ such that $x_0 = y_r$, where r is a positive integer. Let $y_r = x_0$, then

$$x_0 \;=\; y_r < y_{r-1} < \cdots < y_1 < y_0$$

is a saturated chain of P.

In this case, let $\phi(\sigma) = \tau$: for any $y \in P - x_0$,

$$\tau(y) \;=\; \begin{cases} \sigma(y) & \text{if } y \neq y_i \\ \sigma(y_{i+1}) & \text{if } y = y_i, \end{cases}$$

where $i = 0, 1, 2, \ldots, r-1$.

Now we will prove that ϕ is a bijective map from $E(P)$ to $\cup_{x \in A} E(P-x)$.

Claim (a). ϕ *is a surjective map*:

Take $\tau \in E(P - x_0)$, $x_0 \in A$.

(1) If x_0 is a maximal element of P, then define a map σ from P to $\boldsymbol{n}_{\leq}$ as follows:

$$\sigma(y) \;=\; \begin{cases} n & if\;\; y = x_0 \\ \tau(y) & if\;\; y \in P - x_0 \end{cases}$$

Then $\sigma \in E(P)$ and $\phi(\sigma) = \tau$.

(2) If x_0 is not a maximal element of P, then let

$$\tau(\overline{y}_1) = \min\{\tau(y) : y \in P \text{ and } y \text{ is a cover of } x_0\}\;,$$
$$\tau(\overline{y}_2) = \min\{\tau(y) : y \in P \text{ and } y \text{ is a cover of } \overline{y}_1\}\;,$$
$$\cdots\quad\cdots$$
$$\tau(\overline{y}_l) = \min\{\tau(y) : y \in P \text{ and } y \text{ is a cover of } \overline{y}_{l-1}\}\;,$$
$$\cdots\quad\cdots$$

Hence we have a saturated chain:

$$x_0 \;=\; \overline{y_0} < \overline{y}_1 < \cdots < \overline{y}_l < \cdots$$

Since P is finite, there must exist a positive integer s such that $\overline{y}_s$ is a maximal element of P. In this case, define a map σ from P to $\boldsymbol{n}_{\leq}$ as follows:

$$\sigma(y) \;=\; \begin{cases} n & \text{if } y = \overline{y}_s \\ \tau(\overline{y}_{i+1}) & \text{if } y = \overline{y}_i \; i = 0, 1, \ldots, s-1;\; \overline{y}_0 = x_0 \\ \tau(y) & \text{otherwise.} \end{cases}$$

Then obviously $\sigma \in E(P)$, and we can show $\phi(\sigma) = \tau$ as follows:

Suppose $\phi(\sigma) = \tau'$. Let

$$\overline{y}_s = y_0,\; \overline{y}_{s-1} = y_1,\; \ldots,\; \overline{y}_1 = y_{s-1},\; \overline{y}_0 = y_s = x_0.$$

Since for any $y \in P$ and $y \neq y_1$ and y_0 is a cover of y we have

$$\sigma(y) = \tau(y) < \tau(y_0) = \tau(\overline{y}_s) = \sigma(\overline{y}_{s-1}) = \sigma(y_1).$$

Thus $\sigma(y_1) = \max\{\sigma(y) : y \in P \text{ and } y_0 \text{ is a cover of } y\}$.

Similarly,we have

$$\sigma(y_2) = \max\{\sigma(y) : y \in P \text{ and } y_1 \text{ is a cover of } y\}.$$
$$\cdots \quad \cdots$$
$$\big(\sigma(x_0) = \big)\sigma(y_s) = \max\{\sigma(y) : y \in P \text{ and } y_{s-1} \text{ is a cover of } y\}.$$

Hence for any $y \in P - x_0$, if $y \neq y_i$, then $\tau^{'}(y) = \sigma(y) = \tau(y)$. If $y = y_i$, then $\tau^{'}(y) = \sigma(y_{i+1}) = \sigma(\overline{y}_{s-i-1}) = \tau(\overline{y}_{s-i}) = \tau(y_i) = \tau(y)$.

That is, for any $y \in P - x_0$, $\tau^{'}(y) = \tau(y)$, which implies $\tau^{'} = \tau$. Therefore ϕ is surjective.

Claim (b). *ϕ is a injective map*:

Let $\phi : \sigma \longmapsto \tau$ and $\overline{\sigma} \longmapsto \overline{\tau}$.

Suppose $\tau = \overline{\tau} \in E(P - x_0)$.

By the definition of ϕ, we have the followings:

(1) If x_0 is a maximal element of P, then $\sigma(x_0) = \overline{\sigma}(x_0) = n$. While for every $y \in P - x_0$, $\sigma(y) = \tau(y) = \overline{\tau}(y) = \overline{\sigma}(y)$. Thus $\sigma = \overline{\sigma}$.

(2) Assure x_0 is not a maximal element of P. Let $\sigma(y_0) = n,\ y_0 \in P - A$; $\sigma(y_i) = \max\{\sigma(y) : y \in P \text{ and } y_{i-1} \text{ is a cover of } y\}$, where $i = 1, 2, \cdots, r$; $y_r = x_0$.
Then
$$x_0 = y_r < y_{r-1} < \cdots < y_1 < y_0$$
is a saturated chain of P.

Samely, let $\overline{\sigma}(\overline{y}_0) = n$, $\overline{y}_0 \in P - A$; $\overline{\sigma}(\overline{y}_j) = \max\{\sigma(y) : y \in P$ and $\overline{y}_{j-1}$ is a cover of $y\}$, where $j = 1, 2, \cdots, s$; $\overline{y}_s = x_0$.

Then
$$x_0 \ = \ \overline{y}_s < \overline{y}_{s-1} < \cdots < \overline{y}_1 < \overline{y}_0$$
is a saturated chain of P.

Thus for $y \in P - x_0$ we have

$$\tau(y) \ = \ \begin{cases} \sigma(y) & \text{if } y \neq y_i \\ \sigma(y_{i+1}) & \text{if } y = y_i,\ i = 0, 1, \ldots, r-1 \end{cases}$$

and

$$\overline{\tau}(y) \ = \ \begin{cases} \overline{\sigma}(y) & \text{if } y \neq \overline{y}_i \\ \overline{\sigma}(\overline{y}_{j+1}) & if\ \ y = \overline{y}_j,\ j = 0, 1, \ldots, s-1. \end{cases}$$

If $y_{r-1} \neq \overline{y}_{s-1}$, then

$$\begin{cases} \tau(y_{r-1}) = \sigma(y_r) = \sigma(x_0) \\ \overline{\tau}(y_{r-1}) = \sigma(y_{r-1}) \end{cases}$$

and

$$\begin{cases} \tau(\overline{y}_{s-1}) = \sigma(\overline{y}_{s-1}) \\ \overline{\tau}(\overline{y}_{s-1}) = \overline{\sigma}(\overline{y}_s) = \overline{\sigma}(x_0). \end{cases}$$

Since $\tau = \overline{\tau}$, we have $\sigma(x_0) = \overline{\sigma}(y_{r-1})$ and $\sigma(\overline{y}_{s-1}) = \overline{\sigma}(x_0)$. Hence

$$\overline{\sigma}(x_0) = \sigma(\overline{y}_{s-1}) > \sigma(x_0) = \overline{\sigma}(y_{r-1}) > \overline{\sigma}(x_0),$$

which is a contradiction.

Thus $y_{r-1} = \overline{y}_{s-1}$.

By induction, we obtain

$$r = s,\ y_i = \overline{y}_i \quad \text{for} \quad i = 0, 1, \ldots, r-1.$$

Hence $\sigma(y) = \overline{\sigma}(y)$ holds for every $y \in P$, that is, $\sigma = \overline{\sigma}$ if $\tau = \overline{\tau}$. Therefore ϕ is injective.

From Claim (a) and Claim (b), we know ϕ is bijective. Therefore

$$|E(P)| \ = \ \left| \bigcup_{x \in A} E(P-x) \right|,$$

that is

$$e(P) \ = \ \sum_{x \in A} e(P-x).$$

□

For any finite poset P, obviously its all maximal chains does not always have the same cardinality. But there are many finite posets whose maximal chains have the same cardinality. For example, modular lattices and partition lattice Π_n of $[n]$ are both such posets. Especially, all maximal chains of Π_n have the length n.

Let P be a finite poset with its all maximal chains having the fixed length l. Let P have k maximal chains C_i:

$$x_{i,1} < x_{i,2} < \cdots < x_{i,l}, \quad i = 1, 2, \ldots, k.$$

Suppose A_j be the subset consisted by the all different elements in the set $\{x_{i,j} \mid i = 1, 2, \ldots, k\}$ for every $j = 1, 2, \ldots, l$. Then obviously, every A_j is an antichain cutset and P is the disjoint union of A_j.

Corollary 2.2. *Preserve the above notations and suppositions, then we have*

$$l \ = \ \frac{\sum_{x \in P} e(P-x)}{e(P)}.$$

Proof. Since A_j is an antichain cutset of P, by the above theorem we have

$$e(P) \ = \ \sum_{x \in A_j} e(P-x)$$

holds for every j: $1 \le j \le l$.

Thus

$$\sum_{1 \le j \le l} e(P) \ = \ \sum_{1 \le j \le l} \sum_{x \in A_j} e(P-x),$$

that is,

$$l \cdot e(P) = \sum_{x \in \bigcup_{1 \le j \le l} A_j} e(P-x) = \sum_{x \in P} e(P-x)$$

or

$$l = \frac{\sum_{x \in P} e(P-x)}{e(P)} .$$

□

Example 2.3. Let $P = \{2, 3, 4, 6, 12, 18\}$ and P is partially ordered by divisibility relation. Then P is a poset and has five maximal chains with the fixed length 3:

$$\{2,\ 4,\ 12\};\ \{2,\ 6,\ 12\};\ \{2,\ 6,\ 18\};\ \{3,\ 6,\ 12\};\ \{3,\ 6,\ 18\}.$$

$A = \{2,\ 3\}$ is an antichain of P which intersects every maximal chain.

We use the matrix expression to denote the extension of P to the totally ordered set $\mathbf{6}_{\leq}$ (here $n = 6$). For example, the extension σ:

$$2 \longmapsto 1;\ 3 \longmapsto 3;\ 4 \longmapsto 2;\ 6 \longmapsto 4;\ 12 \longmapsto 5;\ 18 \longmapsto 6.$$

is expressed as:

$$\begin{pmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 1 & 3 & 2 & 4 & 5 & 6 \end{pmatrix}.$$

Hence we have

$$E(P) =$$
$$\Big\{\sigma_1 = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 1 & 3 & 2 & 4 & 5 & 6 \end{smallmatrix}\right),\ \sigma_2 = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 1 & 3 & 2 & 4 & 6 & 5 \end{smallmatrix}\right),\ \sigma_3 = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{smallmatrix}\right),$$
$$\sigma_4 = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 1 & 2 & 3 & 4 & 6 & 5 \end{smallmatrix}\right),\ \sigma_5 = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 1 & 2 & 4 & 3 & 5 & 6 \end{smallmatrix}\right),\ \sigma_6 = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 1 & 2 & 4 & 3 & 6 & 5 \end{smallmatrix}\right),$$
$$\sigma_7 = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 1 & 2 & 5 & 3 & 6 & 4 \end{smallmatrix}\right),\ \sigma_8 = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{smallmatrix}\right),\ \sigma_9 = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 2 & 1 & 3 & 4 & 6 & 5 \end{smallmatrix}\right),$$
$$\sigma_{10} = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 2 & 1 & 4 & 3 & 5 & 6 \end{smallmatrix}\right),\ \sigma_{11} = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{smallmatrix}\right),\ \sigma_{12} = \left(\begin{smallmatrix} 2 & 3 & 4 & 6 & 12 & 18 \\ 2 & 1 & 5 & 3 & 6 & 4 \end{smallmatrix}\right)\Big\}.$$

$$E(P-3) =$$
$$\Big\{\tau_1 = \left(\begin{smallmatrix} 2 & 4 & 6 & 12 & 18 \\ 1 & 2 & 3 & 5 & 4 \end{smallmatrix}\right),\ \tau_2 = \left(\begin{smallmatrix} 2 & 4 & 6 & 12 & 18 \\ 1 & 2 & 3 & 4 & 5 \end{smallmatrix}\right),\ \tau_3 = \left(\begin{smallmatrix} 2 & 4 & 6 & 12 & 18 \\ 1 & 3 & 2 & 5 & 4 \end{smallmatrix}\right),$$
$$\tau_4 = \left(\begin{smallmatrix} 2 & 4 & 6 & 12 & 18 \\ 1 & 3 & 2 & 4 & 5 \end{smallmatrix}\right),\ \tau_5 = \left(\begin{smallmatrix} 2 & 4 & 6 & 12 & 18 \\ 1 & 4 & 2 & 5 & 3 \end{smallmatrix}\right)\Big\}.$$

$$E(P-2) =$$
$$\Big\{\tau_6 = \left(\begin{smallmatrix} 3 & 4 & 6 & 12 & 18 \\ 2 & 1 & 3 & 4 & 5 \end{smallmatrix}\right),\ \tau_7 = \left(\begin{smallmatrix} 3 & 4 & 6 & 12 & 18 \\ 2 & 1 & 3 & 5 & 4 \end{smallmatrix}\right),\ \tau_8 = \left(\begin{smallmatrix} 3 & 4 & 6 & 12 & 18 \\ 1 & 3 & 2 & 5 & 4 \end{smallmatrix}\right),$$
$$\tau_9 = \left(\begin{smallmatrix} 3 & 4 & 6 & 12 & 18 \\ 1 & 3 & 2 & 4 & 5 \end{smallmatrix}\right),\ \tau_{10} = \left(\begin{smallmatrix} 3 & 4 & 6 & 12 & 18 \\ 1 & 4 & 2 & 5 & 3 \end{smallmatrix}\right),\ \tau_{11} = \left(\begin{smallmatrix} 3 & 4 & 6 & 12 & 18 \\ 1 & 2 & 3 & 4 & 5 \end{smallmatrix}\right),$$
$$\tau_{12} = \left(\begin{smallmatrix} 3 & 4 & 6 & 12 & 18 \\ 1 & 2 & 3 & 5 & 4 \end{smallmatrix}\right)\Big\}.$$

Thus the map ϕ from $E(P)$ to $E(P-2) \cup E(P-3)$ determined as in the theorem is that:

$$\sigma_i \longmapsto \tau_i; \quad \text{where} \quad i = 1, 2, \ldots, 12.$$

For example, since $\sigma_1(18) = 6$, we have $y_0 = 18$ and

$$\sigma_1(y_1) = \max\{\sigma_1(y) : y \in P \text{ and } y_0 \text{ is a cover of } y\} = \sigma_1(6), \text{ that is, } y_1 = 6.$$

$$\sigma_1(y_2) = \max\{\sigma_1(y) : y \in P \text{ and } y_1 \text{ is a cover of } y\} = \sigma_1(3), \text{ that is, } y_2 = 3.$$

Because $y_2 = 3 \in A$, $\phi : \sigma_1 \longmapsto \tau$, where $\tau \in E(P - x_0) = E(P - 3)$ and

$$\tau(y) = \begin{cases} \sigma_1(y) & \text{if } y \neq 18,\, 6 \\ \sigma_1(6) = 4 & \text{if } y = 18 \\ \sigma_1(3) = 3 & \text{if } y = 6 \end{cases}$$

for $y \in P - 3$, that is

$$\tau = \begin{pmatrix} 2 & 4 & 6 & 12 & 18 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} = \tau_1.$$

Besides, it is easy to verify $\frac{\sum_{x \in P} e(P-x)}{e(P)} = 3$.

References

1. R. Stanley, *Enumerative Combinatorics* [M], California, Wadsworth, Inc., 1986.
2. W.T. Trotter, *Partially ordered sets*, Handbook of combinatorics [M], Vol. 1, 2, pp. 433–480, Amsterdam, New York, Elsevier, Cambridge, Mass., MIT Press, 1995.
3. R. Maltby, *When is every minimal cutset an antichain* [J], Acta Sci. Math. **59** (3–4) (1994), 381–403.
4. R. Ahlswede, P.L. Erdős and N. Graham, *A spilitting property of maximal antichains* [J], Combinatorica, **15** (4) (1995), 475–480.
5. M.H. El-zahar and N. Zaguria, *Antichains and cutsets* [J], Contemporary Mathematics, **57** (1986), 227–261.

Lin You
Department of Applied Mathematics
Dalian University of Technology
Dalian 116023
P. R. CHINA
mryoulin@263.net

Tianming Wang
Department of Applied Mathematics
Dalian University of Technology
Dalian 116023
P. R. CHINA
wangtm@dlut.edu.cn