⊕-SUPPLEMENTED MODULES RELATIVE TO A TORSION THEORY

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Abstract. Let $R$ be ring and $M$ a right $R$-module. This article introduces the concept of $τ$-⊕-supplemented modules as follows: Given a hereditary torsion theory in $\text{Mod-}R$ with associated torsion functor $τ$ we say that a module $M$ is $τ$-⊕-supplemented when for every submodule $N$ of $M$ there exists a direct summand $K$ of $M$ such that $M = N + K$ and $N \cap K$ is $τ$-torsion, and $M$ is called completely $τ$-⊕-supplemented if every direct summand of $M$ is $τ$-⊕-supplemented. We present here some fundamental properties of these class of modules and study the decompositions of $τ$-⊕-supplemented modules under certain conditions on modules. The question of which direct sum of $τ$-⊕-supplemented modules are $τ$-⊕-supplemented is treated here. The ring $R$ is called right $τ$-perfect if every right $R$-module has a $τ$-projective cover, and the module $M$ is strongly $τ$-⊕-supplemented when for every submodule $N$ of $M$ there exists a direct summand $K$ of $M$ such that $M = N + K$ and $N \cap K$ is small $τ$-torsion submodule of $M$. It is shown that $R$ is right $τ$-perfect if and only if every projective right $R$-module is strongly $τ$-⊕-supplemented, and $R$ is right $τ(R)$-perfect if and only if every projective right $R$-module is $τ$-⊕-supplemented.

1. Introduction

Let $τ$ be a class of right modules over a ring. Motivated by the notion $τ$-complemented modules studied in [11], $τ$-supplemented modules are introduced in [7] and [8]. Here we introduce and study $τ$-⊕-supplemented modules. In what follows $R$ will denote any ring with identity and all modules will be unital right $R$-modules. $τ$ will denote the torsion functor associated with an arbitrary torsion theory on the category $\text{Mod-}R$ of all right $R$-modules. A module $M$ is $⊕$-supplemented if for any given $A \leq M$ there exists a direct summand $B$ of $M$ such that $M = A + B$ and $A \cap B$ is small in $B$(see namely [4], [5], [6], [8], [9]). This article introduces the concept of $τ$-⊕-supplemented modules as follows. Let $τ = (T, F)$ be a torsion theory. Then $τ$ is uniquely determined by its associated class $T$ of $τ$-torsion modules $T = \{ M \in \text{Mod-}R \mid τ(M) = M \}$ where for a module $M$, $τ(M) = \{ N \mid N \leq M, N \in T \}$ and $F$ is referred as $τ$-torsion free class and $F = \{ M \in \text{Mod-}R \mid τ(M) = 0 \}$. A module in $T$(or $F$) is called $τ$-torsion module(or $τ$-torsionfree module). Every torsion class $T$ determines in every module $M$ a unique maximal $T$-submodule $τ(M)$, the $τ$-torsion submodule of $M$, and $τ(M/τ(M)) = 0$. In what follows $τ$ will represent a hereditary torsion theory, that is, if $τ = (T, F)$ then the class $T$ is closed under taking submodules, direct sums, images and extensions by short exact sequences, equivalently the class

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\( \mathcal{F} \) is closed under submodules, direct products, injective hulls and isomorphic copies. We refer the reader to [2] and [12] as torsion theoretic sources sufficient for our purposes and [9] and [15] for the other notations in this paper.

Given a hereditary torsion theory \( \tau = (T, \mathcal{F}) \) in \( \text{Mod}-R \) we say that a module \( M \) is \( \tau - \oplus \)-supplemented if for each submodule \( A \) of \( M \) there exists a direct summand \( B \) of \( M \) such that \( M = A + B \) and \( A \cap B \) is \( \tau \)-torsion. \( M \) is called \( \tau \)-supplemented if for every submodule \( A \) of \( M \) contains a direct summand \( C \) of \( M \) with \( A/C \) \( \tau \)-torsion.

Every \( \tau \)-supplemented module is \( \tau - \oplus \)-supplemented. Some further properties of \( \tau \)-supplemented modules were studied in [7] and [8]. For the torsion class \( \text{Mod}-R \) we denote the corresponding torsion functor by \( \chi \) and if the torsion class is the class of zero modules we denote the corresponding torsion functor by \( \xi \). In this notation \( \xi = (0, \text{Mod} - R) \) and \( \chi = (\text{Mod} - R, 0) \) where 0 denotes the class of zero modules. The torsion functor for the dual Goldie torsion theory will be denoted by \( \tau_\ast \). Then the dual Goldie torsion theory \( \tau_\ast = (T_\ast, \mathcal{F}_\ast) \) is generated by the class of small \( R \)-modules. If \( M = Z^\ast(M) \) where \( Z^\ast(M) = \{ n \in M : nR \text{ is small}\} \) (See [10]), then \( M \) is \( \tau_\ast \)-torsion.

**Example 1.1.** Let \( R \) be any ring. Then

(i) Every \( R \)-module is \( \xi - \oplus \)-supplemented.

(ii) An \( R \)-module \( M \) is \( \xi - \oplus \)-supplemented if and only if \( M \) is semisimple.

(iii) Every \( \oplus \)-supplemented \( R \)-module is \( \tau_\ast - \oplus \)-supplemented.

**Proof.** (i) and (iii) Clear. (ii) By [1, Theorem 9.6]. \( \Box \)

**Example 1.2.** Let \( I \) be an ideal generated as a right ideal \( I = Re = ReR \) by an idempotent element \( e \) of an arbitrary ring \( R \). Let \( \tau_I \) denote the hereditary torsion theory defined by \( I \) with torsion class \( T_I = \{ N \in \text{Mod} - R \mid NI = 0 \} \). Then an \( R \)-module \( M \) is \( \tau_I - \oplus \)-supplemented if and only if \( NI \) is a direct summand of \( M \) for each submodule \( N \) of \( M \).

**Proof.** Notice that any module \( N \) is \( \tau_I \)-torsion if and only if \( Ne = 0 \). Now the proof is clear. \( \Box \)

### 2. \( \tau - \oplus \)-Supplemented Modules

We will start with some simple characterizations of \( \tau - \oplus \)-supplemented modules.

**Lemma 2.1.** Let \( M \) be a \( \tau - \oplus \)-supplemented module. Then

(i) Any submodule \( N \) of \( M \) with \( \tau(N) = 0 \) is a direct summand.

(ii) \( M/\tau(M) \) is semisimple module.

**Proof.** (i) Let \( N \) be any submodule of \( M \) with \( \tau(N) = 0 \). There exists a direct summand \( K \) of \( M \) such that \( M = N + K = K' \oplus K \) and \( N \cap K \) is \( \tau \)-torsion. Since \( \tau(N) = 0 \), \( N \cap K = 0 \). Hence \( M = N \oplus K \).

(ii) Let \( T = \tau(M) \). If \( N/T \) is any submodule of \( M/T \), then there exists a direct summand \( K \) of \( M \) such that \( M = N + K \) with \( N \cap K \leq T \). Then \( M/T = (N/T) \oplus (K+T)/T \). Hence \( M/T \) is semisimple. \( \Box \)

**Corollary 2.2.** Let \( M \) be a \( \tau \)-torsionfree module. Then the following statements are equivalent.

1. \( M/\tau(M) \) is a direct summand of \( M \).
2. \( M/\tau(M) \) is semisimple.
(i) $M$ is $\tau - \oplus$-supplemented module.

(ii) $M$ is semisimple module.

**Proof.** Let $M$ be $\tau$-torsionfree module. Every submodule of $M$ is $\tau$-torsionfree. The proof is clear by Lemma 2.1. □

**Theorem 2.3.** Any finite direct sum of $\tau - \oplus$-supplemented modules is $\tau - \oplus$-supplemented.

**Proof.** Let $M_i$ be a $\tau - \oplus$-supplemented for each $1 \leq i \leq n$. Let $M = \bigoplus_{i=1}^{n} M_i$. To prove that $M$ is $\tau - \oplus$-supplemented, it is sufficient by induction on $n$ to prove this is the case when $n = 2$. Let $L$ be any submodule of $M$. By hypothesis there exists a direct summand $H$ of $M_2$ such that $M_2 = H \oplus H_1 = H + (M_2 \cap (M_1 + L))$ and $H \cap (M_1 + L)$ is $\tau$-torsion. Also there exists a direct summand $K$ of $M_1$ such that $M_1 = K \oplus K_1 = K + (M_1 \cap (L + H))$ and $K \cap (L + H)$ is $\tau$-torsion. Since $M = M_1 \oplus M_2 = K \oplus K_1 \oplus H \oplus H_1$, $K \oplus H$ is a direct summand of $M$. Now, $M = K + H + L$ is $\tau$-torsion and $H \cap (K + L) \subset H \cap (M_2 + H)$ is $\tau$-torsion, $(K + H) \cap L$ is $\tau$-torsion by $(K + H) \cap L \subset H \cap (K + L) + K \cap (H + L)$. Hence $M$ is $\tau - \oplus$-supplemented. □

**Proposition 2.4.** Let $M_i$ ($1 \leq i \leq n$) be any finite collection of relatively projective modules. Then the module $M = M_1 \oplus \ldots \oplus M_n$ is $\tau - \oplus$-supplemented if and only if $M_i$ is $\tau - \oplus$-supplemented for each $1 \leq i \leq n$.

**Proof.** $\Rightarrow$: We only prove $M_1$ to be $\tau - \oplus$-supplemented. Let $A \leq M_1$. There exists $B \leq M$ such that $M = A + B = B \oplus B_1$ and $A \cap B$ is $\tau$-torsion. Since $M = A + B = M_1 + B$, by [9, Lemma 4.47], there exists $B_2 \leq B$ such that $M = M_1 \oplus B_2$. Then $B = B_2 \oplus (M_1 \cap B)$. Note that $M_1 = A + (M_1 \cap B)$ and $M_1 \cap B$ is a direct summand of $M_1$. Therefore $A \cap B = A \cap (M_1 \cap B)$ is $\tau$-torsion. Hence $M_1$ is $\tau - \oplus$-supplemented. $\Leftarrow$: By Theorem 2.3. □

Let $M$ be $\tau - \oplus$-supplemented module with $\tau(M)$ small in $M$. Then $M$ is $\oplus$-supplemented. We don’t know if any infinite direct sum of $\tau - \oplus$-supplemented modules is $\tau - \oplus$-supplemented.

**Theorem 2.5.** Let $M$ be any $R$-module such that $M = \bigoplus_{i \in I} M_i$ where $M_i$ is a $\tau$-supplemented module for each $i \in I$. Suppose further that $\tau(M)$ is small in $M$. Then $M$ is $\tau - \oplus$-supplemented.

**Proof.** Let $N$ be a submodule of $M$. For each $i \in I$, let $T_i = \tau(M_i)$. If $T = \tau(M)$ then $T = \bigoplus_{i \in I} T_i$. For $i \in I$, $T_i = T \cap M_i$ and hence $M/T_i \cong (M_i + T)/T$ and so is semisimple by Lemma 2.1. Now $M/T = \sum_{i \in I} (M_i + T)/T$. Hence $M/T = ((N + T)/T) \oplus \bigoplus_{j \in \Lambda} (L_j + T)/T$ for some submodule $L_j$ of $M_j$ and an index set $\Lambda \subseteq I$. By [7, Lemma 2.1], for $j \in \Lambda$, there exists a direct summand $K_j$ of $M_j$ such that $K_j \leq L_j \leq K_j + T_j$. Let $K = \bigoplus_{j \in \Lambda} K_j$. Then $K$ is a direct summand of $M$. Note that $M = N + (\sum_{j \in \Lambda} L_j) + T \subseteq N + K + T$ so that $M = N + K + T$ and hence $M = N + K$ since $\tau(M) = T$ is small in $M$. Next, $N \cap K \subseteq (N + T) \cap (\sum_{j \in \Lambda} L_j + T) \subseteq T$. It follows that $N \cap K$ is $\tau$-torsion. Hence $M$ is $\tau - \oplus$-supplemented. □

A torsion theory $\tau$ is called stable if the class of $\tau$-torsion right $R$-modules is closed under essential extensions, equivalently, it is closed under injective hulls. We
recall that the singular submodule \( Z(M) \) of a module \( M \) is defined by \( Z(M) = \{ m \in M : \exists mI = 0 \text{ for some essential right ideal } I \text{ of } R \} \). If \( Z(M) = M \) (\( Z(M) = 0 \)) then \( M \) is called singular (non-singular) module. Let \( \tau_G \) denote the Goldie torsion theory. The \( \tau_G \)-torsion submodule \( \tau_G(M) \) of the module \( M \) is the second singular submodule \( Z_2(M) \) of \( M \) is a submodule \( M \) containing \( Z(M) \) such that \( Z_2(M)/Z(M) \) is the singular submodule of \( M/Z(M) \):

\[
Z_2(M)/Z(M) = Z(M/Z(M)).
\]

The Goldie torsion theory is stable [12, page 153, Proposition 7.3].

**Proposition 2.6.** Let \( M \) be a \( \tau - \oplus \)-supplemented module. Then

(i) \( M = K \oplus L \), where \( \tau(K) = 0 \) and \( \tau(L) \) is essential in \( L \).

(ii) If \( \tau \) is stable, then \( M = K \oplus \tau(M) \).

**Proof.** (i) Let \( M \) be a \( \tau - \oplus \)-supplemented module and \( K \) be a maximal submodule of \( M \) such that \( \tau(M) \cap K = 0 \). There exists a direct summand \( L \) of \( M \) such that \( M = K + L \) and \( K \cap L \) is \( \tau \)-torsion. Since \( \tau(M) \cap K = 0 \), \( K \cap L = 0 \). Hence \( M = K \oplus L \). Since \( \tau(M) \oplus K \) is an essential submodule of \( M \) and \( \tau(M) = \tau(K) \oplus \tau(L) = \tau(L) \), \( \tau(L) \) is an essential submodule of \( L \).

(ii) Assume that \( \tau \) is stable. Then \( \tau(M) = \tau(L) = L \) in the decomposition of \( M \) in (i).

**Corollary 2.7.** A module \( M \) is \( \tau_G - \oplus \)-supplemented if and only if \( M = Z_2(M) \oplus K \) where \( K \) is semisimple nonsingular submodule of \( M \) and \( Z_2(M) \) and \( K \) are \( \tau_G - \oplus \)-supplemented.

**Proof.** Necessity is clear by Theorem 2.6. Sufficiency is by Theorem 2.3.

The module \( M \) will be called completely \( \tau - \oplus \)-supplemented module if any direct summand of \( M \) is \( \tau - \oplus \)-supplemented. Recall that;

\((D_3)\) If \( A \) and \( B \) are direct summands of \( M \) with \( M = A + B \) then \( A \cap B \) is a direct summand of \( M \).

**Proposition 2.8.** Let \( M \) be a \( \tau - \oplus \)-supplemented module with \((D_3)\). Then \( M \) is completely \( \tau - \oplus \)-supplemented module.

**Proof.** Let \( N \) be a direct summand of \( M \) and \( K \) be any submodule of \( N \). By hypothesis, there exists a direct summand \( L \) of \( M \) such that \( M = K + L \) and \( K \cap L \) is \( \tau \)-torsion. Then \( N = K + (N \cap L) \) and \( K \cap (N \cap L) = K \cap L \) is \( \tau \)-torsion. Since \( M \) has \((D_3)\), \( N \cap L \) is direct summand of \( M \) and \( N \). It follows that \( N \) is \( \tau - \oplus \)-supplemented module.

**Theorem 2.9.** For a module \( M \) with \((D3)\), the following statements are equivalent.

(i) \( M \) is \( \tau - \oplus \)-supplemented.

(ii) \( M \) is completely \( \tau - \oplus \)-supplemented,

(iii) \( M = M_1 \oplus M_2 \), where \( M_1 \) is a semisimple module and \( M_2 \) is a \( \tau - \oplus \)-supplemented module with \( \tau(M_2) \) essential in \( M_2 \).
Proof. (i) ⇒ (ii) It follows from Proposition 2.8.  
(ii) ⇒ (i) Clear from definition. (ii) ⇒ (iii) By Proposition 2.6. (iii) ⇒ (i) By Theorem 2.3.

A module $M$ is said to have the finite exchange property if for any finite index set $I$, whenever $M \oplus N = \oplus_{i \in I} K_i$ for modules $N$ and $K_i$, then $M \oplus N = M \oplus (\oplus_{i \in I} L_i)$ for submodules $L_i \subseteq K_i$.

**Lemma 2.10.** Let $M_i$ be a module such that $M_i$ is completely $\tau - \oplus$-supplemented and has the finite exchange property for each $i = 1, \ldots, n$. Then $\oplus_{i=1}^n M_i$ is completely $\tau - \oplus$-supplemented.

**Proof.** By Theorem 2.3, $\oplus_{i=1}^n M_i$ is $\tau - \oplus$-supplemented. Let $N_1$ be a direct summand of $\oplus_{i=1}^n M_i$ and suppose that $\oplus_{i=1}^n M_i = N_1 \oplus N_2$ for some submodule $N_2$ of $M$. By [9, Lemma 3.20], $\oplus_{i=1}^n M_i$ and $N_2$ have the finite exchange property. So, $N_1 \oplus N_2 = \oplus_{i=1}^n M_i = N_2 \oplus (\oplus_{i=1}^n K_i)$, where $K_i$ is a direct summand of $M_i$. By hypothesis, every $K_i$ is $\tau - \oplus$-supplemented, and so $\oplus_{i=1}^n K_i$ is $\tau - \oplus$-supplemented by Theorem 2.3. Therefore $\oplus_{i=1}^n K_i \cong N_1$ is $\tau - \oplus$-supplemented. Hence every direct summand of $\oplus_{i=1}^n M_i$ is $\tau - \oplus$-supplemented. □

In Section 4, we shall show that there are some $\tau$-supplemented modules but not $\tau - \oplus$-supplemented modules.

**Theorem 2.11.** If a module $M$ is $\tau - \oplus$-supplemented such that $\tau(M)$ is small in $M$ and whenever $M = M_1 \oplus M_2$ then $M_1$ and $M_2$ are relatively projective, we have:

(i) $M$ is $\tau$-supplemented and,

(ii) $M$ has $(D_3)$.

**Proof.** (i) Let $N$ be any submodule of $M$. Since $M$ is $\tau - \oplus$-supplemented, there exists a decomposition $M = N + K_1 = K_1 \oplus K_2$ such that $N \cap K_1$ is $\tau$-torsion. By hypothesis, $K_1$ is $K_2$-projective. It follows from [9, Lemma 4.47] that $M = N_1 \oplus K_1$ for some submodule $N_1$ of $M$ such that $N_1 \subseteq N$. Hence $M$ is $\tau$-supplemented by [7, Lemma 2.1].

(ii) Let $M_1$ and $M_2$ be direct summands of $M$ and $M = M_1 + M_2$. Then $M = M_1 \oplus K_1 = M_2 \oplus K_2 = M_1 + M_2$ for some $K_1 \subseteq M_2$ and $K_2 \leq M_1$ by [9, Lemma 4.47]. Note that $M_2 = K_1 \oplus (M_2 \cap M_1)$. Hence $M_2 \cap M_1$ is direct summand of $M_2$ and so of $M$. Therefore $M$ has $(D_3)$.

Let $M$ be a module. $M$ has summand sum property (SSP) if the sum of any two direct summands of $M$ is direct summand of $M$.

**Theorem 2.12.** Let $M$ be any $R$-module.

(i) Assume that $M$ is $\tau - \oplus$-supplemented and $N$ is a submodule of $M$. If for every direct summand $K$ of $M$, $(N + K)/N$ is a direct summand of $M/N$, then $M/N$ is $\tau - \oplus$-supplemented. In particular, if $M$ has the summand sum property then $M$ is completely $\tau - \oplus$-supplemented.

(ii) Assume that $M$ is $(D_3)$. Let $K$ and $L$ be direct summands of $M$ such that $M/(K \cap L)$ is $\tau$-torsionfree. If $M/L$ is $\tau - \oplus$-supplemented, then $(K + L)/L$ is a direct summand of $M/L$. 
(iii) Assume that $M$ is $\tau$-torsionfree with $(D_3)$. If $M$ is completely $\tau$-$\oplus$-supplemented then it has the summand sum property.

**Proof.** (i) Let $T/N \leq M/N$, where $N \leq T \leq M$. Since $M$ is $\tau$-$\oplus$-supplemented, there exists a direct summand $M_1$ of $M$ such that $M = T + M_1 = M_1 \oplus M_2$ and $T \cap M_1$ is $\tau$-torsion. $M/N = (T/N) + ((M_1 + N)/N)$. By hypothesis, $(M_1 + N)/N$ is a direct summand of $M/N$. Since $(T/N) \cap ((M_1 + N)/N) = (T \cap (M_1 + N))/N = (N + (T \cap M_1))/N$ and $T \cap M_1$ is $\tau$-torsion, $(N + (T \cap M_1))/N$ is $\tau$-torsion. Hence $M/N$ is $\tau$-$\oplus$-supplemented. Assume that $M$ has SSP. Let $N$ be a direct summand of $M$ and $M = N \oplus N_1$. We want to show that $M/N_1$ is $\tau$-$\oplus$-supplemented. Let $K$ be a direct sum of $M$. By SSP, $M = (K + N_1)/L$ for some submodule $L$ of $M$. Hence $M/N_1 = ((K + N_1)/L)/N_1 \oplus ((L + N_1)/N_1)$. Therefore $M/N_1$ is $\tau$-$\oplus$-supplemented and so $N$ is $\tau$-$\oplus$-supplemented. (ii) Since $M/L$ is $\tau$-$\oplus$-supplemented, there exists a decomposition $M/L = ((K + L)/L) + (L_1/L) = (L_1/L) \oplus (L_2/L)$ with $((K + L)/L) \cap (L_1/L) = ((K \cap L_1) + L)/L \tau$-torsion. Clearly $L = L_1 \cap L_2$ and $M = L_1 + L_2 = K + L_1$. We consider the monomorphism $((K \cap L_1) + L_2)/L_1 \to ((K \cap L_1) + L)/L$ defined by $x + L_2 \to x + L$ where $x \in K \cap L_1$, which implies that $((K \cap L_1) + L_2)/L_2$ is $\tau$-torsion, and then $((K \cap L_1) + L_2)/L_2 \cong (K \cap L_1)/(K \cap L)$ is $\tau$-torsion. Since $K \cap L_1 \leq M/(K \cap L)$, $(K \cap L_1)/(K \cap L)$ is $\tau$-torsionfree. Hence $K \cap L_1 = K \cap L$. Since $M$ has $(D_3), K \cap L$ is a direct summand of $M$ and so $K \cap L_1$ is a direct summand of $M$. By modularity, $(K + L) \cap L_1 = (K \cap L_1) + L = (K \cap L) + L = L$. Hence $(K + L)/L$ is a direct summand of $M/L$. (iii) Let $K$ and $L$ be direct summands of $M$. It is clear that $M/(K \cap L)$ is $\tau$-torsionfree. Since $M$ is completely $\tau$-$\oplus$-supplemented, $M/L$ is $\tau$-$\oplus$-supplemented. Then $(K + L)/L$ is a direct summand of $M/L$ by (ii). Assume that $M/L = ((K + L)/L) \oplus (T/L)$ for some submodule $T$ of $M$ with $L \subseteq T$. Now $M = K + T$. Since $L$ is a direct summand of $M$, $M = L \oplus L_1$ for some submodule $L_1$ of $M$. By modularity, $T = L \oplus (T \cap L_1)$ and so $M = (K + L) + (T \cap L_1)$. Since $(K + L) \cap (T \cap L_1) = ((K + L) \cap T) \cap L_1 = L \cap L_1 = 0$, $K + L$ is a direct summand of $M$. Thus $M$ has the SSP. 

The module $M$ will be called $\tau$-local if the $\tau$-torsion submodule $\tau(M)$ is the unique maximal submodule.

**Lemma 2.13.** Let $M$ be any indecomposable module. Then $M$ is $\tau$-$\oplus$-supplemented module if and only if $M$ is $\tau$-torsion or $\tau$-local.

**Proof.** Sufficiency is clear. Conversely suppose that $M$ is $\tau$-$\oplus$-supplemented. Assume that $M \neq \tau(M)$. Let $x \in M \setminus \tau(M)$. There exist submodules $A$ and $B$ of $M$ such that $M = (xR) + B = A \oplus B$ and $(xR) \cap B$ is $\tau$-torsion. Since $M$ is indecomposable, $A = 0$ or $B = 0$. If $A = 0$, then $xR$ is $\tau$-torsion. This is not the case. Hence $B = 0$ and so $M = xR$ and $M$ is $\tau$-local.

**Theorem 2.14.** The following are equivalent for a projective module $M$.

(i) $M$ is a direct sum of $\tau$-$\oplus$-supplemented modules and $\tau(M)$ has finite Goldie dimension.

(ii) $M = M_1 \oplus M_2 \oplus M_3$ for some semisimple module $M_1$, module $M_2$ has finite Goldie dimension and it is a (finite) direct sum of $\tau$-local submodules and $M_3$ is $\tau$-torsion module and it has finite Goldie dimensions.
Proof. (i) ⇒ (ii) Assume $M = \oplus_{i \in I} M_i$, $M_i$ is $\tau - \oplus$-supplemented and $\tau(M)$ has finite Goldie dimension. Let $T_i = \tau(M_i)$ and $T = \tau(M)$. Since $T = \oplus_{i \in I} T_i$, there is a finite subset $J$ of $I$ such that $T_i = 0$ for all $i \in I - J$. Therefore $M_i$ is semisimple for all $i \in I - J$ by Corollary 2.2. Hence there is a semisimple submodule $M_1$ of $M$ such that $M = M_1 \oplus (\oplus_{j \in J} M_j)$ with $T_j \neq 0$. By Proposition 2.6, we may assume $T_j$ is essential in $M_j$. Then $M_j$ has finite Goldie dimension by [3, Proposition 3.20]. Let $N = M_j$ for any $j \in J$ with $\tau(N) \neq N$. Assume that $N$ is indecomposable. By Lemma 2.13, $N$ is $\tau$-local. Let $n > 1$ be a positive integer and assume each $M_j$ having Goldie dimension $k$, $1 \leq k < n$, is $\tau$-local or finite direct sum of $\tau$-local submodules. Let $j \in J$ and $N = M_j$ and $N$ has Goldie dimension $n$. Assume that $N$ is not $\tau$-local. Let $x \in N \setminus \tau(N)$ such that $N \neq xR$. Since $N$ is $\tau - \oplus$-supplemented, there exists a submodule $K$ of $N$ such that $N = xR + K = K \oplus K_1$ and $xR \cap K$ is $\tau$-torsion. Then $K_1 \neq 0$. Also $K \neq 0$. Since every projective module satisfy $(D_3)$, $K$ and $K_1$ are $\tau - \oplus$-supplemented by Proposition 2.8. By induction on Goldie dimension, $K$ and $K_1$ are $\tau$-local or finite direct sum of $\tau$-local submodules. 

(ii) ⇒ (i) By Theorem 2.3.\hfill \qed

We recall that a module $M$ is $\tau$-projective if and only if it is projective with respect to every $R$-epimorphism having a $\tau$-torsion kernel [2].

Lemma 2.15. Let $M$ be a module and $L$ a direct summand of $M$ and $K$ a submodule of $M$ such that $M/K$ is $\tau$-projective and $M = L + K$ and $L \cap K$ is $\tau$-torsion. Then $L \cap K$ is direct summand of $M$.

Proof. Let $M = L \oplus L'$ and $\alpha : M/L' \rightarrow L$ be the isomorphism and $\beta : L \rightarrow M/K \cong L/(L \cap K)$ the epimorphism that having $L \cap K$ as kernel. Then we have epimorphism $\beta_0 : M/L' \rightarrow M/K$ having kernel $((L \cap K) \oplus L')/L' \cong L \cap K$ which is $\tau$-torsion. Since $M/K$ is $\tau$-projective, there exists $g : M/K \rightarrow M/L'$ such that $1 = \beta_0 g$. Hence $L \cap K$ is direct summand.\hfill \qed

Proposition 2.16. Let $M$ be a $\tau - \oplus$-supplemented module and $L$ a submodule of $M$ such that $M/L$ is $\tau$-projective. Then $L$ is $\tau - \oplus$-supplemented.

Proof. Let $L'$ be submodule of $L$. Since $M$ is $\tau - \oplus$-supplemented, there exists a direct summand $K$ of $M$ such that $M = L' + K$ and $L' \cap K$ is $\tau$-torsion. Then $L = L' + L \cap K$. By Lemma 2.15, $L \cap K$ is a direct summand of $M$ and $L' \cap (L \cap K) = L' \cap K$ is $\tau$-torsion. Hence $L$ is $\tau - \oplus$-supplemented.\hfill \qed

In [7], we called module $M$ is almost torsion if every proper submodule of $M$ is $\tau$-torsion, $\tau$-torsion modules and $\tau$-local modules are almost torsion and almost torsion modules are $\tau$-supplemented and $\tau - \oplus$-supplemented.

Proposition 2.17. Let $M$ be any uniform module. Then $M$ is $\tau - \oplus$-supplemented module if and only if $M$ is almost $\tau$-torsion.

Proof. ⇒ Let $N$ be a proper submodule of $M$. There exists direct summand $K$ of $M$ such that $M = N + K = K \oplus K_1$ and $N \cap K$ is $\tau$-torsion. By hypothesis $K \neq 0$ and $K_1 = 0$. Then $M = K$ and $N \cap K = N$ is $\tau$-torsion.

⇐: It is clear that every torsion module is $\tau - \oplus$-supplemented. Let $x \in M \setminus \tau(M)$. Then $M = xR$. Hence $M$ is $\tau$-local and so $\tau - \oplus$-supplemented.\hfill \qed
Lemma 2.18. Let $M$ be an indecomposable module. Then $M$ is almost $\tau$-torsion if and only if $M$ is completely $\tau-\oplus$-supplemented.

Proof. Follows from Lemma 2.13 and Theorem 2.3. □

Proposition 2.19. Let $M = U \oplus V$ such that $U$ and $V$ have local endomorphism rings. Then $M$ is a completely $\tau-\oplus$-supplemented module if and only if $U$ and $V$ are almost $\tau$-torsion.

Proof. Necessity follows from Lemma 2.18. Conversely, let $K$ be a direct summand of $M$. If $K = M$ then $K$ is $\tau-\oplus$-supplemented by Lemma 2.18 and Theorem 2.3. Assume $K \neq M$. Then either $K \cong U$ or $K \cong V$ by Krull-Schmidt-Azumaya Theorem [1, Corollary 12.7]. Hence $K$ is $\tau-\oplus$-supplemented. □

Theorem 2.20. Let $M$ be a non-zero module with finite Goldie dimension. Then the following statements are equivalent.

(i) Every direct summand of $M$ is a finite direct sum of almost $\tau$-torsion modules and a semisimple module.

(ii) $M$ is completely $\tau-\oplus$-supplemented module.

Proof. (i) $\Rightarrow$ (ii) Let $N$ be a direct summand of $M$. By (i), $N = (\oplus_i N_i) \oplus K$, where $N_i$ is almost $\tau$-torsion for all $i \in I$, $I$ finite and $K$ is semisimple. Hence $N$ is $\tau-\oplus$-supplemented by Lemma 2.18 and Theorem 2.3. (ii) $\Rightarrow$ (i) Every torsion free submodule of $M$ is semisimple module and direct summand of $M$ by Lemma 2.1. Let $K$ denote the largest $\tau$-torsion free submodule of $M$. Then $M = K \oplus N$ for some $N \leq M$. By hypothesis $N$ has finite Goldie dimension, say $n$. We complete the proof by induction on $n$. If $n = 1$, then $N$ is uniform. By Proposition 2.17 $N$ is almost $\tau$-torsion. Assume $n > 1$. Let $U$ be a uniform submodule of $N$. There exists direct summand $V$ of $N$ such that $N = U + V = V' \oplus V$ with $U \cap V$ $\tau$-torsion. We may assume that $V'$ and $V$ are proper submodules. Then $V'$ and $V$ have Goldie dimensions less than $n$. By induction hypothesis they are finite direct sums of almost $\tau$-torsion modules. □

Corollary 2.21. Let $M$ be a linearly compact module. Then the following statements are equivalent.

(i) Every direct summand of $M$ is a finite direct sum of almost $\tau$-torsion modules and a semisimple module.

(ii) $M$ is completely $\tau-\oplus$-supplemented module.

Proof. By [13, Proposition 3.4], $M$ has finite Goldie dimension. Therefore, proof is clear from Theorem 2.20. □

3. $\tau$-Perfect Rings

Let $\tau$ be a torsion theory. An epimorphism $f : P \to M$ is called a $\tau$-projective cover of $M$ if $P$ is $\tau$-projective and Ker($f$) is small $\tau$-torsion submodule of $P$ (see [2, page 117]). We call the ring $R$ right $\tau$-perfect if every right $R$-module has a $\tau$-projective cover (compare with Remark 4.5 in [16]). We call the module $M$ is strongly $\tau-\oplus$-supplemented if for any submodule $N$ of $M$ there exists a direct summand $K$ of $M$ with $M = N + K$ and $N \cap K$ is small $\tau$-torsion in $K$. Every
right \(\tau\)-perfect ring is right perfect, and any strongly \(\tau - \oplus\)-supplemented module is \(\tau - \oplus\)-supplemented. In [14] for any ideal \(I\), the ring \(R\) is said to be right \(I\)-perfect if for any submodule \(X\) of a projective module \(P\), \(X = A \oplus B\) where \(A\) is a direct summand of \(P\) and \(B \leq PI\).

**Lemma 3.1.** Let \(M = A + B\). If \(M/A\) has a \(\tau\)-projective cover then \(B\) contains a submodule \(C\) of \(M\) such that \(M = A + C\), \(A \cap C\) small \(\tau\)-torsion in \(C\).

**Proof.** Let \(f : P \to M/A\) be a \(\tau\)-projective cover of \(M\) with \(\text{Ker}(f)\) small \(\tau\)-torsion in \(P\), and \(\pi : B \to M/A\) the canonical epimorphism. Since \(P\) is projective, there exists \(g : P \to B\) such that \(\pi g = f\). Then \(B = g(P) + (A \cap B)\) and \(\text{Ker}(\pi) = A \cap B\). \(g(P) \cap A\) is small \(\tau\)-torsion in \(B\) since \(g(P) \cap A \leq g(\text{Ker}(f))\) \(\Box\).

**Lemma 3.2.** Let \(M\) be a quasi-projective \(R\)-module. If \(M = A + B\) and \(A \cap B\) is small in \(A\) and \(B\), then \(M = A \oplus B\).

**Proof.** Consider the epimorphism \(f : A \oplus B \to M\) with \(\text{Ker}(f) = A \cap B\) which is small by assumption. But \(f\) splits by quasi-projective. Therefore \(A \cap B = 0\). \(\Box\)

**Lemma 3.3.** Let \(N\) be any submodule of the module \(M\). Assume that \(M/N\) has a \(\tau\)-projective cover. Then there exists a submodule \(L\) of \(M\) such that \(M = N + L\) with \(N \cap L\) small and \(\tau\)-torsion in \(L\).

**Proof.** Let \(f : P \to M/N\) be a \(\tau\)-projective cover and \(\pi : M \to M/N\) the canonical epimorphism with kernel \(A\). Since \(P\) is projective, there exists \(g : P \to M\) such that \(\pi g = f\). Then \(M = g(P) + N\) and \(g(P) \cap N = g(\text{Ker}(f))\). Since \(\text{Ker}(f)\) is small \(\tau\)-torsion in \(P\), \(g(\text{Ker}(f))\) is small and \(\tau\)-torsion in \(g(P)\). \(\Box\)

**Theorem 3.4.** Let \(P\) be a projective \(R\)-module. Then the following are equivalent:

(i) \(P\) is \(\tau\)-supplemented.

(ii) \(P\) is \(\tau - \oplus\)-supplemented.

**Proof.** (i) \(\Rightarrow\) (ii) Clear from definitions. (ii) \(\Rightarrow\) (i) Let \(N\) be submodule of \(P\). By (ii) there exists a direct summand \(K\) of \(P\) such that \(P = N + K = K' \oplus K\) and \(N \cap K\) is \(\tau\)-torsion. By [Lemma 4.47] there exists a direct summand \(L\) of \(P\) such that \(P = L \oplus K\) and \(L \leq N\). Since \(N/L\) is isomorphic to \(N \cap K\), \(N/L\) is \(\tau\)-torsion. (i) follows. \(\Box\)

**Lemma 3.5.** Let \(P\) be a projective module. Then \(\tau(P) = P \cdot \tau(R)\).

**Proof.** For any module \(P\), \(P \cdot \tau(R) \leq \tau(P)\) always holds. Assume that \(P\) is any projective right \(R\)-module and \((x_i, f_i)_{i \in I}\) a dual basis for \(P\). Let \(x \in \tau(P)\). Then \(f_i(x) \in \tau(R)\) for each \(i \in I\). Hence \(x = \sum_{i=1}^{n} x_i f_i(x) \in P.\tau(R)\). Thus \(\tau(P) \leq P \cdot \tau(R)\). It follows that \(\tau(P) = P \cdot \tau(R)\). \(\Box\)

**Theorem 3.6.** Let \(R\) be a ring. Then the following are equivalent.

(i) \(R\) is right \(\tau\)-perfect.

(ii) Every projective right \(R\)-module is \(\tau\)-supplemented.

(iii) Every projective right \(R\)-module is \(\tau - \oplus\)-supplemented.
Proof. (i) ⇒ (ii) By definitions. (ii) ⇔ (iii) By Theorem 3.4. (ii) ⇒ (i) Let $N$ be any submodule of a projective module $P$. By [Lemma 2.1] $N$ has a decomposition $N = K \oplus L$ where $K$ is a direct summand of $P$ and $L \leq \tau(P)$. By Lemma 3.5, $L \leq P.\tau(R)$. Hence $R$ is $\tau(R)$-perfect. □

Theorem 3.7. Let $R$ be a ring. Then the following are equivalent.

(i) $R$ is right $\tau$-perfect ring.

(ii) Every projective $R$-module is strongly $\tau$-⊕-supplemented.

Proof. (i) ⇒ (ii) Let $N$ be submodule of the projective module $M$. By (i) $M/N$ has $\tau$-projective cover. By Lemma 3.3 there exists a submodule $L$ of $M$ such that $M = N + L$ with $N \cap L$ is small and $\tau$-torsion in $L$. By Lemma 3.1 $N$ contains a submodule $K$ such that $M = K + L$ with $K \cap L$ is small and $\tau$-torsion in $K$. By Lemma 3.2, $K \cap L = 0$. Hence $M = N + L = K \oplus L$ and $N \cap L$ is small and $\tau$-torsion in $L$. It follows that $M$ is strongly $\tau$-⊕-supplemented.

(ii) ⇒ (i) Let $M$ be any $R$-module, $P$ a projective module and $f$ an epimorphism $f : P \rightarrow M$. By (ii) $P$ has direct summands $K$ and $K'$ so that $P = \ker(f) + K = K' \oplus K$ with $\ker(f) \cap K$ small and $\tau$-torsion in $K$. Hence $K$ is the required $\tau$-projective cover of $M$. □

4. Examples

There are torsion theories $\tau$ and modules $M$ relative to which $M$ is $\tau$-⊕-supplemented but not $\tau$-supplemented:

(1) Let $F$ be a field and $R$ the subring

$$R = \begin{pmatrix} F & 0 & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$$

of all 3 by 3 matrices over $F$ and

$$X = \begin{pmatrix} F & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\tau_X$ the torsion theory with torsion class $\tau = \{ N \in \text{Mod} - R \mid NX = 0 \}$ associated to the idempotent ideal $X$. Let $M'$ denote the right module

$$M' = \begin{pmatrix} 0 & 0 & F \\ 0 & F & F \\ F & F & F \end{pmatrix}$$

and $M''$ the right $R$-module $R$. We consider the right $R$-module $M = M' \oplus M''$. Then $\tau_X$ is a hereditary torsion theory. We prove that $\tau_X$ is not stable. Let

$$N_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & F & F \end{pmatrix}$$

and

$$N_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & F & F \end{pmatrix}$$
be submodules of \( M' \). Then \( N_2 \) is \( \tau_X \)-torsion and \( N_2 \) is an essential submodule of \( N_1 \). Since \( N_1X \neq 0 \), \( N_1 \) is not \( \tau_X \)-torsion. Hence \( \tau_X \) is not stable. \( M' \) and \( M'' \) are \( \tau_X \)-supplemented modules as direct sum of \( \tau_X \)-torsion and almost \( \tau_X \)-torsion modules. By Theorem 2.3 \( M \) is \( \tau_X \)-\( \oplus \)-supplemented. We prove that \( M \) is \( \tau \)-supplemented. Let \( e_{ij} \) denote the matrix units with \((i, j)\) entry is 1 and all other entries 0. Let \( N' = e_{31}R \leq M' \) and \( N'' = e_{13}R \leq M'' \) and \( N = N' \oplus N'' \). \( N' \) is uniform and essential submodule of a direct summand of \( M' \) and not \( \tau_X \)-torsion. \( N'' \) is simple and essential submodule of a direct summand of \( M'' \) and \( \tau_X \)-torsion. Hence \( N' \) and \( N'' \) are not a direct summand of \( M \). We show that \( N \) is not direct summand of \( M \). Assume that \( M = N \oplus K \) for some \( K \leq M \). Let \( m = (e_{32}, 0) \in M \) where \( e_{32} \in M' \) and 0 is the zero matrix of \( M'' \). Then \( m = n + k \) where \( n \in N \) and \( k \in K \). There exist \( a, b, c \in F \) such that \( k = (ae_{31} + e_{32} + be_{13}, ce_{13}) \). Since \( ke_{11} \in K \cap N \) and \( ke_{33} \in K \cap N \), \( a = b = c = 0 \). Hence \( m = k \in K \) and so \( mR \leq K \). This is a contradiction since \( mR \cap N \neq 0 \).

There are \( \tau \)-\( \oplus \)-supplemented modules in which torsion submodules are not direct summands.

(2) Let \( F \) be any field and let \( S \) be any \( F \)-algebra. Let \( R \) denote the subring of the ring of all 2 by 2 matrices over \( S \) consisting of all 2 by 2 matrices with second column having entries from \( S \), with (1,1) entry from \( F \) and with (2,1) entry 0. Let \( I \) denote the ideal generated as a right ideal by the idempotent

\[
e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then

\[
I = \begin{pmatrix} F & S \\ 0 & 0 \end{pmatrix}.
\]

Let \( \tau \) denote the torsion theory where a module \( M \) is \( \tau \)-torsion if \( MI = 0 \). Then

\[
\tau(R) = \begin{pmatrix} 0 & S \\ 0 & S \end{pmatrix}.
\]

It can be shown that any submodule of the right \( R \)-module \( R \) is \( \tau \)-torsion or contains \( e \). It follows that \( R \) is \( \tau \)-\( \oplus \)-supplemented \( R \)-module but \( \tau(R) \) is not a direct summand of \( R \).

There are modules \( M \), torsion theories \( \tau_X \) and \( \tau_Y \) such that \( M \) \( \tau_X \)-supplemented but not \( \tau_Y \)-supplemented.

(3) Let \( F \) be any field and

\[
R = \begin{pmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}
\]

the subring of all 3 by 3 matrices over \( F \),

\[
X = \begin{pmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
and $\tau_X$ the torsion theory with torsion class $\{N \in \text{Mod} - R \mid NX = 0\}$. Then

$$\tau_X(R) = \begin{pmatrix} 0 & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}$$

is the unique maximal submodule and so $M$ is $\tau_X - \oplus$-supplemented. Now let

$$Y = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in F \right\}$$

and $\tau_Y$ with the torsion class $\{N \in \text{Mod} - R \mid NY = 0\}$. Then

$$\tau_Y(M) = \begin{pmatrix} 0 & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Let

$$K = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Then $KY \neq 0$ and there is no direct summand $L$ of $M$ such that $M = K + L$ and $K \cap L$ is $\tau_Y$-torsion. Hence $M$ is not $\tau_Y - \oplus$-supplemented.

(4) Let $F$ be any field and

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & f \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d, f \in F \right\}$$

the subring of all 3 by 3 matrices over $F$ and

$$I = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & d & f \\ 0 & 0 & 0 \end{pmatrix} \mid b, c, d, f \in F \right\}.$$ 

Let $\tau$ be the torsion theory with torsion class $\{N \in \text{Mod} - R \mid \text{Hom}(N, R/I) = 0\}$. Let $M$ denote the right $R$-module $R$. Then $\tau(M) = 0$. Since $M$ is not semisimple, by Corollary 2.2 $M$ is not $\tau - \oplus$-supplemented.

(5) Let $F$ be a field and

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$

the ring with matrix operations and

$$I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$$

and $\tau_I = \{N \in \text{Mod} - R \mid NI = 0\}$. Let $M$ denote the right $R$-module $R$. By Proposition 2.19 $M$ is completely $\tau - \oplus$-supplemented module.
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