

ON RIEMANN'S THEOREM ABOUT CONDITIONALLY CONVERGENT SERIES

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Abstract. We extend Riemann's rearrangement theorem on conditionally convergent series of real numbers to multiple instead of simple sums.

1. Introduction and Statement of Results

By a well-known theorem due to B. Riemann, each conditionally convergent series of real numbers can be rearranged in such a way that the new series converges to some arbitrarily given real value or to ∞ or $-\infty$ (see, for example, [1, § 32]). As to series of vectors in \mathbb{R}^n , in 1905 P. Lévy [2] and in 1913 E. Steinitz [5] showed the following interesting extension (see also [3] for a simplified proof).

Theorem A. (Lévy-Steinitz Theorem) *The set of all sums of rearrangements of a given series of vectors in \mathbb{R}^n is either the empty set or a translate of a linear subspace (i.e., a set of the form $v + M$ where v is a given vector and M is a linear subspace).*

Here, of course, M is the zero space if and only if the series is absolutely convergent. For a further generalization of the Lévy-Steinitz theorem to spaces of infinite dimension, see [4].

In this paper, we extend Riemann's result in a different direction, turning from simple to multiple sums which provides many more possibilities of rearranging a given sum. First of all, we have to introduce some notations.

By $\text{Sym}(n)$ we denote the symmetric group of the set $\{1, \dots, n\}$, i.e., the group of all permutations of $\{1, \dots, n\}$.

If $(a_m)_m$ is a sequence of elements of a non-empty set X , J is an infinite subset of \mathbb{N}^n and if $\tau : J \rightarrow \mathbb{N}$ is a bijection and

$$b(j_1, \dots, j_n) := a_{\tau(j_1, \dots, j_n)} \quad \text{for each } (j_1, \dots, j_n) \in J,$$

then we say that the mapping $b : J \rightarrow X$, $(j_1, \dots, j_n) \mapsto b(j_1, \dots, j_n)$ is a rearrangement of $(a_m)_m$. We write

$$\left(b(j_1, \dots, j_n) \mid (j_1, \dots, j_n) \in J \right)$$

for such a rearrangement (which is a more convenient notation for our purposes than the notation $(b_{j_1, \dots, j_n})_{(j_1, \dots, j_n) \in J}$ one would probably expect). Instead of

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$(b(j_1, \dots, j_n) \mid (j_1, \dots, j_n) \in \mathbb{N}^n)$, we also write $(b(j_1, \dots, j_n) \mid j_1, \dots, j_n \geq 1)$ and also use notations like $(b(j_1, \dots, j_n) \mid j_1 \geq k_1, \dots, j_n \geq k_n)$ which should be self-explanatory now.

With these notations, we can state our main result as follows.

Theorem 1. *Let $n \geq 2$ be a natural number and let $\sum_{m=1}^{\infty} a_m$ be a conditionally convergent series of real numbers a_m . For each $\sigma \in \text{Sym}(n)$, let $(s_k^{(\sigma)})_{k \geq 1}$ be a sequence of real numbers. Then there exists a rearrangement $(b(j_1, \dots, j_n) \mid j_1, \dots, j_n \geq 1)$ of $(a_m)_m$ such that for each $\sigma \in \text{Sym}(n)$ and each $k \geq 1$, one has*

$$\sum_{j_1=1}^k \sum_{j_2=1}^{\infty} \cdots \sum_{j_n=1}^{\infty} b(j_{\sigma(1)}, \dots, j_{\sigma(n)}) = s_k^{(\sigma)}.$$

Corollary 2. *Let $n \geq 1$ be a natural number and let $\sum_{m=1}^{\infty} a_m$ be a conditionally convergent series of real numbers a_m . For each $\sigma \in \text{Sym}(n)$, let $s^{(\sigma)}$ be a real number or $\pm\infty$. Then there exists a rearrangement $(b(j_1, \dots, j_n) \mid j_1, \dots, j_n \geq 1)$ of $(a_m)_m$ such that for each $\sigma \in \text{Sym}(n)$, one has*

$$\sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \cdots \sum_{j_n=1}^{\infty} b(j_{\sigma(1)}, \dots, j_{\sigma(n)}) = s^{(\sigma)}.$$

Proof. For $n = 1$, this is just Riemann's theorem. For $n \geq 2$, it is an immediate consequence of Theorem 1. \square

By moving to continuous functions on \mathbb{R}^n , we can construct an example of a continuous function in the “positive part” $Q := [0, \infty)^n$ of \mathbb{R}^n whose iterated integrals exist for each order of integration, but all of them have different values. This is a kind of “ultimate” counterexample to show that the assumptions in Fubini's theorem are inevitable.

Corollary 3. *Let $n \geq 2$ be a natural number. For each $\sigma \in \text{Sym}(n)$, let $s^{(\sigma)}$ be a real number or $\pm\infty$. Then there exists a function $f \in C^\infty(Q)$ such that*

$$\int_0^\infty \cdots \int_0^\infty f(x_1, \dots, x_n) dx_{\sigma(1)} dx_{\sigma(2)} \cdots dx_{\sigma(n)} = s^{(\sigma)} \quad \text{for each } \sigma \in \text{Sym}(n). \quad (1)$$

Proof. Let $\sum_{m=1}^{\infty} a_m$ be some conditionally convergent series. By Corollary 2, there exists a rearrangement $(b(j_1, \dots, j_n) \mid j_1, \dots, j_n \geq 1)$ of $(a_m)_m$ such that for each $\sigma \in \text{Sym}(n)$, one has

$$\sum_{k_n=1}^{\infty} \cdots \sum_{k_1=1}^{\infty} b(k_{\sigma^{-1}(1)}, \dots, k_{\sigma^{-1}(n)}) = s^{(\sigma)}.$$

We set $I = [-0.49; 0.49]^n$ and define the function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\varphi(x) := \begin{cases} Ae^{-1/(0.49-\|x\|)^2} & \text{for } \|x\| < 0.49 \\ 0 & \text{for } \|x\| \geq 0.49, \end{cases}$$

where $A > 0$ and $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^n . Then $\varphi \in C^\infty(\mathbb{R}^n)$, and φ vanishes outside the compact set I . So φ is integrable with respect to the Lebesgue

measure λ , and by choosing an appropriate A we can obtain

$$\int_{\mathbb{R}^n} \varphi(x) d\lambda(x) = 1.$$

In particular, by Fubini's theorem the last integral can be written in any order of integration, i.e.

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \varphi(x_1, \dots, x_n) dx_{\sigma(1)} dx_{\sigma(2)} \dots dx_{\sigma(n)} = 1 \quad \text{for each } \sigma \in \text{Sym}(n).$$

Since φ vanishes outside I , for any $j_1, \dots, j_n \geq 1$, we also have

$$\int_{-j_n}^{\infty} \dots \int_{-j_1}^{\infty} \varphi(x_1, \dots, x_n) dx_{\sigma(1)} dx_{\sigma(2)} \dots dx_{\sigma(n)} = 1 \quad \text{for each } \sigma \in \text{Sym}(n). \quad (2)$$

Now we define $f : Q \rightarrow \mathbb{R}$ by

$$f(x_1, \dots, x_n) := \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \dots \sum_{j_n=1}^{\infty} b(j_1, \dots, j_n) \cdot \varphi(x_1 - j_1, \dots, x_n - j_n).$$

For each $x = (x_1, \dots, x_n) \in Q$, at most one of the terms $\varphi(x_1 - j_1, \dots, x_n - j_n)$ is non-zero, so the multiple sum in the definition of f reduces to just one term, and we conclude that $f \in C^\infty(Q)$. Let $\sigma \in \text{Sym}(n)$ be given. Then we obtain by (2)

$$\begin{aligned} & \int_0^{\infty} \dots \int_0^{\infty} f(x_1, \dots, x_n) dx_{\sigma(1)} dx_{\sigma(2)} \dots dx_{\sigma(n)} \\ &= \sum_{j_{\sigma(n)}=1}^{\infty} \dots \sum_{j_{\sigma(1)}=1}^{\infty} b(j_1, \dots, j_n) \int_0^{\infty} \dots \int_0^{\infty} \varphi(x_1 - j_1, \dots, x_n - j_n) dx_{\sigma(1)} \dots dx_{\sigma(n)} \\ &= \sum_{k_n=1}^{\infty} \dots \sum_{k_1=1}^{\infty} b(k_{\sigma^{-1}(1)}, \dots, k_{\sigma^{-1}(n)}) \cdot 1 \\ &= s^{(\sigma)}, \end{aligned}$$

hence (1). \square

Observe that in the case $n = 2$, by Corollary 3 we get the existence of a function $f \in C^\infty([0, \infty)^2)$ such that

$$\int_0^{\infty} \int_0^{\infty} f(x, y) dx dy = +\infty \quad \text{and} \quad \int_0^{\infty} \int_0^{\infty} f(x, y) dy dx = -\infty.$$

For the functions f from Corollary 3, in general, the improper integral

$$\int_Q f(x_1, \dots, x_n) d(x_1, \dots, x_n)$$

(in the sense of Riemann) does not exist in the extended sense¹. A necessary condition for the existence of this integral is that $s^{(\sigma)} = s^{(\tau)}$ for every $\sigma, \tau \in \text{Sym}(n)$. However, it can be shown that this condition is not sufficient for the convergence of the improper integral.

¹We say that the improper integral $\int_Q f(x_1, \dots, x_n) d(x_1, \dots, x_n)$ exists in the extended sense if for arbitrary exhaustions $(K_m)_m$ of Q with compact sets K_m , the limits $\lim_{m \rightarrow \infty} \int_{K_m} f(x_1, \dots, x_n) d(x_1, \dots, x_n)$ exist and are equal.

It is obvious that, by modifying the definition of f (such that its “peaks” are at the points $(\frac{1}{2^{j_1}}, \dots, \frac{1}{2^{j_n}})$ rather than at the points (j_1, \dots, j_n)), one can replace Q by $(0, 1]^n$ in Corollary 3, i.e., we can find a function $f \in C^\infty((0, 1]^n)$ whose iterated integrals exist for every order of integration, but each time give different values. Of course, this is not possible for the compact cube $[0, 1]^n$, since continuous functions on compact sets are Lebesgue-integrable, so by Fubini’s Theorem their integrals are independent of the order of integration.

2. Proofs

It is well known that a *convergent* series $\sum_{m=1}^{\infty} a_m$ of real numbers is conditionally convergent if and only if

$$\sum_{a_m > 0} a_m = \infty \quad \text{and} \quad \sum_{a_m < 0} a_m = -\infty. \quad (3)$$

This property is a bit more general than the property of conditional convergence: It may also hold for series which are not convergent themselves. It turns out that this is the property we actually deal with in the proof of our main result. This gives rise to the following definition.

Definition. We say that a series $\sum_{m=1}^{\infty} a_m$ of real numbers is **conditionally convergible** if $\lim_{m \rightarrow \infty} a_m = 0$ and if (3) holds.

As the proof of Riemann’s theorem shows, a series is conditionally convergible if and only if it has some rearrangement which is conditionally convergent.

The main advantage of this newly introduced notion is the following: Conditional convergibility is invariant under rearrangements while conditional convergence is not.

Lemma 4. Let $\sum_{m=1}^{\infty} a_m$ be a conditionally convergible series of real numbers a_m . Then there is a disjoint partition $\mathbb{N} = \bigcup_{t=1}^{\infty} I_t$ of \mathbb{N} into infinite subsets I_t such that for each $t \in \mathbb{N}$ the series $\sum_{m \in I_t} a_m$ is conditionally convergible².

Proof. I. Let $(\beta_m)_m$ be a sequence of non-negative numbers such that

$$\sum_{m=1}^{\infty} \beta_m = \infty.$$

Then it is evident that one can decompose \mathbb{N} into two infinite disjoint subsets $I_1, I^{(2)}$ such that $1 \in I_1$ and

$$\sum_{m \in I_1} \beta_m = \infty \quad \text{and} \quad \sum_{m \in I^{(2)}} \beta_m = \infty.$$

Let us assume that we have already found subsets $I_1, \dots, I_t, I^{(t+1)} \subseteq \mathbb{N}$ such that $\mathbb{N} = I_1 \cup \dots \cup I_t \cup I^{(t+1)}$ is a disjoint union,

$$\sum_{m \in I_s} \beta_m = \infty \quad (s = 1, \dots, t) \quad \text{and} \quad \sum_{m \in I^{(t+1)}} \beta_m = \infty$$

²In notations like $\sum_{j \in I_t} a_j$, the order of summation is of course understood to be in the natural order of increasing indices j . On the other hand, since conditional convergibility is invariant under rearrangements, we do not have to specify the order of summation at all, at least not for the purpose of Lemma 4.

and such that $\min(\mathbb{N} \setminus (I_1 \cup \dots \cup I_{s-1})) \in I_s$ for $s = 1, \dots, t$. Then we can find a disjoint decomposition $I^{(t+1)} = I_{t+1} \cup I^{(t+2)}$ such that

$$\sum_{m \in I_{t+1}} \beta_m = \infty \quad \text{and} \quad \sum_{m \in I^{(t+2)}} \beta_m = \infty$$

and such that $\min(\mathbb{N} \setminus (I_1 \cup \dots \cup I_t)) \in I_{t+1}$.

In this way, inductively we construct subsets $I_t \subseteq \mathbb{N}$ such that $\sum_{m \in I_t} \beta_m = \infty$ for all t . It is evident that $\bigcup_{t=1}^{\infty} I_t = \mathbb{N}$ and that this union is disjoint. (Observe that it is crucial to put the smallest element from $\mathbb{N} \setminus (I_1 \cup \dots \cup I_{t-1})$ into I_t in each step, in order to guarantee that each natural number appears in some I_t , i.e., that it is not forgotten “forever”.)

II. Let $\sum_{m=1}^{\infty} a_m$ be a conditionally convergible series of real numbers and let

$$P := \{m \in \mathbb{N} \mid a_m \geq 0\}, \quad N := \{m \in \mathbb{N} \mid a_m < 0\}.$$

Then we have

$$\sum_{m \in P} a_m = +\infty, \quad \sum_{m \in N} a_m = -\infty.$$

By I. there exist disjoint decompositions $P = \bigcup_{t=1}^{\infty} P_t$ and $N = \bigcup_{t=1}^{\infty} N_t$ of P and N into infinite subsets P_t, N_t such that

$$\sum_{m \in P_t} a_m = \infty \quad \text{and} \quad \sum_{m \in N_t} a_m = -\infty$$

for all t . If we set

$$I_t := P_t \cup N_t,$$

then for every t the series $\sum_{m \in I_t} a_m$ is conditionally convergible, and $\mathbb{N} = \bigcup_{t=1}^{\infty} I_t$ is a disjoint decomposition. This proves the assertion. \square

Since the proof of the general case of Theorem 1 is quite abstract, we start with a discussion of the case $n = 2$ to give the reader an idea of what is really going on.

Proof of the Case $n = 2$ of Theorem 1. Here, $\text{Sym}(2)$ consists of two elements $\sigma = (1 \ 2) = id_{\{1,2\}}$ and $\tau = (2 \ 1)$.

According to Lemma 4, there exists a disjoint partition $\mathbb{N} = \bigcup_{t=1}^{\infty} I_t$ of \mathbb{N} into infinite subsets I_t such that for each $t \in \mathbb{N}$ the series $\sum_{m \in I_t} a_m$ is conditionally convergible. By Riemann's theorem, we can find a rearrangement $(b(1, k) \mid k \in \mathbb{N})$ of $(a_m)_{m \in I_1}$ such that

$$\sum_{k=1}^{\infty} b(1, k) = s_1^{(\sigma)}.$$

In the same way, we can find a rearrangement $(b(j, 1) \mid j \geq 2)$ of $(a_m)_{m \in I_2}$ such that

$$\sum_{j=2}^{\infty} b(j, 1) = s_1^{(\tau)} - b(1, 1).$$

Next, we choose a rearrangement $(b(2, k) \mid k \geq 2)$ of $(a_m)_{m \in I_3}$ such that

$$\sum_{k=2}^{\infty} b(2, k) = s_2^{(\sigma)} - s_1^{(\sigma)} - b(2, 1)$$

and a rearrangement $(b(j, 2) \mid j \geq 3)$ of $(a_m)_{m \in I_4}$ such that

$$\sum_{j=3}^{\infty} b(j, 2) = s_2^{(\tau)} - s_1^{(\tau)} - b(1, 2) - b(2, 2),$$

and so on. Proceeding in this way, for each $j \geq 2$ we find a rearrangement $(b(j, k) \mid k \geq j)$ of $(a_m)_{m \in I_{2j-1}}$ such that

$$\sum_{k=j}^{\infty} b(j, k) = s_j^{(\sigma)} - s_{j-1}^{(\sigma)} - \sum_{k=1}^{j-1} b(j, k),$$

and for each $k \geq 2$ we find a rearrangement $(b(j, k) \mid j \geq k+1)$ of $(a_m)_{m \in I_{2k}}$ such that

$$\sum_{j=k+1}^{\infty} b(j, k) = s_k^{(\tau)} - s_{k-1}^{(\tau)} - \sum_{j=1}^k b(j, k).$$

In this way, $b(j, k)$ is uniquely defined for all $j, k \in \mathbb{N}$, $(b(j, k) \mid j, k \in \mathbb{N})$ is a rearrangement of $(a_m)_m$, and the $b(j, k)$ satisfy the equations

$$\begin{aligned} \sum_{j=1}^N \sum_{k=1}^{\infty} b(j, k) &= s_1^{(\sigma)} + \sum_{j=2}^N (s_j^{(\sigma)} - s_{j-1}^{(\sigma)}) = s_N^{(\sigma)}, \\ \sum_{k=1}^N \sum_{j=1}^{\infty} b(j, k) &= s_1^{(\tau)} + \sum_{k=2}^N (s_k^{(\tau)} - s_{k-1}^{(\tau)}) = s_N^{(\tau)} \end{aligned}$$

for all $N \in \mathbb{N}$, as asserted. \square

Now we turn to the general case.

Proof of Theorem 1. We prove the theorem by induction. It suffices to show that for each $n \geq 2$, the validity of Corollary 2 for $n-1$ implies the validity of the theorem for n . (Here it is important to note that the corollary also holds for $n=1$ in view of Riemann's theorem.)

So let some $n \geq 2$ be given and assume that Corollary 2 is valid for $n-1$ instead of n . Let $(a_m)_m$ be a sequence of real numbers such that $\sum_{m=1}^{\infty} a_m$ is conditionally convergent.

According to Lemma 4, there exists a disjoint partition $\mathbb{N} = \bigcup_{t=1}^{\infty} I_t$ of \mathbb{N} into infinite subsets I_t such that for each $t \in \mathbb{N}$ the series $\sum_{m \in I_t} a_m$ is conditionally convergent.

For an integer $d \geq 0$, we consider the following assumption.

Assumption A_d . The quantities $b(j_1, \dots, j_n)$ are already defined for all $j_1, \dots, j_n \in \mathbb{N}$ with $\{j_1, \dots, j_n\} \cap \{1, \dots, d\} \neq \emptyset$ such that

$$(b(j_1, \dots, j_n) \mid \{j_1, \dots, j_n\} \cap \{1, \dots, d\} \neq \emptyset)$$

is a rearrangement of $(a_m \mid m \in \bigcup_{t=1}^{nd} I_t)$ and such that for all $k \in \{1, \dots, d\}$, all $\nu \in \{1, \dots, n\}$ and all $\sigma \in \text{Sym}(n)$ with $\sigma(\nu) = 1$ one has

$$\sum_{j_2=1}^{\infty} \cdots \sum_{j_n=1}^{\infty} b(j_{\sigma(1)}, \dots, j_{\sigma(\nu-1)}, k, j_{\sigma(\nu+1)}, \dots, j_{\sigma(n)}) = s_k^{(\sigma)} - s_{k-1}^{(\sigma)}; \quad (4)$$

here, $s_0^{(\sigma)} = 0$ for all $\sigma \in \text{Sym}(n)$.

Here, for $\nu = 1$, the quantity $b(j_{\sigma(1)}, \dots, j_{\sigma(\nu-1)}, k, j_{\sigma(\nu+1)}, \dots, j_{\sigma(n)})$ is of course understood to be just $b(k, j_{\sigma(2)}, \dots, j_{\sigma(n)})$. A similar comment applies to several other notations in the sequel.

We note that this is trivially satisfied for $d = 0$ since in this case the assumption is empty.

Now let some integer $d \geq 0$ be given and assume that A_d is satisfied. We want to show that also A_{d+1} is satisfied. This is done by induction once again: For given $\mu \in \{1, \dots, n+1\}$, we consider the following assumption.

Assumption $B_{d,\mu}$. The quantities $b(j_1, \dots, j_n)$ are already defined for all $j_1, \dots, j_n \in \mathbb{N}$ with $d+1 \in \{j_1, \dots, j_{\mu-1}\}$ such that

$$(b(j_1, \dots, j_n) \mid j_1, \dots, j_n \geq d+1, d+1 \in \{j_1, \dots, j_{\mu-1}\})$$

is a rearrangement of $(a_m \mid m \in \bigcup_{t=nd+1}^{nd+\mu-1} I_t)$ and such that for all $\nu \in \{1, \dots, \mu-1\}$ and all $\sigma \in \text{Sym}(n)$ with $\sigma(\nu) = 1$, one has

$$\sum_{j_2=1}^{\infty} \cdots \sum_{j_n=1}^{\infty} b(j_{\sigma(1)}, \dots, j_{\sigma(\nu-1)}, d+1, j_{\sigma(\nu+1)}, \dots, j_{\sigma(n)}) = s_{d+1}^{(\sigma)} - s_d^{(\sigma)}. \quad (5)$$

Again we note that for $\mu = 1$ the assumption $B_{d,\mu}$ is empty, hence trivially true.

So we let some $\mu \in \{1, \dots, n\}$ be given and assume that $B_{d,\mu}$ holds. For $\sigma \in \text{Sym}(n)$, we set

$$\delta(\sigma, \nu) := \begin{cases} d+2 & \text{if } \nu \in \{\sigma(1), \dots, \sigma(\mu-1)\}, \\ d+1 & \text{if } \nu \in \{\sigma(\mu+1), \dots, \sigma(n)\}. \end{cases}$$

It is not needed to define $\delta(\sigma, \sigma(\mu))$ as we will see in the sequel.

Claim. For all $l = 2, \dots, n$ and all $\sigma \in \text{Sym}(n)$ with $\sigma(\mu) = 1$, the series

$$\sum_{j_2=1}^{\infty} \cdots \sum_{j_{l-1}=1}^{\infty} \sum_{j_l=1}^{\delta(\sigma, l)-1} \sum_{j_{l+1}=\delta(\sigma, l+1)}^{\infty} \cdots \sum_{j_n=\delta(\sigma, n)}^{\infty} b(j_{\sigma(1)}, \dots, j_{\sigma(\mu-1)}, d+1, j_{\sigma(\mu+1)}, \dots, j_{\sigma(n)}) \quad (6)$$

is (well-defined and) convergent.

Proof. Let some $l \in \{2, \dots, n\}$ and some $\sigma \in \text{Sym}(n)$ with $\sigma(\mu) = 1$ be given. In view of $l \neq 1 = \sigma(\mu)$ we have to consider only the following two cases.

Case 1: $l \in \{\sigma(1), \dots, \sigma(\mu-1)\}$.

Then $\delta(\sigma, l) - 1 = d+1$ and there is some $\lambda \in \{1, \dots, \mu-1\}$ such that $l = \sigma(\lambda)$. Now we define a permutation $\tau \in \text{Sym}(n)$ as follows:

$$\tau(i) := \sigma(i) \quad \text{for } i \neq \lambda, \mu, \quad \tau(\lambda) := \sigma(\mu) = 1, \quad \tau(\mu) := \sigma(\lambda). \quad (7)$$

The series (6) is the sum of the $\delta(\sigma, l) - 1 = d+1$ series

$$\sum_{j_2=1}^{\infty} \cdots \sum_{j_{l-1}=1}^{\infty} \sum_{j_{l+1}=\delta(\sigma, l+1)}^{\infty} \cdots \sum_{j_n=\delta(\sigma, n)}^{\infty} b(j_{\tau(1)}, \dots, j_{\tau(\lambda-1)}, j_l, j_{\tau(\lambda+1)}, \dots, j_{\tau(\mu-1)}, d+1, j_{\tau(\mu+1)}, \dots, j_{\tau(n)})$$

where $j_l = 1, \dots, d+1$. This series is convergent by assumption $B_{j_l-1, \lambda+1}$ (see (5)). This shows the convergence of the series in (6).

Case 2: $l \in \{\sigma(\mu+1), \dots, \sigma(n)\}$.

Then $\delta(\sigma, l) - 1 = d$ and there is some $\lambda \in \{\mu+1, \dots, n\}$ such that $l = \sigma(\lambda)$. Now we define τ as in (7). The series (6) is the sum of the $\delta(\sigma, l) - 1 = d$ series

$$\sum_{j_2=1}^{\infty} \cdots \sum_{j_{l-1}=1}^{\infty} \sum_{j_{l+1}=\delta(\sigma, l+1)}^{\infty} \cdots \sum_{j_n=\delta(\sigma, n)}^{\infty} b(j_{\tau(1)}, \dots, j_{\tau(\mu-1)}, d+1, j_{\tau(\mu+1)}, \dots, j_{\tau(\lambda-1)}, j_l, j_{\tau(\lambda+1)}, \dots, j_{\tau(n)})$$

where $j_l = 1, \dots, d$. This latter series is convergent by assumption A_{j_l} (see (4)). So the series in (6) is convergent as well. This proves our claim. \square

According to Corollary 2, one can choose

$$(b(j_1, \dots, j_n) \mid j_1, \dots, j_{\mu-1} \geq d+2, j_{\mu} = d+1, j_{\mu+1}, \dots, j_n \geq d+1)$$

as a rearrangement of $I_{nd+\mu}$ such that for all $\sigma \in \text{Sym}(n)$ with $\sigma(\mu) = 1$, one has

$$\begin{aligned} & \sum_{j_2=\delta(\sigma, 2)}^{\infty} \cdots \sum_{j_n=\delta(\sigma, n)}^{\infty} b(j_{\sigma(1)}, \dots, j_{\sigma(\mu-1)}, d+1, j_{\sigma(\mu+1)}, \dots, j_{\sigma(n)}) \\ &= s_{d+1}^{(\sigma)} - s_d^{(\sigma)} - \sum_{l=2}^n \sum_{j_2=1}^{\infty} \cdots \sum_{j_{l-1}=1}^{\infty} \sum_{j_l=1}^{\delta(\sigma, l)-1} \sum_{j_{l+1}=\delta(\sigma, l+1)}^{\infty} \cdots \sum_{j_n=\delta(\sigma, n)}^{\infty} \\ & \quad b(j_{\sigma(1)}, \dots, j_{\sigma(\mu-1)}, d+1, j_{\sigma(\mu+1)}, \dots, j_{\sigma(n)}). \end{aligned}$$

Here we have used the claim above (see (6)) and the fact that we can identify the subset $\{\sigma \in \text{Sym}(n) \mid \sigma(\mu) = 1\}$ with $\text{Sym}(n-1)$.

Then one can see that

$$\sum_{j_2=1}^{\infty} \cdots \sum_{j_n=1}^{\infty} b(j_{\sigma(1)}, \dots, j_{\sigma(\mu-1)}, d+1, j_{\sigma(\mu+1)}, \dots, j_{\sigma(n)}) = s_{d+1}^{(\sigma)} - s_d^{(\sigma)}$$

for all $\sigma \in \text{Sym}(n)$ with $\sigma(\mu) = 1$.

In this way, we have defined $b(j_1, \dots, j_n)$ for all $j_1, \dots, j_n \in \mathbb{N}$ with $d+1 \in \{j_1, \dots, j_{\mu}\}$ such that

$$(b(j_1, \dots, j_n) \mid j_1, \dots, j_n \geq d+1, d+1 \in \{j_1, \dots, j_{\mu}\})$$

is a rearrangement of $(a_m \mid m \in \bigcup_{t=nd+1}^{nd+\mu} I_t)$ and such that for all $\nu \in \{1, \dots, \mu\}$ and all $\sigma \in \text{Sym}(n)$ with $\sigma(\nu) = 1$, one has

$$\sum_{j_2=1}^{\infty} \cdots \sum_{j_n=1}^{\infty} b(j_{\sigma(1)}, \dots, j_{\sigma(\nu-1)}, d+1, j_{\sigma(\nu+1)}, \dots, j_{\sigma(n)}) = s_{d+1}^{(\sigma)} - s_d^{(\sigma)}.$$

Hence $B_{d, \mu+1}$ holds.

By induction we deduce that $B_{d, n+1}$ holds. But this (together with assumption A_d) just means that A_{d+1} holds. So by induction, we obtain the validity of A_d for all $d \geq 0$. This proves our theorem. \square

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