Abstract. We prove a representation formula for the extended Green function of an open subset $D$ of an open set $E$, in terms of the Green function of $E$. Using this result, we deduce a formula for the greatest thermic minorant on $D$ of any heat potential on $E$, and of any supertemperature on $E$ if the closure of $D$ is a compact subset of $E$. In the latter case, we also determine a necessary and sufficient condition under which the greatest thermic minorant on $D$, of an arbitrary supertemperature $v$ on $E$, is given by the Poisson integral of the restriction of $v$ to $\partial D$.

1. Introduction

Let $E$ be an open subset of Euclidean space $\mathbb{R}^{n+1}$, and let $G_E$ denote its Green function relative to the heat equation ([8, 5, 11]), except that if $E = \mathbb{R}^{n+1}$ we omit the subscript. It is known that, for any point $q \in E$, the function $G_E(\cdot; q)$ can be uniquely extended to a function $G_E^w(\cdot; q)$ on $\mathbb{R}^{n+1}$ with the following properties: (a) $G_E^w(\cdot; q)$ is a nonnegative subtemperature on $\mathbb{R}^{n+1}\{q\}$; (b) $G_E^w(\cdot; q) = 0$ on $\mathbb{R}^{n+1}\setminus E$ and at every finite regular point of the normal boundary $\partial_n E$ of $E$; (c) For each finite point $q_0 \in \partial E$, either there is $c > 0$ such that $\Omega(q_0; c) \cap E = \emptyset$, in which case $G_E^w(q_0; q) = 0$, or there is no such $c$, in which case $G_E^w(q_0; q) = \lim_{c \to 0+} \left( \sup_{\Omega(q_0; c) \cap E} G_E(\cdot; q) \right)$ ([5, p.343], [12, Theorem 2.1]). Here, for any point $q_0 \in \mathbb{R}^{n+1}$ and any positive number $c$, the set

$$\Omega(q_0; c) = \{ p \in \mathbb{R}^{n+1} : G(q_0; p) > (4\pi c)^{-\frac{n}{2}} \}$$

is the heat ball with centre $q_0$ and radius $c$.

Let $D$ be an open subset of $E$. In this paper we prove a representation formula for the extended Green function $G_D^w$ in terms of $G_E$ (Theorem 3), and use it to deduce a formula for the greatest thermic minorant on $D$ of any heat potential on $E$, and of any supertemperature on $E$ if $\overline{D}$ is a compact subset of $E$ (Theorem 7). These results are corrected versions, with completed and corrected proofs, of results in [5, p.344]. We then use Theorem 7, together with results in [12], to determine, for those open sets $D$ with compact closure contained in $E$, a necessary and sufficient condition under which the greatest thermic minorant on $D$, of any supertemperature $v$ on $E$, is equal to the Poisson integral of the restriction of $v$ to $\partial D$.
$\partial D$. The condition is that the Riesz measure associated to $v$ does not charge the complement in $\partial D$ of the set of points that are regular normal boundary points relative to the adjoint heat equation.

Our terminology will follow [11], where further details can be found. For proofs that the corresponding concepts in [1, 4, 5] are equivalent, see [2] or [10].

We work in $\mathbb{R}^{n+1} = \{(x,t) : x \in \mathbb{R}^n, t \in \mathbb{R}\}$, and denote a typical point by $p$ or $(x,t)$ as convenient.

We require a classification of the boundary points of $E$, in which we use the following notations for the upper and lower half-balls. Given $p_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and $r > 0$, we denote by $H(p_0, r)$ the open lower half-ball $\{(x,t) : |x - x_0|^2 + (t - t_0)^2 < r^2, t < t_0\}$, and by $H^*(p_0, r)$ the open upper half-ball $\{(x,t) : |x - x_0|^2 + (t - t_0)^2 < r^2, t > t_0\}$.

**Definitions.** Let $q$ be a boundary point of the open set $E$. We call $q$ a normal boundary point if either

(a) $q$ is the point at infinity, or

(b) $q \in \mathbb{R}^{n+1}$ and for every $r > 0$, $H(q, r) \setminus E \neq \emptyset$. 

Otherwise, we call $q$ an abnormal boundary point; in this case, there is some $r_0 > 0$ such that $H(q, r_0) \subseteq E$. The abnormal boundary points are of two kinds, according to whether they can be approached from above by points in $E$. If there is some $r_1 < r_0$ such that $H^*(q, r_1) \cap E = \emptyset$, then $q$ is called a singular boundary point. In this case, $H(q, r_1) = B(q, r_1) \cap E$. On the other hand if, for every $r < r_0$, we have $H^*(q, r) \cap E \neq \emptyset$, then $q$ is called a semi-singular boundary point.

The set of all normal boundary points of $E$ is denoted by $\partial_n E$, that of all abnormal points by $\partial_a E$, that of all singular points by $\partial_s E$, and that of all semi-singular points by $\partial_{ss} E$. Thus $\partial E = \partial_n E \cup \partial_a E$ and $\partial E = \partial_s E \cup \partial_{ss} E$. The essential boundary $\partial_e E$ is defined by

\[
\partial_e E = \partial_n E \cup \partial_{ss} E = \partial E \setminus \partial_s E.
\]

Let $f$ be an extended real-valued function defined on $\partial E$. For any lower bounded hypertemperature $v$ on $E$, we put $v$ in the class $U_f^E$ if and only if both

\[
\lim\inf_{p \to q} v(p) \geq f(q) \quad \text{for all } q \in \partial_n E,
\]

and

\[
\lim\inf_{(x,t) \to (y,s)+} v(x,t) \geq f(y,s) \quad \text{for all } (y,s) \in \partial_{ss} E.
\]

This is the same as in [8, 11]. We put $v$ in the class $\overline{U}_f^E$ if and only if

\[
\lim\inf_{p \to q} v(p) \geq f(q) \quad \text{for all } q \in \partial E.
\]

This is similar to [1, 4, 5]. Clearly $\overline{U}_f^E \subseteq U_f^E$, so that if $\overline{U}_f^E = \inf\{v : v \in \overline{U}_f^E\}$ and $U_f^E = \inf\{v : v \in U_f^E\}$, then $U_f^E \leq \overline{U}_f^E$ on $E$. It is shown in [14, Theorem 11] that these two infima are equal, so we shall not distinguish between them. They are called the upper solution for $f$ on $E$. The lower solution $L_f^E$ can be defined by the formula $L_f^E = -U_{-f}^E$. 

If \( L_f^E = U_f^E \) and is a temperature on \( E \), we denote it by \( S_f^E \) and say that \( f \) is resolutive for \( E \). We also call \( S_f^E \) the PWB solution to the Dirichlet problem for \( f \) on \( E \). If \( f \) is resolutive for \( E \), then \( S_f^E \) has the representation \( S_f^E(p) = \int_{\partial E} f \, d\omega_p^E \) for all \( p \in E \), where \( \omega_p^E \) denotes the caloric (or parabolic) measure relative to \( E \) and \( p \); see [14] for details. A point \( q \in \partial_n E \) is called regular if \( \lim_{p \to q} S_f^E(p) = f(q) \) for all \( f \in C(\partial E) \). A point \( q = (y, s) \in \partial_n E \) is called regular if \( \lim_{(x,t) \to (y,s)} S_f^E(x, t) = f(y, s) \) for all \( f \in C(\partial E) \).

Let \( u \) be a nonnegative supertemperature on \( E \). If \( A \subseteq E \), then the reduction of \( u \) over \( A \) (relative to \( E \)), denoted by \( \hat{R}_u^A \), is the infimum of the family of nonnegative supertemperatures on \( E \) that majorize \( u \) on \( A \). The lower semicontinuous smoothing \( \hat{R}_u^A \) is called the smoothed reduction of \( u \) over \( A \) (relative to \( E \)).

Every result about the heat equation has an obvious dual for the adjoint heat equation, obtained by reversing the temporal variable. Such results are not stated explicitly here, but are referred to as the coterminal duals of given results on the heat equation itself. We call a reduction relative to the adjoint equation a cothermal reduction; the caloric measure relative to the adjoint equation the cothermal caloric measure; and so on. Sometimes we abbreviate this: a cothermal regular boundary point is called coregular; a set which is cothermal semipolar is called cosemipolar; and so on. Notations pertaining to the adjoint equation are obtained by adding an * to the notations for the heat equation; thus \( \partial^*_a E \) and \( \partial^*_a \hat{P}B \) denote the cothermal normal and abnormal boundaries of \( E \), and \( \omega^*_p \) denotes the cothermal caloric measure with respect to \( E \) and \( p \).

The Green function and coterminal Green function are related by the equation \( G_E(p; q) = G_E^*(q; p) \) for all \( p, q \in E \) ([8, 5, 11]). We can extend \( G_E^*(\cdot; p) \) in the same way as \( G_E(\cdot; q) \). Thus, for any point \( p \in E \), the function \( G_E^*(p; \cdot) \) can be uniquely extended to a function \( G_E^*(p; \cdot) \) on \( \mathbb{R}^{n+1} \), with the following properties: (a) \( G_E^*(p; \cdot) \) is a nonnegative coorthetemperature on \( \mathbb{R}^{n+1} \); (b) \( G_E^*(p; \cdot) = 0 \) on \( \mathbb{R}^{n+1} \); and at every finite coregular point of \( \partial^*_a \hat{P}B \); (c) For each finite point \( q_0 \in \partial E \), either there is \( c > 0 \) such that \( \Omega^*(q_0; c) \cap \hat{E} = \emptyset \), in which case \( G_E^*(p; q_0) = 0 \), or there is no such \( c \), in which case

\[
G_E^*(p; q_0) = \lim_{c \to 0^+} \left( \sup_{\Omega^*(q_0; c) \cap \hat{E}} G_E(p; \cdot) \right).
\]

Here, for any point \( q_0 \in \mathbb{R}^{n+1} \) and any positive number \( c \), the set

\[
\Omega^*(q_0; c) = \{ p \in \mathbb{R}^{n+1} : G(p; q_0) > (4\pi c)^{-\frac{n}{2}} \}
\]

is called the coheat ball with centre \( p_0 \) and radius \( c \).

2. Representation Formula for the Extended Green Function

We need the following analogue of a standard result in Helms [7, Lemma 8.39]. It could be deduced from [14, Lemma 10], but we give a more direct proof.

**Lemma 1.** Let \( E \) be an open set, let \( D \) be an open subset of \( E \), and let \( f \) be a resolutive function on \( \partial E \). If the function \( g \) is defined on \( \partial D \) by

\[
g = \begin{cases} 
S_f^E & \text{on } \partial D \cap E, \\
\frac{1}{f} & \text{on } \partial D \cap \partial E,
\end{cases}
\]

then
then \( g \) is resolutive with \( S^D_g = S^E_f \) on \( D \).

**Proof.** Let \( u \) belong to \( S^E_f \). Then \( u \geq S^E_f \) on \( E \), and so for each point \( q \in \partial D \cap E \) we have

\[
\liminf_{p \to q, \ p \in D} u(p) \geq \liminf_{p \to q, \ p \in D} S^E_f(p) = S^E_f(q) = g(q).
\]

Moreover, for each point \( q \in \partial D \cap \partial E \) we have

\[
\liminf_{p \to q, \ p \in \partial E} u(p) \geq \liminf_{p \to q, \ p \in \partial E} u(p) \geq f(q) = g(q).
\]

Therefore the restriction of \( u \) to \( D \) belongs to \( S^D_g \). It follows that \( U^D_g \leq S^E_f \) on \( D \).

Applying this result to \( -f \) we get \( U^D_g \leq S^E_f \), so that \( L^D_g \geq S^E_f \) on \( D \). This gives \( S^E_f \leq L^D_g \leq U^D_g \leq S^E_f \) on \( D \), which implies that \( g \) is resolutive for \( D \) with \( S^D_g = S^E_f \) on \( D \). \( \square \)

**Lemma 2.** Let \( E \) be an open set, let \( D \) be an open subset of \( E \), and let \( q \in E \). Define a function \( g_E(\cdot; q) \) on \( \partial D \) by putting

\[
g_E(\cdot; q) = \begin{cases} G_E(\cdot; q) & \text{on } \partial D \cap E, \\ 0 & \text{on } \partial D \cap \partial E. \end{cases}
\]

(a) The function \( g_E(\cdot; q) \) is resolutive for \( D \), with

\[
S^D_{g_E(\cdot; q)}(p) = \int_{\partial D} g_E(\cdot; q) \, d\omega^D_p = R^E_{g_E(\cdot; q)}(p)
\]

for all \( p \in D \).

(b) If \( q \in D \) then

\[
G_D(\cdot; q) = G_E(\cdot; q) - S^D_{g_E(\cdot; q)}
\]

on \( D \), and \( S^D_{g_E(\cdot; q)} \) is the greatest thermic minorant of \( G_E(\cdot; q) \) on \( D \).

(c) If \( q \in E \setminus \overline{D} \), then

\[
S^D_{g_E(\cdot; q)} = G_E(\cdot; q)
\]

on \( D \).

**Proof.** (a) It follows from [14, Lemma 8 Corollary] that \( g_E(\cdot; q) \) is resolutive for \( D \) with \( S^D_{g_E(\cdot; q)}(p) = \int_{\partial D} g_E(\cdot; q) \, d\omega^D_p \). The fact that \( S^D_{g_E(\cdot; q)} = R^E_{g_E(\cdot; q)} \) follows from [14, Theorem 9] (or [1, Satz 4.1.4] if \( \overline{D} \) is a compact subset of \( E \)).

(b) If \( q \in E \) and \( f = G(\cdot; q) \) on \( \partial E \), where \( G \) denotes the Green function for \( \mathbb{R}^{n+1} \) (defined to be 0 at infinity), then \( f \) is resolutive for \( E \) and \( G_E(\cdot; q) = G(\cdot; q) - S^E_G(\cdot; q) \) on \( E \), by [11, Theorem 8.53(a)]. Applying Lemma 1 with this choice of \( f \), we see that the function

\[
\phi(\cdot; q) = G(\cdot; q) - g_E(\cdot; q) = \begin{cases} S^E_{G(\cdot; q)} & \text{on } \partial D \cap E, \\ G(\cdot; q) & \text{on } \partial D \cap \partial E, \end{cases}
\]

for all \( q \in \partial D \cap E \), and \( \phi(\cdot; q) \) is resolutive for \( D \) with \( S^D_{\phi(\cdot; q)}(p) = 0 \) for all \( p \in D \) and \( q \in \partial D \cap E \). Therefore, by [14, Theorem 9], \( S^D_{\phi(\cdot; q)} = G_E(\cdot; q) \) on \( D \).
is resolutive with $S^D_{\phi(\cdot;q)} = S^E_{G(\cdot;q)}$ on $D$. Therefore, if $q \in D$,
\[
G_E(\cdot;q) - S^D_{gD(\cdot;q)} = \left( G(\cdot;q) - S^E_{G(\cdot;q)} \right) - S^D_{gE(\cdot;q)} \\
= \left( G(\cdot;q) - S^D_{G(\cdot;q)} \right) - S^E_{G(\cdot;q)} + S^D_{gE(\cdot;q)} \\
= G_D(\cdot;q)
\]
on $D$, which proves (1). Moreover, denoting by $GM_E$ the operation of taking the greatest thermic minorant over $E$, we use [11, Theorem 3.66] to obtain
\[
G_D(\cdot;q) = G(\cdot;q) - GM_D(G(\cdot;q)) \\
= G(\cdot;q) - GM_D(G_E(\cdot;q) + GM_E(G(\cdot;q))) \\
= G(\cdot;q) - GM_D(G_E(\cdot;q)) - GM_D(GM_E(G(\cdot;q))) \\
= G(\cdot;q) - GM_D(G_E(\cdot;q)) - GM_E(G(\cdot;q)) \\
= G(\cdot;q) - GM_D(G_E(\cdot;q)),
\]
and so comparison with (1) shows that $GM_D(G_E(\cdot;q)) = S^D_{gE(\cdot;q)}$.

(c) If $q \in E \setminus \overline{D}$, then $G(\cdot;q)$ is a temperature on an open superset of $\overline{D} \cap \mathbb{R}^{n+1}$, so that $S^D_{gE(\cdot;q)} = G(\cdot;q)$ on $D$. Hence
\[
S^D_{gE(\cdot;q)} = S^D_{gE(\cdot;q)} - S^D_{\phi(\cdot;q)} = G(\cdot;q) - S^E_{G(\cdot;q)} = G_E(\cdot;q)
\]
on $D$.

Remark. In [5, p.332], Doob observed that a more general result than Lemma 2(a) is true, and that Lemma 2(b) is true, but gave no proof of either.

Theorem 3. Let $E$ be an open set, and let $D$ be an open subset of $E$. The function $G^*_D$ has the representation
\[
G^*_D(p; q) = G_E(p; q) - \int_{\partial D \cap E} G_E(\cdot;q) \, d\nu^D_p
\]
for all $p \in D$ and $q \in E$. Therefore, for each point $q \in E$, the function $G^*_D(\cdot;q)$ is a temperature on $D \setminus \{q\}$.

Proof. Let $p \in D$. In the case where $q \in D$ also, we have, by Lemma 2(b),
\[
G_D(p; q) = G_E(p; q) - S^D_{gE(\cdot;q)}(p).
\]
In view of Lemma 2(a), this is equation (2) in this case. In the case where $q \in E \setminus \overline{D}$, we have $G^*_D(p; q) = 0$, by the cothermal dual of [5, 1.XVIII.9] or of [12, Theorem 2.1], and so Lemma 2(c) gives
\[
G^*_D(p; q) = 0 = G_E(p; q) - S^D_{gE(\cdot;q)}(p),
\]
which is (2). Hence (2) holds for all $q \in E \setminus \partial D$.

We now consider the case where $q$ is a coregular point of $\partial^*_D D$ in $E$. Here again we have $G^*_D(p; q) = 0$, by the cothermal dual of [5, 1.XVIII.9] or of [12, Theorem 2.1], so that equation (2) reduces to
\[
0 = G_E(p; q) - S^D_{gE(\cdot;q)}(p).
\]
By Lemma 2(a), this is equivalent to the equation
\[ G_E(p; q) = R_{G_E(p; q)}^{E \setminus D}(p) = \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(p), \]
since \( p \) is an interior point of \( D \). In order to prove (3), we first show that
\[ \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(p) = \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(q) \]
whenever \( p \in D \) and \( q \in E \), where \( \hat{\hat{R}} \) denotes a smoothed cothermal reduction. We consider first the case where \( q \in D \). Here Lemma 2(b) and its cothermal dual hold, and imply that
\[ S^D_{g^E(\cdot; q)}(p) = G_E(p; q) - G_D(p; q) = G_E^*(q; p) - G_D^*(q; p) = S^D_{g^E(\cdot; p)}(q), \]
where \( S^D \) denotes a cothermal PWB solution on \( D \). Now equation (4) follows, because
\[ S^D_{g^E(\cdot; q)}(p) = R_{G_E(p; q)}^{E \setminus D}(p) = \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(p) \]
by Lemma 2(a), and dually
\[ S^D_{g^E(\cdot; p)}(q) = R_{G_E(p; q)}^{E \setminus D}(q) = \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(q) = \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(q), \]
whenever \( p, q \in D \). Now we consider the case where \( q \in E \setminus D \). For any positive integer \( k \) such that \( \frac{1}{k} \) is less than the distance from \( q \) to the boundary of \( E \), we put \( D_k = D \cup B(q, \frac{1}{k}) \). Then \( D_k \) is an open subset of \( E \) such that \( p, q \in D_k \), so that we can apply (4) with \( D_k \) instead of \( D \). Thus
\[ \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(p) = \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(q). \]
As \( k \to \infty \), the sequence \( \{E \setminus D_k\} \) expands to the union \( E \setminus (D \cup \{q\}) \). It therefore follows from [11, Theorem 9.33] and its cothermal dual that
\[ \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(p) = \lim_{k \to \infty} \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(p) = \lim_{k \to \infty} \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(q) = \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(q). \]
Since \( \{q\} \) is a polar set, it now follows from [11, Theorem 7.27(c)] and its cothermal dual that equation (4) holds in this case too. Hence (4) is valid whenever \( p \in D \) and \( q \in E \). Putting (4) into (3), we see that we need to prove that
\[ G_E(p; q) = \hat{\hat{R}}_{G_E(p; \cdot)}^{E \setminus D}(q). \]
We know that, for any fixed point \( p \in D \), the equation
\[ G_E(p; q) = R_{G_E(p; q)}^{E \setminus D}(q) \]
holds for all \( q \in E \setminus D \). Furthermore, by the dual of [11, Theorem 9.27], any cothermal reduction differs from its lower semicontinuous smoothing on at most a cosenipolar subset of \( E \). Therefore equation (5) holds for all points \( q \in E \setminus D \) except a cosenipolar subset of \( E \), and hence on a cothermal fine dense subset of \( E \setminus D \), by the dual of a result in [5, p.310]. Any smoothed cothermal reduction on \( E \) is cothermal fine continuous on \( E \), by the dual of [11, Lemma 9.3], so it follows that equation (5) holds at every cothermal fine limit point of \( E \setminus D \) in \( E \), and hence at any coregular point \( q \in \partial^*_n D \) in \( E \), by the dual of [11, Theorem 9.40].

We have now shown that, for any point \( p \in D \), equation (2) is true whenever \( q \) lies either in \( E \setminus \partial D \) or in the set of coregular points of \( \partial^*_n D \) in \( E \). Thus (2) holds
outside a cothermal semipolar subset of $\mathbb{R}^{n+1}$, by [11, Corollary 9.47], and hence on a set which is cothermal fine dense in $E$. Since both sides of (2) define cothermal fine continuous functions of $q$ on $E$, it follows that (2) holds for all $q \in E$.

For the last part we take any point $q \in E$ and note that, by Lemma 2(a), formula (2) can be written as

$$G_D^2(p; q) = G_E(p; q) - S^D_{g_E(\cdot ; q)}(p),$$

which implies the result. 

Remark. For any given point $p \in D$, the integral in (2), namely

$$\int_{\partial D \cap E} G_E(\cdot ; q) \, d\omega^D_p = \int_{\partial D \cap E} G^*_E(q; \cdot) \, d\omega^D_p,$$

is the value at $q$ of the coheat potential of the restriction of the measure $\omega^D_p$ to $\partial D \cap E$. Since equation (2) shows that

$$\int_{\partial D \cap E} G_E(\cdot ; q) \, d\omega^D_p \leq G_E(p; q),$$

this potential is finite-valued.

**Corollary 4.** Let $E$ be an open set, and let $D$ be an open subset of $E$. If $p, q \in D$, then the symmetry relation

$$\int_{\partial D \cap E} G_E(\cdot ; q) \, d\omega^D_p = \int_{\partial D \cap E} G_E(p; \cdot) \, d\omega^*_q$$

holds, where $\omega^*_q$ denotes the cothermal caloric measure relative to $D$ at $q$.

**Proof.** If $p, q \in D$, then the cothermal dual of Theorem 3 yields

$$G_D(p; q) = G^*_D(q; p) = G^*_E(q; p) - \int_{\partial D \cap E} G^*_E(\cdot ; p) \, d\omega^*_q$$

$$= G_E(p; q) - \int_{\partial D \cap E} G_E(p; \cdot) \, d\omega^*_q.$$

This, together with Theorem 3 itself, gives (6). 

According to [5, Theorem 1.XVIII.9(b)], the function $G^*_D$ has the representation $G^*_D(p; q) = G_E(p; q) - \int_{\partial D} G_E(\cdot ; q) \, d\omega^D_p$ for all $p \in D$ and $q \in E$. This is not generally the case, as Theorem 3 and the following example show.

**Example 5.** We give an example in which

$$\int_{\partial D \cap \partial E} G_E(\cdot ; q) \, d\omega^D_p > 0.$$ 

Since $G_E(\cdot ; q) = G_E(\cdot ; q)$ on $\partial D \cap E$, this will imply that

$$\int_{\partial D} G_E(\cdot ; q) \, d\omega^D_p > \int_{\partial D \cap E} G_E(\cdot ; q) \, d\omega^D_p.$$

Let $E = \{(x, t) : t \neq 0\}$ and $D = \{(x, t) : 0 < |t| < 1\}$, so that $\partial D \cap E = \mathbb{R}^n \times \{0\}$. Let $q = (y, s)$ with $-1 < s < 0$. Then $S^E_{g_E(\cdot ; q)}(x, t) = 0$ whenever $t < 0$, so that

$$G_E(\cdot ; q) = G(\cdot ; q) - S^E_{g_E(\cdot ; q)} = G(\cdot ; q)$$

Thus for all $p \in D$, and

$$G_E(\cdot ; q) = G(\cdot ; q) - S^E_{g_E(\cdot ; q)} = G(\cdot ; q)$$

implies

$$\int_{\partial D \cap \partial E} G_E(\cdot ; q) \, d\omega^D_p > 0.$$
on $\mathbb{R}^n \times [-\infty, 0]$. Furthermore, because $G_E^n(z; q)$ is a subtemperature on $\mathbb{R}^{n+1}\setminus \{q\}$, for any point $z \in \mathbb{R}^n$ we have, by [11, Lemma 3.16],

$$G_E^n((z, 0); q) = \limsup_{(x, t) \to (z, 0)-} G_E^n((x, t); q) = \limsup_{(x, t) \to (z, 0)-} G_E((x, t); q)$$

$$= \lim_{(x, t) \to (z, 0)-} G((x, t); q) = G((z, 0); q) > 0.$$ 

Therefore, for any point $p \in \mathbb{R}^n \times [0, 1] \subseteq D$, we have

$$\int_{\partial D \cap \partial E} G_E^n(z; q) d\omega_p \geq \int_{\mathbb{R}^n \times \{0\}} G_E^n(z; q) d\omega_p > 0.$$

3. Representation Formulas for the Greatest Thermic Minorant

We use the following notations. If $\nu$ is a nonnegative Borel measure on an open set $E$, and $A$ is a $\nu$-measurable set, we denote the restriction of $\nu$ to $A$ by $\nu_A$. If $D$ is an open subset of $E$, we define the function $G_D^n \nu$ by

$$G_D^n \nu(p) = \int_E G_D^n(p; q) d\nu(q)$$

for all $p \in D$.

The proof of our next theorem requires the following technical result.

**Lemma 6.** Let $E$ be an open set, let $D$ be an open subset of $E$, let $G_E \nu$ be a heat potential on $E$, and let $Y$ be the support of $\nu$. Then $G_D^n \nu$ is a temperature on $D \setminus Y$.

**Proof.** We shall use [11, Theorem 1.29], which requires that $G_D^n \nu$ satisfy a mean value theorem and be locally integrable on $D \setminus Y$. Let $m$ be an integer such that $m \geq 5$, and for each point $p$ let $\Omega_m(p; c)$ denote the corresponding modified heat ball with centre $p$ and radius $c$ ([11, p.19]). For each point $q \in Y \cap E$, Theorem 3 tells us that $G_D^n(\cdot; q)$ is a temperature on $D \setminus \{q\}$, so that, by [11, Theorem 1.25],

$$G_D^n(p; q) = \nu_m(G_D^n(\cdot; q); p; c)$$

whenever $\Omega_m(p; c) \subseteq D \setminus Y$. It follows that

$$G_D^n(p) = \int_{Y \cap E} G_D^n(p; q) d\nu(q) = \int_{Y \cap E} \nu_m(G_D^n(\cdot; q); p; c) d\nu(q) = \nu_m(G_D^n \nu; p; c)$$

whenever $\Omega_m(p; c) \subseteq D \setminus Y$.

Let $p \in D$. If $q \in D$ also, then $G_D^n(p; q) = G_D(p; q) \leq G_E(p; q)$. If $q \in E \setminus \overline{D}$, then $G_D(p; q) = 0 \leq G_E(p; q)$. If $q \in \partial D \cap E$, then either $G_D(p; q) = 0$, or

$$G_D^n(p; q) = \lim_{c \to 0^+} \sup_{(r^*(q; c) \cap D} G_D(p; \cdot) \leq \lim_{c \to 0^+} \sup_{(r^*(q; c) \cap E} G_E(p; \cdot) = G_E(p; q).$$

It follows that $G_D^n \nu \leq G_E \nu$. Therefore, because $G_E \nu$ is locally integrable on $D$, the same is true of $G_D^n \nu$. Now [11, Theorem 1.29] shows that $G_D^n \nu$ is a temperature on $D \setminus Y$.

**Theorem 7.** Let $E$ be an open set, let $D$ be an open subset of $E$, and let $Y$ be the complement in $\partial D \cap E$ of the set of coregular points of $\partial^n D$.
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(a) If \( v = G_{EV} \) is a heat potential on \( E \), then the greatest thermic minorant \( u \) of \( v \) on \( D \) has the representation

\[
u(p) = \int_{\partial D \cap E} v \, d\omega_p^D + G_D^v \nu_Y(p)
\]

for all \( p \in D \).

(b) If \( \overline{D} \) is a compact subset of \( E \), \( v \) is any supertemperature on \( E \), and \( \nu \) is the Riesz measure associated to \( v \), then equation (7) again holds for all \( p \in D \).

**Proof.** Let \( p \in D \). We denote by \( R \) the set of coregular points of \( \partial_r D \) in \( E \). Then, since \( G_D^v(p; \cdot) = 0 \) on \( (E \setminus \overline{D}) \cup R \), we have

\[
G_D^v(p) = G_D^v \nu_Y(p) + G_D \nu_Y(p).
\]

By applying (2), we obtain

\[
\begin{align*}
G_D^v \nu(p) &= \int_E G_E(p; q) \, d\nu(q) - \int_E d\nu(q) \int_{\partial D \cap E} G_E(q; \cdot) \, d\omega_p^D \\
&= G_E \nu(p) - \int_{\partial D \cap E} G_E \nu \, d\omega_p^D \\
&= v(p) - \int_{\partial D \cap E} v \, d\omega_p^D.
\end{align*}
\]

The last integral represents a temperature on \( D \), by [14, Lemma 8 Corollary]. We now combine (9) and (8) to obtain

\[
v(p) = G_D^v \nu(p) + \int_{\partial D \cap E} v \, d\omega_p^D = G_D^v \nu_Y(p) + G_D \nu_Y(p) + \int_{\partial D \cap E} v \, d\omega_p^D.
\]

By Lemma 6, \( G_D^v \nu_Y \) is a temperature on \( D \). Therefore, taking the greatest thermic minorant on \( D \) of each term, we find that (7) holds for all \( p \in D \).

We now consider the case where \( \overline{D} \) is a compact subset of \( E \), and \( v \) is any supertemperature on \( E \) with associated Riesz measure \( \nu \). Let \( C \) be an open superset of \( \overline{D} \) such that \( \overline{C} \) is a compact subset of \( E \). Then \( \nu_C \) is a finite measure, so that \( G_{EV_C} \) is a heat potential on \( E \). Applying the Riesz decomposition theorem [11, Theorem 6.34] on \( C \), we find that \( G_{C \nu_C} \) is a heat potential, and that both \( v = G_{C \nu_C} + h \) and \( G_{EV_C} = G_{C \nu_C} + h' \) on \( C \), where \( h \) and \( h' \) are temperatures. It follows that \( v = G_{EV_C} + w \) on \( C \), where \( w = h - h' \) is a temperature. Hence, on \( D \), \( u \) is equal to the greatest thermic minorant of \( G_{EV_C} \) on \( D \), plus \( w \). Therefore, by the first part and the fact that \( w \) is a temperature on \( C \supseteq \overline{D} \), we have

\[
u(p) = \int_{\partial D} G_{EV_C} \, d\omega_p^D + G_D^v \nu_Y(p) + \int_{\partial D} w \, d\omega_p^D \\
= \int_{\partial D} v \, d\omega_p^D + G_D^v \nu_Y(p)
\]

as required. \( \square \)

Formula (7) is analogous to one due to Frostman [6], [3, p. 116], for superharmonic functions in the case where \( \overline{D} \subseteq E \). The general case of that result can be found in [5, p. 133].

The following example illustrates both that the integral in (7) can be identically zero when the greatest thermic minorant of \( v \) is positive, and a contrast with the superharmonic case.
Therefore we put $k$ for all $G$, Lindelöf Theorem. For each $\{J\}$ we can find a countable subclass then it belongs to the normal boundary thereof. We first note that, if $E$ is strictly increasing. By a polygonal path, we mean a path which is a union of there is a polygonal path $E$ on the set where $v > c$ is the constant function 1, not $c$.

In the context of Theorem 7(b), it is clear that $u$ is the Poisson integral of the restriction of $v$ to $\partial D$ if $\nu(Y) = 0$. In our final theorem we shall prove the converse. To do this we need the following result, which was mentioned briefly in [9]. We use the following notation. Given an open set $E$ and a point $p_0 \in E$, we denote by $E^*(p_0; E)$ the set of points $p$ that are higher than $p_0$ relative to $E$, in the sense that there is a polygonal path $\gamma \subseteq E$ joining $p_0$ to $p$, along which the temporal variable is strictly increasing. By a polygonal path, we mean a path which is a union of finitely many line segments.

**Theorem 9.** Let $E$ be an open set, and let $q_0$ be a finite point of $\partial_n E$. Then $q_0$ is regular for $E$ if and only if it is a regular point of $\partial_n E^*(p; E)$ whenever it belongs to $\partial E^*(p; E)$.

**Proof.** We first note that, if $q_0$ belongs to the boundary of $E^*(p; E)$ for some $p \in E$, then it belongs to the normal boundary thereof.

If $q_0$ is regular for $E$, and also belongs to $\partial E^*(p; E)$, then the above note and [11, Corollary 8.47] show that it is a regular point of $\partial_n E^*(p; E)$. We now suppose, conversely, that $q_0$ is regular point of $\partial_n E^*(p; E)$ whenever it belongs to $\partial E^*(p; E)$. The class of open sets $\{E^*(p; E) : p \in E\}$ covers $E$, and so we can find a countable subclass $\{E^*(p_k; E) : k \in \mathbb{N}\}$ which also covers $E$, by the Lindelöf Theorem. For each $k \in \mathbb{N}$, we can similarly choose a sequence $\{q_{k,j}\}$ in $E^*(p_k; E)$ such that

$$\bigcup_{j=1}^{\infty} E^*(p_k,j; E) = E^*(p_k; E),$$

so that

$$\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} E^*(p_k,j; E) = E. \quad (10)$$

We put $J = \{k \in \mathbb{N} : q_0 \in \partial E^*(p_k; E)\}$, and observe that, by hypothesis, $q_0$ is a regular point of $\partial_n E^*(p_k; E)$ for all $k \in J$. Therefore

$$\lim_{p \to q_0} G_{E^*(p_k; E)}(p; p_k,j) = 0 \quad (11)$$

for all $k \in J$ and $j \in \mathbb{N}$, by [11, Theorem 8.53(b)]. By [11, Theorem 6.7], each $G_{E^*(p_k; E)}$ is the restriction to $\Lambda^*(p_k; E) \times E^*(p_k; E)$ of $G_E$, and $G_E(\cdot; p_k,j) = 0$ on
\( E \setminus \Lambda^*(p_k; E) \supseteq E \setminus \Lambda^*(p_k; E) \). It therefore follows from (11) that
\[
0 = \lim_{p \to q_0, p \in \check{\Lambda}^*(p_k; E)} G_E(p; p_k, j) = \lim_{p \to q_0, p \in E} G_E(p; p_k, j)
\]
for all \( k \in J \) and \( j \in \mathbb{N} \). On the other hand, for each \( k \in \mathbb{N} \setminus J \), there is a neighbourhood \( V_k \) of \( q_0 \) that does not meet \( \Lambda^*(p_k; E) \supseteq \Lambda^*(p_k, j; E) \) for any \( j \in \mathbb{N} \), so that \( G_E(\cdot; p_k, j) = 0 \) on \( E \cap V_k \), and hence
\[
\lim_{p \to q_0} G_E(p; p_k, j) = 0.
\]
Thus (12) holds whenever \( k, j \in \mathbb{N} \) so that, in view of (10) and [11, Theorem 8.53(c)], \( q_0 \) is regular for \( E \).

**Corollary 10.** Let \( E \) be an open set, let \( q_0 \) be a finite point of \( \partial_n E \), and let \( \{p_k\} \) be a sequence of points in \( E \) such that
\[
\bigcup_{k=1}^{\infty} \Lambda^*(p_k; E) = E.
\]
If \( q_0 \) is irregular for \( E \), then there is a positive integer \( k' \) such that \( q_0 \) is an irregular normal boundary point for \( \Lambda^*(p_{k'}; E) \).

**Proof.** Theorem 9 shows that there is a point \( p' \in E \) such that \( q_0 \in \partial_s \Lambda^*(p'; E) \) and \( q_0 \) is irregular for \( \Lambda^*(p'; E) \). Our condition on the sequence \( \{p_k\} \) shows that \( p' \in \Lambda^*(p_{k'}; E) \) for some positive integer \( k' \), so that \( \Lambda^*(p'; E) \subseteq \Lambda^*(p_{k'}; E) \). Since \( q_0 \in \partial E \cap \partial \Lambda^*(p'; E) \), we see that \( q_0 \in \partial \Lambda^*(p_{k'}; E) \). Since \( q_0 \in \partial_n E \), we see that \( q_0 \in \partial_n \Lambda^*(p_{k'}; E) \). Since \( q_0 \) is irregular for \( \Lambda^*(p'; E) \), it is irregular for \( \Lambda^*(p_{k'}; E) \).

**Lemma 11.** Let \( E \) be an open set, let \( q_0 \in \partial_n E \), and let \( p \) be a point in \( E \) such that \( q_0 \in \partial_n \Gamma^*(p; E) \). Then \( G_E^\circ(q_0; p) > 0 \).

**Proof.** We put \( q_0 = (y_0, s_0) \). We choose an open ball \( B \) in \( \mathbb{R}^n \), and real numbers \( a \) and \( b \), such that \( a < s_0 < b \) and \( B \times [a, s_0] \subseteq \Lambda^*(p; E) \). We denote by \( w \) the nonnegative subtemperature \( G_E^\circ(\cdot; p) \) on \( \mathbb{R}^{n+1} \setminus \{p\} \), and note that \( w \) is a positive temperature on \( \Lambda^*(p; E) \). By [11, Theorem 3.21], the Poisson integral \( u \) of the restriction of \( w \) to the normal boundary of \( B \times [a, b] \), is a nonnegative temperature on \( B \times [a, b] \). Since \( w \) is a temperature on an open superset of \( \overline{B} \times [a, s_0] \), we see that \( u = w \) on \( B \times [a, s_0] \), so that [11, Lemma 3.16] gives
\[
w(q_0) = \limsup_{q \to q_0} w(q) = \lim_{q \to q_0} u(q) = u(q_0).
\]
Since \( u = w > 0 \) on \( B \times [a, s_0] \), the minimum principle shows that \( u(q_0) > 0 \). Hence \( G_E^\circ(q_0; p) = w(q_0) > 0 \).

We are now ready to prove our final theorem. The corresponding result for the superharmonic case was given by Brelot [3, p. 116].

**Theorem 12.** Let \( D \) and \( E \) be open sets such that \( \overline{D} \) is a compact subset of \( E \), and let \( Y \) be the complement in \( \partial D \) of the set of coregular points of \( \partial_n^* D \). Let \( \nu \) be any supertemperature on \( E \) and \( \nu \) its associated Riesz measure. Then the greatest thermic minorant \( \nu \) of \( \nu \) on \( D \) has the representation
\[
u(p) = \int_{\partial D} v \, d\omega_p^D
\]
for all \( p \in D \), if and only if \( \nu(Y) = 0 \).

**Proof.** If \( \nu(Y) = 0 \), then the representation (13) follows from Theorem 7.

In the proof of the converse, we shall use the following notation. For any open subset \( A \) of \( D \), we denote by \( Z_A \) the set of points of \( \partial^*_A \cap \partial^*_D \) that are not coregular for \( A \). Note that \( Z_A \subseteq Z_D \) and \( Y = Z_D \cup \partial^*_D \). We shall assume that \( \nu(Y) > 0 \), and deduce that the representation (13) fails at some point of \( D \).

Let \( \{p_k\} \) be a sequence of points in \( D \) such that

\[
\bigcup_{k=1}^{\infty} \Lambda(p_k; D) = D. \tag{14}
\]

If \( q_0 \in \partial^*_0 D \), then there is an open half-ball \( H^*(q_0; \epsilon) \subseteq D \), and for any point \( p \in H^*(q_0; \epsilon) \) we have \( q_0 \in \partial^*_A(p; D) \). We choose such a point and label it \( p' \).

Condition (14) shows that \( p' \in \Lambda(p_k; D) \) for some positive integer \( k' \), so that \( \Lambda(p'; D) \subseteq \Lambda(p_{k'}; D) \). It follows that \( \partial^*_0 D \subseteq \bigcup_{k=1}^{\infty} (\partial^*_A(p_k; D) \cap \partial^*_D D) \).

Therefore, if \( \nu(\partial^*_0 D) > 0 \) there is a point \( \tilde{p} \in D \) such that \( \nu(\partial^*_0 \Lambda(p; D) \cap \partial D) > 0 \). On the other hand, if \( q_0 \in Z_D \) then the cothermal dual of Corollary 10 shows that there is a positive integer \( k' \) such that \( q_0 \in Z_{\Lambda(p_{k'}; D)} \). It follows that

\[
Z_D = \bigcup_{k=1}^{\infty} Z_{\Lambda(p_k; D)},
\]

Therefore, if \( \nu(Z_D) > 0 \) there is a point \( \tilde{p} \in D \) such that \( \nu(Z_{\Lambda(\tilde{p}; D)}) > 0 \).

In either case, we now put \( \Lambda = \Lambda(\tilde{p}; D) \) and choose a sequence \( \{\pi_j\} \) of points in \( \Lambda \) such that

\[
\bigcup_{j=1}^{\infty} \Lambda(\pi_j; \Lambda) = \Lambda \tag{15}
\]

If \( q \in \partial^*_0 \Lambda \), we choose a point \( p_q \in \Lambda \) such that \( q \in \partial^*_A(p_q; \Lambda) \). Condition (15) shows that \( p_q \in \Lambda(\pi_{j_q}; \Lambda) \) for some positive integer \( j_q \), so that \( \Lambda(p_q; \Lambda) \subseteq \Lambda(\pi_{j_q}; \Lambda) \), and hence \( q \in \partial^*_A(\pi_{j_q}; \Lambda) \). Therefore

\[
G^\alpha_{\Lambda}(\pi_{j_q}; q) > 0, \tag{16}
\]

by the cothermal dual of Lemma 11. On the other hand, if \( q \in Z_\Lambda \) it follows from the cothermal dual of [12, Theorem 2.2] that there is a positive integer \( j_q \) such that \( \Omega^*(q; c) \) meets \( \Lambda(\pi_{j_q}; \Lambda) \) for every \( c > 0 \) and

\[
\lim_{c \to 0^+} \left( \sup_{\Omega^*(q; c) \cap \Lambda} G_{\Lambda}(\pi_{j_q}; \cdot) \right) > 0,
\]

which implies that (16) holds in this case too.

By Theorem 3, for each point \( q \in \partial \Lambda \) the function \( G^\alpha_{\Lambda}(\cdot; q) \) is a nonnegative temperature on \( \Lambda \), so that the inequality (16) and the minimum principle together
imply that $G_A^m(\cdot;q) > 0$ on $\Lambda^*(\pi_{j_i};\Lambda)$. Moreover, for any point $p \in \Lambda^*(\pi_{j_i};\Lambda)$ we have

$$G_A^m(p;q) = \lim_{c \to 0^+} \left( \sup_{\Omega^*(q,c) \cap \Lambda} G_A(p;\cdot) \right) \leq \lim_{c \to 0^+} \left( \sup_{\Omega^*(q,c) \cap D} G_D(p;\cdot) \right) = G_D^m(p;q),$$

so that $G_D^m(p;q) > 0$. If $q \in \partial^*\Lambda \cap \partial D$, then $G_D^m(\cdot;q)$ is a nonnegative temperature on $D$, so that the minimum principle shows that $G_D^m(\cdot;q) > 0$ on $\Lambda^*(\pi_{j_i};D)$, and in particular $G_D^m(\hat{p};q) > 0$. This holds for each $q \in (\partial^*_n \Lambda \cap \partial D) \cup Z_\Lambda$, so that

$$G^m_D(\cdot;q) > 0.$$ 

It now follows from Theorem 7 that the representation (13) does not hold at $\hat{p}$. 

**Corollary 13.** Let $D$ and $E$ be open sets such that $D$ is a compact subset of $E$, and let $q_0 \in \partial D$. Then

$$G_E(p;q_0) = \int_{\partial D} G_E(\cdot;q_0) \, d\omega_p^D$$

(17)

for all $p \in D$, if and only if $q_0$ is a coregular point of $\partial^*_n D$.

**Proof.** Let $Y$ be the complement in $\partial D$ of the set of coregular points of $\partial^*_n D$. Since $q_0 \in \partial D$, the restriction of $G_E(\cdot;q_0)$ to $D$ is a temperature. Moreover, the Riesz measure $\nu$ associated with $G_E(\cdot;q_0)$ is the unit mass at $q_0$. Therefore Theorem 12 shows that equation (17) holds if and only if $\nu(Y) = 0$, which occurs if and only if $q_0 \notin Y$. 

**Corollary 14.** Let $D$ and $E$ be open sets such that $D$ is a compact subset of $E$, and let $Y$ be the complement in $\partial D$ of the set of coregular points of $\partial^*_n D$. Then the greatest thermic minorant on $D$ of an arbitrary supertemperature $v$ on $E$, is given by

$$\int_{\partial D} v \, d\omega_p^D$$

(18)

for all $p \in D$ if and only if $Y = \emptyset$.

**Proof.** If $Y = \emptyset$, then the representation (18) follows from Theorem 12.

On the other hand, if $Y \neq \emptyset$ and $q_0 \in Y$, then Corollary 13 shows that (18) does not hold for $G_E(\cdot;q_0)$. 

In [13, Theorem 3.4], it was shown that any heat ball with closure in $E$ has this property.

**References**


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