EXTENSIONS OF 2-POINT SELECTIONS

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Abstract. We consider a special order-like relation on the subsets of a given space \(X\), which is generated by a continuous selection \(f\) for at most 2-point subsets of \(X\). The relation allows to define a “minimal” set of any non-empty compact subset of \(X\), which is then used to construct continuous extensions of \(f\) over families of non-empty finite subsets of \(X\). For instance, we show that \(f\) can be extended to a continuous selection for at most 3-point subsets if and only if the hyperspace of at most 3-point subsets has a continuous selection. Other possible applications are demonstrated as well.

Dedicated to Professor Takao Hoshina on the occasion of his 60th birthday

1. Introduction

Let \(X\) be a topological space, and let \(\mathcal{F}(X)\) be the set of all non-empty closed subsets of \(X\). Also, let \(\mathcal{D} \subseteq \mathcal{F}(X)\). A map \(f : \mathcal{D} \to X\) is a selection for \(\mathcal{D}\) if \(f(S) \in S\) for every \(S \in \mathcal{D}\). A selection \(f : \mathcal{D} \to X\) is continuous if it is continuous with respect to the relative Vietoris topology \(\tau_V\) on \(\mathcal{D}\). Let us recall that \(\tau_V\) is generated by all collections of the form

\[
\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X) : S \subseteq \bigcup \mathcal{V} \text{ and } S \cap V \neq \emptyset, \text{ whenever } V \in \mathcal{V} \right\},
\]

where \(\mathcal{V}\) runs over the finite families of open subsets of \(X\). Sometimes, for reasons of convenience, we will also say that \(f\) is Vietoris continuous to stress the attention that \(f\) is continuous with respect to the topology \(\tau_V\).

In the sequel, all spaces are assumed to be at least Hausdorff. In this note, we are interested of continuous selections for \(\mathcal{D}\), where \(\mathcal{D}\) is a family of finite subsets of \(X\). To this end, let

\[
\mathcal{F}_n(X) = \{ S \in \mathcal{F}(X) : |S| \leq n \}, \quad n \geq 1.
\]

Note that we may identify \(X\) with the set \(\mathcal{F}_1(X)\), and, in fact, \(X\) is homeomorphic to the space \((\mathcal{F}_1(X), \tau_V)\). The latter means that the Vietoris topology is admissible, see [4].

It should be mentioned that there are spaces \(X\) (for instance, one can take \(X\) to be the real line \(\mathbb{R}\)), which have a continuous selection for \(\mathcal{F}_n(X)\) for every \(n \geq 2\), but they have no continuous selection for \(\mathcal{F}(X)\), see [1]. On the other hand, we don’t know if there exists a space \(X\) which has a continuous selection for \(\mathcal{F}_n(X)\) for some \(n \geq 2\), but it has no continuous selection for \(\mathcal{F}_{n+1}(X)\), see [3].
In the present paper we are mainly interested in the above problem when \( n = 2 \). Briefly, we show that a continuous selection \( f \) for \( \mathcal{F}_2(X) \) can be continuously extended to a selection for \( \mathcal{F}_3(X) \) if and only if \( \mathcal{F}_3(X) \) has a continuous selection, see Corollary 4.2. We also demonstrate that, for a space \( X \) with only one non-isolated point, the hyperspace \( \mathcal{F}_3(X) \) has a continuous selection if and only if \( \mathcal{F}_3(X) \) has a continuous selection, see Corollary 5.4. The technique developed to achieve these results is based on an order-like relation on the subsets of a given space \( X \) that is generated by a continuous selection for \( \mathcal{F}_2(X) \), see the next section. In particular, it culminates in an extension result (Theorem 3.2) that may have an independent interest. Finally, we also consider a local version of this selection-extension problem for hyperspaces, see Section 5.

2. An Order-like Relation on Subsets

Suppose that \( f : \mathcal{F}_2(X) \rightarrow X \) is a selection. Then, it defines a natural order-like relation \( \preceq \) on \( X \) by letting \( x \preceq y \) if and only if \( f(\{x, y\}) = x \), see [4]. For convenience, we will write that \( x < y \) if \( x \preceq y \) and \( x \neq y \).

The relation is very similar to a linear order on \( X \) in that it is both reflexive and antisymmetric, but, unfortunately, it may fail to be transitive. In the present paper, we extend this relation to all subsets of \( X \). Namely, if \( B \) and \( C \) are subsets of \( X \) (not necessarily non-empty), then we shall write that \( B \preceq C \) if \( y \preceq z \) for every \( y \in B \) and \( z \in C \). As before, we will write that \( B < C \) if \( y < z \) for every \( y \in B \) and \( z \in C \); equivalently, if \( B \preceq C \) and \( B \cap C = \emptyset \).

Here are some basic properties of this relation.

**Proposition 2.1.** Let \( X \) be a space, \( f : \mathcal{F}_2(X) \rightarrow X \) be a selection, and let \( \preceq \) be the order-like relation generated by \( f \). Also, let \( B, C \in \mathcal{F}(X) \) be such that \( B \preceq C \) and \( C \preceq B \). Then, both \( B \) and \( C \) are singletons, and \( B = C \).

**Proof.** The observation is almost obvious. Namely, take points \( y \in B \) and \( z \in C \). Then, by definition, \( y \preceq z \) and \( z \preceq y \), so \( y = z \). That is, \( C = \{y\} = \{z\} = B \). \( \square \)

**Proposition 2.2.** Let \( X \) be a space, \( f : \mathcal{F}_2(X) \rightarrow X \) be a selection, and let \( \preceq \) be the order-like relation generated by \( f \). Also, let \( B, C \in \mathcal{F}(X) \) be such that \( B \preceq C \) and \( B \cap C \neq \emptyset \). Then, \( B \cap C \) is a singleton.

**Proof.** Suppose that \( y, z \in B \cap C \). Then, by definition, \( y \preceq z \) and \( z \preceq y \), so \( y = z \). \( \square \)

**Proposition 2.3.** Let \( X \) be a space, \( f : \mathcal{F}_2(X) \rightarrow X \) be a selection, and let \( \preceq \) be the order-like relation generated by \( f \). Also, let \( S \in \mathcal{F}(X) \), and let \( B, C \subseteq S \) be such that \( B \preceq S \setminus B \) and \( C \preceq S \setminus C \). Then, either \( B \subseteq C \) or \( C \subseteq B \).

**Proof.** Suppose, if possible, that this fails. Then, \( B \setminus C \neq \emptyset \) and \( C \setminus B \neq \emptyset \), so there is a point \( y \in B \setminus C \) and a point \( z \in C \setminus B \). However, this implies that \( z \preceq y \) because \( y \in S \setminus C \), and \( y \preceq z \) because \( z \in S \setminus B \). Hence, \( y = z \), but \( y \neq z \). This is a contradiction, which completes the proof. \( \square \)

We are now ready for our main result concerning this relation. Towards this end, we introduce the following concept. Let \( f : \mathcal{F}_2(X) \rightarrow X \) be a selection, \( \preceq \) be the
corresponding order-like relation generated by $f$, and let $S \in \mathcal{F}(X)$. We shall say that a subset $B \subset S$, $B \in \mathcal{F}(X)$, is an $f$-minimum of $S$ if

1. $B \preceq S \setminus B$,
2. $C \preceq S \setminus B$, and $C \preceq S \setminus C$, then $B \subset C$.

In this case we will write that $B = \min_f S$.

**Lemma 2.4.** Let $X$ be a space, $f : \mathcal{F}_2(X) \to X$ be a selection, and let “$\preceq$” be the order-like relation generated by $f$. Then, every non-empty compact subset $S \subset X$ has a unique $f$-minimum.

**Proof.** Let $S \in \mathcal{F}(X)$ be compact, and let $B = \min_f S$ and $C = \min_f S$. Then, by definition, $B \preceq S \setminus B$ and $C \preceq S \setminus C$. Hence, by Proposition 2.3, either $B \subset C$ or $C \subset B$. According once again to the definition of an $f$-minimal set, we get that $B = C$.

Turning to the existence of $f$-minimal sets, consider the family

$$D = \{B \in \mathcal{F}(S) : B \preceq S \setminus B\}.$$ 

Note that $S \in D$ because $S \preceq S \setminus S = \emptyset$, so $D \neq \emptyset$. On the other hand, $D$ consists of compact sets, and, by Proposition 2.3, it has the finite intersection property. Hence, $D = \bigcap D \in \mathcal{F}(X)$. In fact, $D \in D$. Indeed, if $D = S$, this was mentioned before. If $D \neq S$, take points $y \in D$ and $z \in S \setminus D$. Then, there exists $B \subset D$, with $z \not\in B$. However, $y \in D \subset B$, and therefore $y \preceq z$. That is, $D \preceq S \setminus D$, which completes the proof.

3. Selection-regular Selections

Let $X$ be a space, and let $\mathcal{K}(X) = \{S \in \mathcal{F}(X) : |S| < \omega \}$. Also, let $f : \mathcal{F}_2(X) \to X$ be a continuous selection, and let $D \subset \mathcal{K}(X)$ be such that $\min_f S \in D$ for every $S \in D$. We shall say that a selection $h : D \to X$ is $f$-regular if $h(S) = h(\min_f S)$ for every $S \in D$.

Let us observe that if $D = \mathcal{F}_2(X)$, then $h$ is $f$-regular if and only if $h = f$. That is, any $f$-regular selection $h$ provides an extension of $f$ to the elements of $D$ in sense that $h(S) = f(S)$ for every $S \in D \cap \mathcal{F}_2(X)$. In particular, this also implies that there are continuous selections for $\mathcal{F}_2(X)$ which are not $f$-regular. On the other hand, we have the following general example of continuous selections $g : \mathcal{F}_3(X) \to X$ which are not $g \mid \mathcal{F}_2(X)$-regular.

**Example 3.1.** Let $X$ be a space, $\mathcal{C}$ be a disjoint cover of $X$ consisting of non-empty clopen subsets of $X$, with $|\mathcal{C}| \geq 3$, and let $h : \mathcal{F}_3(X) \to X$ be a continuous selection such that $|\min_f S| = 1$ for every $S \in \mathcal{F}_3(X)$, where $f = h \mid \mathcal{F}_2(X)$. Then, there exists a continuous selection $g : \mathcal{F}_3(X) \to X$ which is not $f$-regular, but $f = g \mid \mathcal{F}_2(X)$.

**Proof.** By hypothesis, $X$ has a cover of pairwise disjoint non-empty clopen sets $C_1, C_2, C_3 \subset X$ and points $x_i \in C_i$, $1 \leq i \leq 3$, such that $\min_f \{x_1, x_2, x_3\} = x_1$. Let $\mathcal{U} = \{C_1, C_2, C_3\}$ \cap $\mathcal{F}_3(X)$, which is a $\tau_1$-clopen subset of $\mathcal{F}_3(X)$, with $\mathcal{U} \cap \mathcal{F}_2(X) = \emptyset$. Next, define a continuous selection $g : \mathcal{F}_3(X) \to X$ by letting that $g(S) \in S \setminus C_3$ if $S \in \mathcal{U}$, and $g(S) = h(S)$ otherwise. Then, $g \mid \mathcal{F}_2(X) = f$ because $\mathcal{U} \cap \mathcal{F}_2(X) = \emptyset$. However, $g$ is not $f$-regular because $g(\{x_1, x_2, x_3\}) = x_3$, while $g(\min_f \{x_1, x_2, x_3\}) = g(x_1) = x_1$. 

\[\square\]
In our next considerations, to any family $\mathcal{D} \subset \mathcal{K}(X)$ we associate the family
\[
\min_f(\mathcal{D}) = \{\min_f S : S \in \mathcal{D}\}.
\]
Note that $|\min_f S| = 1$ or $|\min_f S| \geq 3$, but $|\min_f S| = 2$ is impossible. On the other hand, with respect to selections, it suffices to consider at least 2-point sets. Namely, any selection is continuous on the singletons of $X$. Thus, the substantial part of $\min_f(\mathcal{D})$ are the non-singletons, i.e. the following family
\[
\min_f^*(\mathcal{D}) = \{B \in \min_f(\mathcal{D}) : |B| > 1\}.
\]

**Theorem 3.2.** Let $X$ be a space, $f : \mathcal{F}_2(X) \to X$ be a continuous selection, and let $\mathcal{D} \subset \mathcal{K}(X)$ be such that $\min_f(\mathcal{D}) \subset \mathcal{D}$. Then, the following are equivalent.

(a) $\mathcal{D}$ has a continuous $f$-regular selection,
(b) $\mathcal{D}$ has a continuous selection,
(c) $\min_f^*(\mathcal{D})$ has a continuous selection.

To prepare for the proof of Theorem 3.2, we need the following simple criterion for continuity in $\mathcal{F}_2(X)$, see [2, Theorem 3.1].

**Proposition 3.3.** Let $X$ be a space, $f : \mathcal{F}_2(X) \to X$ be a selection, and let $\preceq$ be the order-like relation generated by $f$. Also, and let $x, y \in X$ be such that $x \prec y$. Then, $f$ is continuous at $(x, y)$ if and only if there are open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $U \prec V$.

In fact, relying on this criterion, we have the following crucial result concerning the proof of Theorem 3.2.

**Lemma 3.4.** Let $X$ be a space, $f : \mathcal{F}_2(X) \to X$ be a continuous selection, and let $\preceq$ be the order-like relation generated by $f$. Then, whenever $S \in \mathcal{K}(X)$, there is a disjoint family $\{V_x : x \in S\}$ of open subsets of $X$ such that

(a) $x \in V_x$, for every $x \in S$,
(b) if $T \in \langle\{V_x : x \in S\}\rangle$, then $\min_f T \in \langle\{V_x : x \in \min_f S\}\rangle$.

Proof. By Proposition 3.3, there exists a disjoint family $\{V_x : x \in S\}$ of open subsets of $X$ such that $x \in V_x$, $x \in S$, and if $x, y \in S$ and $x \prec y$, then $V_x \prec V_y$. This family is as required. Indeed, take a $T \in \langle\{V_x : x \in S\}\rangle$, and let $B = \bigcup\{V_x \cap T : x \in \min_f S\}$. Then,
\[
T \setminus B = \bigcup\{T \cap V_y : y \in S \setminus \min_f S\},
\]
and therefore $B \preceq T \setminus B$ because $V_x \prec V_y$, for every $x \in \min_f S$ and $y \in S \setminus \min_f S$. Hence, by definition,
\[
\min_f T \subset B \subset \bigcup\{V_x : x \in \min_f S\}. \tag{1}
\]
Take now points $x, y \in S$, with $\min_f T \cap V_x \neq \emptyset = \min_f T \cap V_y$. Then, $z \prec t$ for every $z \in \min_f T \cap V_x$ and $t \in V_y \cap T$. So, according to the properties of the family $\{V_x : x \in S\}$, we have that $V_x \prec V_y$, i.e. that $x \prec y$. In particular, this implies that
\[
\min_f S \subset \{x \in S : V_x \cap \min_f T \neq \emptyset\}. \tag{2}
\]
Thus, according to (1) and (2), we finally get that $x \in \min_f S$ if and only if $\min_f T \cap V_x \neq \emptyset$, i.e. that $\min_f T \in \langle\{V_x : x \in \min_f S\}\rangle$. \qed
Proof of Theorem 3.2. Since \( \min_f(D) \subseteq \min_f(D) \subseteq D \), the implications (a) \( \Rightarrow (b) \Rightarrow (c) \) are obvious. So, we are going to prove only that (c) \( \Rightarrow (a) \). Suppose that \( g^* : \min_f(D) \rightarrow X \) is a continuous selection. Then, \( g^* \) defines a continuous selection \( g : \min_f(D) \rightarrow X \) by letting \( g(B) = g^*(B) \) if \( B \in \min_f(D) \), and \( g(B) \in B \) otherwise. Next, we define an \( f \)-regular selection \( h \) for \( D \) by \( h(S) = g(\min_f S) \) for every \( S \in D \). It remains to show that \( h \) is continuous. To this end, take an \( S \in D \), and a neighbourhood \( U \) of \( h(S) \). Since \( g(\min_f S) \in U \) and \( g \) is continuous, there exists a finite family \( \{W_x : x \in \min_f S\} \) of disjoint open subset of \( X \) such that \( x \in W_x, x \in \min_f S, \) and \( g(\{W_x : x \in \min_f S\}) \subseteq U \). On the other hand, by Lemma 3.4, there exists a disjoint family \( \{V_x : x \in S\} \) of open subset of \( X \) such that \( x \in V_x, x \in S, \) and \( \min_f T \in \{V_x : x \in \min_f S\} \) for every \( T \in \{V_x : x \in S\} \). Take \( U_x = W_x \cap V_x \) if \( x \in \min_f S, \) and \( U_x = V_x \) otherwise. Then, \( \min_f T \in \{U_x : x \in \min_f S\} \) provided \( T \in \{U_x : x \in \min_f S\} \cap D \), so \( h(T) = g(\min_f T) \in U \). □

4. Extensions of 2-point Selections

In this section we provide some possible applications of the extension theorem in the previous section. To this end, let us recall that

\[ F_n(X) = \{S \in \mathcal{F}(X) : |S| \leq n\}. \]

Also, for \( 1 \leq m \leq n \), we let \( F_{(m,n)}(X) = \{S \in \mathcal{F}(X) : m \leq |S| \leq n\} \).

**Corollary 4.1.** Let \( X \) be a space, and let \( f : F_2(X) \rightarrow X \) and \( g : F_{(3,3)}(X) \rightarrow X \) be continuous selections. Then, there exists a continuous selection \( h \) for \( F_3(X) \) such that \( h \upharpoonright F_2(X) = f \), i.e. \( f \) can be extended to a continuous selection for \( F_3(X) \).

**Proof.** Let \( D = F_3(X) \). As it was mentioned before, \( \min_f S \notin F_{(2,2)}(X) \) for every \( S \in D \). So, in this case, we have that \( B \in F_{(3,3)}(X) \) for every \( B \in \min_f(D) \), i.e. that \( \min_f(D) \subseteq F_{(3,3)}(X) \). Hence, by hypothesis, \( \min_f(D) \) has a continuous selection, take, for instance, \( g \upharpoonright \min_f(D) \). Therefore, by Theorem 3.2, \( D = F_3(X) \) has a continuous \( f \)-regular selection \( h \). In particular, \( h \) is an extension of \( f \), which completes the proof. □

Related to this consequence, we have the following natural question.

**Question 1.** Let \( X \) be a space such that, for some \( n \geq 2 \), both families \( F_n(X) \) and \( F_{(n+1,n+1)}(X) \) have continuous selections. Is it true that \( F_{n+1}(X) \) also has a continuous selection?

Corollary 4.1 is interesting also from another point of view. Namely, for a space \( X \) and a continuous selection \( f : F_2(X) \rightarrow X \), we may ask if \( f \) can be extended to a continuous selection for \( F_3(X) \). On the other hand, we may only be interested if there exists a continuous selection for \( F_3(X) \). It turns out that these two properties are equivalent.

**Corollary 4.2.** Let \( X \) be a space, and let \( f : F_2(X) \rightarrow X \) be a continuous selection. Then, \( f \) can be extended to a continuous selection for \( F_3(X) \) if and only if \( F_3(X) \) has a continuous selection.

**Proof.** Note that, by Corollary 4.1, \( f \) can be extended to a continuous selection for \( F_3(X) \) if and only if \( F_{(3,3)}(X) \) has a continuous selection. So, it is an immediate consequence of Corollary 4.1. □
5. One-point Extensions

In this section we consider a local version of the previous extension problem. Namely, let \( f : \mathcal{F}_3(X) \to X \) be a continuous selection, and let \( p \in X \) be a fixed point. Now, we become interested if \( f \) can be extended to a continuous selection for \( \mathcal{F}_3(X \setminus \{p\}) \) has a continuous selection.

Turning to this problem, we consider the order-like relation “\( \preceq \)” generated by \( f \), and we let \( L_p = \{ x \in X : x \prec p \} \) and \( R_p = \{ y \in X : p \prec y \} \).

Also, we consider the families
\[
\mathcal{P} = \{ S \in \mathcal{F}_{(3,3)}(X) : S \cap L_p \neq \emptyset \neq R_p \cap S \},
\]
and
\[
\Omega = \{ S \in \mathcal{F}_{(3,3)}(X) : S \cap L_p = \emptyset \text{ or } S \cap R_p = \emptyset \}.\]

**Proposition 5.1.** Let \( X \) be a space, \( p \in X \), and let \( f : \mathcal{F}_2(X) \to X \) be a continuous selection. Then, \( \min f \) is a singleton for every \( S \in \mathcal{P} \), with \( p \in S \).

**Proof.** Suppose that \( S \in \mathcal{P} \), and let \( S = \{ x,y,p \} \) for some points \( x, y \in X \setminus \{ p \} \), with \( x \prec y \). If \( \{ x,y,p \} \subseteq L_p \), then \( x \prec p \) and \( y \prec p \), so \( \min f \{ x,y,p \} = \{ x \} \). In the same way, if \( \{ x,y \} \subseteq R_p \), then \( p \prec x \) and \( p \prec y \), so \( \min f \{ x,y,p \} = \{ p \} \). This complete the proof. \( \square \)

**Proposition 5.2.** Let \( X \) be a space, \( p \in X \), and let \( f : \mathcal{F}_2(X) \to X \) be a continuous selection. Then, there exists a continuous selection \( g : \mathcal{P} \to X \).

**Proof.** Whenever \( S \in \mathcal{P} \), let us observe that \( 1 \leq |S \cap L_p| \leq 2 \) and \( 1 \leq |S \cap R_p| \leq 2 \). Then, for every \( S \in \mathcal{P} \), let \( \ell_S \) be the minimal element of \( S \cap L_p \) with respect to the order-like relation “\( \preceq \)” generated by \( f \), and let \( r_S \) be the corresponding maximal element of \( S \cap R_p \). Now, define a selection \( g : \mathcal{P} \to X \) by \( g(S) = f(\{ \ell_S,r_S \}) \), \( S \in \mathcal{P} \).

To check the continuity of \( g \), take an \( S \in \mathcal{P} \), and a neighbourhood \( U \) of \( g(S) \). Since \( f \) is continuous, there are open subsets \( V,W \subset X \) such that \( \ell_S \in V \subset L_p \), \( r_S \in W \subset R_p \), and \( f(\{ V,W \}) \subset U \). We distinguish the following cases for \( x \in S \setminus \{ \ell_S,r_S \} \). If \( p \neq x \), then either \( x \in L_p \) or \( x \in R_p \), say \( x \in L_p \). In this case, \( \ell_S \prec x \) and, by Proposition 3.3, there are open subsets \( V_1,V_2 \subset L_p \), such that \( \ell_S \in V_1 \subset V, x \in V_2, \) and \( V_1 \prec V_2 \). Thus, we get a \( \tau_1 \)-neighbourhood \( \{ V_1,V_2,W \} \) of \( S \) such that \( T \in \{ V_1,V_2,W \} \) implies \( \ell_T \in V_1 \subset V \) and \( r_T \in W \). Hence, \( \{ \ell_T,r_T \} \in \{ V,W \} \), and therefore \( g(T) = f(\{ \ell_T,r_T \}) \in U \). The case \( x \in R_p \) is symmetric. Suppose finally that \( x = p \). Then \( S = \{ \ell_S,p,r_S \} \) and \( \ell_S \prec p \prec r_S \).

Hence, there are open sets \( V_0,O,W_0 \) such that \( \ell_S \in V_0 \subset V, p \in O, r_S \in W_0 \subset W, \) and \( V_0 \prec O \prec W_0 \). Thus, just like before, we get a \( \tau_1 \)-neighbourhood \( \{ V_0,O,W_0 \} \) of \( S \) such that \( T \in \{ V_0,O,W_0 \} \) implies \( \ell_T \in V_0 \subset V \) and \( r_T \in W_0 \subset W \). Hence, \( g(T) = f(\{ \ell_T,r_T \}) \in U \) which completes the proof. \( \square \)

We are now ready for our main result in this section.

**Theorem 5.3.** Let \( X \) be a space, \( p \in X \), and let \( f : \mathcal{F}_2(X) \to X \) and \( g : \mathcal{F}_3(X \setminus \{ p \}) \to X \) be continuous selections. Then, \( f \) can be extended to a continuous selection for \( \mathcal{F}_3(X) \).
Proof. Let \( D = \mathcal{F}_3(X) \), and let \( L_p, R_p \subset X \) and \( \mathcal{P}, \mathcal{Q} \subset \mathcal{F}_{(3,3)}(X) \) be defined as at the beginning of this section. By Theorem 3.2, it suffices to show that \( \min^*_3(D) \) has a continuous selection. To this end, note that \( S \in \min^*_3(D) \) if and only if \( S \) has no minimal element. In particular, if \( S \in \min^*_3(D) \), then \( |S| = 3 \).

Now, from one hand, we have that \( \mathcal{P} \cap \mathcal{Q} = \emptyset \) and \( \mathcal{P} \cup \mathcal{Q} = \mathcal{F}_{(3,3)}(X) \), hence \( \min^*_3(D) \subset \mathcal{P} \cup \mathcal{Q} \). From another hand, by Proposition 5.1, \( S \notin \min^*_3(D) \) provided \( p \in S \in \mathcal{Q} \). Hence, \( \mathcal{Q} \cap \min^*_3(D) \subset \mathcal{H} \), where

\[ \mathcal{H} = \{ S \in \mathcal{F}_{(3,3)}(X) : S \subset L_p \text{ or } S \subset R_p \}. \]

Finally, let us observe that both \( \mathcal{P} \) and \( \mathcal{H} \) are \( \tau_3 \)-open, and clearly \( \mathcal{P} \cap \mathcal{H} = \emptyset \).

Therefore, \( \mathcal{P} \cap \min^*_3(D) \) and \( \mathcal{Q} \cap \min^*_3(D) \) define a \( \tau_3 \)-clopen partition for \( \min^*_3(D) \). Thus, we can define a continuous selection for \( \min^*_3(D) \) by pointing out how to define continuous selections for \( \mathcal{P} \) and \( \mathcal{H} \). To this end, let us observe that \( \mathcal{H} \in \mathcal{F}_3(X \setminus \{p\}) \), hence \( g \upharpoonright \mathcal{H} \) is a continuous selection for \( \mathcal{H} \). On the other hand, by Proposition 5.2, \( \mathcal{P} \) has also a continuous selection. This, in fact, completes the proof. \( \square \)

Corollary 5.4. Let \( X \) be a space with only one non-isolated point \( p \in X \), and let \( f : \mathcal{F}_3(X) \to X \) be a continuous selection. Then, \( f \) can be extended to a continuous selection for \( \mathcal{F}_3(X) \).

Proof. By Theorem 3.5, it suffices to show that \( \mathcal{F}_3(X \setminus \{p\}) \) has a continuous selection. However, \( X \setminus \{p\} \) is a discrete space, and any selection for \( \mathcal{F}_3(X \setminus \{p\}) \) will be continuous. \( \square \)

Note that, in Corollary 5.4, the space \( X \) has a dense set of isolated points. Related to this, we were recently informed by M. Hrusak that he and J. Steprans proved that if a space \( X \) has a countable dense set of isolated points and a continuous selection for \( \mathcal{F}_2(X) \), then it has a continuous selection for \( \mathcal{F}_3(X) \) as well; their paper is in process.

Corollary 5.5. Let \( X \) be a collectionwise normal zero-dimensional space which has a continuous selection \( f : \mathcal{F}_3(X) \to X \), \( P \subset X \) be a closed discrete set, and let \( g \) be a continuous selection for \( \mathcal{F}_3(X \setminus P) \). Then, \( f \) can be extended to a continuous selection for \( \mathcal{F}_3(X) \).

Proof. By Corollary 4.2, it suffices to show that \( \mathcal{F}_3(X) \) has a continuous selection. Since \( X \) is collectionwise normal and zero-dimensional, there exists a clopen discrete cover \( \{X_p : p \in P\} \) of \( X \) such that \( p \in X_p, p \in P \). Then, by hypothesis, each \( \mathcal{F}_3(X_p \setminus \{p\}), p \in P \), has a continuous selection. Hence, by Theorem 5.3, \( f \upharpoonright \mathcal{F}_2(X_p) \) can be extended to a continuous selection \( f_p : \mathcal{F}_3(X_p) \to X_p \) for every \( p \in P \). Now, we consider a well-ordering \( \leq \) on \( P \), and then, for every \( S \in \mathcal{F}_3(X) \), we let \( p(S) = \min\{p \in P : S \cap X_p \neq \emptyset\} \). Finally, we define a continuous selection \( h \) for \( \mathcal{F}_3(X) \) by letting \( h(S) = f_{p(S)}(S \cap X_{p(S)}) \) for every \( S \in \mathcal{F}_3(X) \). \( \square \)

References


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