THE TRANSFORM FORMULA FOR SUBMODULES OF MULTIPLICATION MODULES

MAJID M. ALI

(Received October 21, 2010)

Abstract. Let $R$ be an integral domain with quotient field $Q(R)$ and $M$ a faithful multiplication $R$-module. For a submodule $N$, let $T(N) = \bigcup_{n \geq 0} B_n$ where

$$B_n = \{x \in Q(R) : x[N : M]^n N \subseteq M \text{ for some positive integer } n\}.$$  

We study the problem of determining for which multiplication modules over integral domains we have the equality $T([K : M] N) = T(K) + T(N)$ for all submodules, or all finitely generated submodules, or all cyclic submodules $K$ and $N$ of $M$. We show that a faithful multiplication Prufer module over an integral domain satisfies $T([K : M] N) = T(K) + T(N)$ for all finitely generated submodules $K$ and $N$ of $M$.

Introduction

Throughout this paper, all rings are assumed commutative with identity, and all modules are unital. Let $R$ be a ring and $M$ an $R$-module. Then $M$ is called multiplication if every submodule $N$ of $M$ has the form $IM$ for some ideal $I$ of $R$, [7]. Note that $I \subseteq [N : M]$ and hence $N = IM \subseteq [N : M]M \subseteq N$, so that $N = [N : M]M$. Anderson [5], defined $\theta(M) = \sum_{m \in M} [Rm : M]$ and showed the usefulness of this ideal in studying multiplication modules. He proved for example that if $M$ is multiplication then $M = \theta(M) M$, and a finitely generated module is multiplication if and only if $\theta(M) = R$, [5, Proposition 1 and Theorem 1]. An ideal $I$ of a ring $R$ is called pure if it is locally either zero or $R$. Equivalently, $I$ is pure if for all ideals $J \subseteq I$, $J = JI$. If $M$ is a faithful multiplication $R$-module then $\theta(M)$ is a pure ideal of $R$, [6, Theorem 2.3]. Assume $R$ is an integral domain and $M$ a faithful multiplication $R$-module. Then $\theta(M)$ is a pure ideal of $R$ and hence for all $a \in \theta(M)$, $Ra = a\theta(M)$, so there exists $b \in \theta(M)$ such that $a = ab$. Hence $b = 1$, and hence $\theta(M) = R$. So $M$ is finitely generated.


2010 Mathematics Subject Classification 13C13, 13A15.

Key words and phrases: Transform formula, Multiplication module, Prufer module.
is multiplication (resp. finitely generated), [15, Theorem 10] and [4, Proposition 2.2]. Multiplication modules have recently received considerable attention, see for example [4]-[7] and [15].

Let $R$ be a ring and $S$ the set of nonzero divisors of $R$ and $R_S$ the total quotient ring of $R$. For a nonzero ideal $I$ of $R$, let $I^{-1} = \{ x \in R_S : xI \subseteq R \}$. $I$ is called an invertible ideal of $R$ if $II^{-1} = R$. Let $M$ be an $R$-module. Let

$$
T = \{ t \in S : \text{ for all } m \in M, tm = 0 \text{ implies } m = 0 \} .
$$

Then $T$ is a multiplicatively closed subset of $S$, and if $M$ is torsion-free, then $T = S$. In particular, $T = S$ if $M$ is a faithful multiplication $R$-module, [7, Lemma 1.4]. Also $T = S$ if $M$ is an ideal of $R$. Let $N$ be a nonzero submodule of $M$ and let $N^{-1} = \{ x \in R_T : xN \subseteq M \}$. Then $N^{-1}$ is an $R$-submodule of $R_T$, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. $N$ is said to be invertible in $M$ if $NN^{-1} = M$. Clearly $M$ is invertible in $M$. Naoum and Al-Alwan, [13] introduced the invertibility of submodules generalizing the concept of invertible ideals and gave several properties of such submodules. An $R$-module $M$ is called a Dedekind (resp. Prufer) module if and only if every nonzero (resp. nonzero finitely generated) submodule of $M$ is invertible. It is shown [13, Theorems 3.5 and 3.6] that if $R$ is an integral domain and $M$ a faithful multiplication $R$-module then $M$ is Dedekind (resp. Prufer) if and only if $R$ is a Dedekind (resp. Prufer) domain. The author investigated invertible submodules of multiplication modules and considered several other characterizations of Prufer and Dedekind modules, [1] and [2].

Let $R$ be an integral domain with quotient field $Q(R)$ and let $I$ be an ideal of $R$. According to [9], $T(I)$ the ideal transform of $I$, is $T(I) = \bigcup_{n \geq 1} \left[ R : Q(R) I^n \right]$. It is clear that $I^{-1} \subseteq T(I)$. $T(I)$ is a ring, called the $I$-transformation of $R$. In [9], the authors study the problem of determining for which integral domains one has the equality $T(IJ) = T(I) + T(J)$ for all ideals, or all finitely generated ideals, or all principal ideals $I$ and $J$ of $R$. Here $T(I) + T(J) = \{ \alpha + \beta \in T(I) \text{ and } \beta \in T(J) \}$, so that $T(I) + T(J)$ is not always a ring. The authors show for example, that in a Prufer domain $T(IJ) = T(I) + T(J)$ for all finitely generated ideals $I$ and $J$, and that a Noetherian domain $R$ has the property that $T(IJ) = T(I) + T(J)$ for all ideals, or all principal ideals, if and only if the Krull dimension of $R$ is at most 1. If $R$ is an almost Dedekind domain, then it needs not be true that $T(IJ) = T(I) + T(J)$ for all ideals $I$ and $J$ of $R$.

In this paper we generalize the transform formula for ideals of a ring $R$ to submodules of faithful multiplication modules over integral domains. Let $R$ be an integral domain and $M$ a faithful multiplication module over $R$. Let $N$ be a submodule of $M$ and define $T(N) = \bigcup_{n \geq 0} \left[ M : R_T [N : M]^n N \right]$. Obviously, $N^{-1} \subseteq T(N)$ and $T(M) = [M : R_T M]$. Moreover, since $M$ is a faithful multiplication module, $R_T = Q(R)$, [7, Lemma 4.1]. So

$$
T(N) = \bigcup_{n \geq 0} \left[ M : Q(R) [N : M]^n N \right] = \bigcup_{n \geq 0} \left[ M : Q(R) [N : M]^{n+1} M \right] \\
= \bigcup_{n \geq 0} \left[ R : Q(R) [N : M]^{n+1} \right] = T([N : M]) .
$$

Consider the following conditions on $M$:

(T1) $T([K : M] N) = T(K) + T(N)$ for all submodules $K$ and $N$ of $M$. 


(T_2) \( T([K : M]N) = T(K) + T(N) \) for all finitely generated submodules \( K \) and \( N \) of \( M \).

(T_3) \( T([K : M]N) = T(K) + T(N) \) for all cyclic submodules \( K \) and \( N \) of \( M \).

Modules with \( T_1 \)-property are said to satisfy the transform formula for submodules or simply the transform formula. Theorems 1, 2 and 8 give several properties of the transform formula. We show in Proposition 4 that a faithful multiplication Prufer module over an integral domain satisfies \( T_2 \)-property. Since faithful multiplication Dedekind modules over integral domains are Noetherian and Prufer, [2, Theorem 3.4], these modules satisfy the transform formula. Several other properties of the transform formula for submodules of faithful multiplication modules are considered.

All rings in this paper are integral domains. For the basic concepts used, we refer the reader to [8], [10] and [11].

Transform Formula

In this section we consider the transform formula for submodules of faithful multiplication modules. We start with a result generalizing some facts about the transform formula for ideals.

**Theorem 1.** Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module. Let \( K \) and \( N \) be submodules of \( M \).

1. If \( K \subseteq N \), then \( T(N) \subseteq T(K) \).
2. \( T(K) + T(N) \subseteq T([K : M]N) \).
3. \( T(K \cap N) = T([K : M]N) \).
4. \( T(K) \cap T(N) = T(K + N) \).

**Proof.** (1) Let \( K \subseteq N \). Then for every positive integer \( n \), \( [K : M]^n K \subseteq [N : M]^n N \). Hence

\[
T(N) = \bigcup_{n \geq 0} [M : Q(R)] [N : M]^n N \subseteq \bigcup_{n \geq 0} [M : Q(R)] [K : M]^n K = T(K).
\]

(2) Since \([K : M] N \subseteq N\), it follows by (1) that \( T(N) \subseteq T([K : M] N) \). Next, \([K : M] N = [N : M] K \subseteq K\). Hence \( T(K) \subseteq T([K : M] N) \), and this gives that

\[
T(K) + T(N) \subseteq T([K : M] N).
\]

(3) As \([K : M] N = [N : M] K \subseteq K \cap N\), we infer from (1) that \( T(K \cap N) \subseteq T([K : M] N) \). Next

\[
([K \cap N] : M)^2 = ([K : M] \cap [N : M])^2 \subseteq [K : M] [N : M],
\]

so,

\[
T([K : M] [N : M]) \subseteq T([K \cap N] : M)^2.
\]

It is clear that for any ideal \( I \) of \( R \), \( T(I) = T(I^m) \) for every positive integer \( m \). Hence \( T(IM) = T(I^m M) \), and hence

\[
T([K : M] N) = T([K : M] [N : M] M) \subseteq T([K \cap N] : M)^2 M
\]

\[
= T([K \cap N] : M) = T(K \cap N).
\]

This finally gives that \( T(K \cap N) = T([K : M] N) \).
Observe, $T(K + N) \subseteq T(K) \cap T(N)$. Now, let $x \in T(K) \cap T(N)$. Then $x[K : M]^n K \subseteq M$ and $x[N : M]^m N \subseteq M$ for some positive integers $m$ and $n$. It follows that $x[K : M]^{n+1} \subseteq R$ and $x[N : M]^{m+1} \subseteq R$. So
\[ x([K : M] + [N : M])^{m+n+2} \subseteq R. \]

\[ [(K + N) : M] = [K : M] + [N : M], \]
and hence, $x(K + N) : M)^{m+n+2} \subseteq R$. It follows that $x(K + N) : M)^{m+n+1}(K + N) \subseteq M$, and hence $x \in T(K + N)$. So $T(K) \cap T(N) \subseteq T(K + N)$, and hence the result follows. \hfill $\Box$

**Theorem 2.** Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module. Let $K, N$ and $H$ submodules of $M$.

1. If $T(K + N)(K + N) = T(K + N)M$, then for all positive integers $n$,
   \[ ([K : M]^n K + [N : M]^n N) T(K + N) = T(K + N)M. \]
2. If $K$ and $N$ are comaximal submodules of $M$ then
3. If $K$ is invertible, then
   \[ T([K : M]M) = T(K)T(N). \]
4. Let $H$ be finitely generated. If $T(N) \subseteq T(K)$ then $T([H : M]N) \subseteq T([H : M]K)$.
5. $T(H) = T([H : M]^nH)$ for each positive integer $n$.

**Proof.** (1) Since
\[ T(K + N)K + T(K + N)N = T(K + N)M, \]
we infer that $T(K + N)K$ and $T(K + N)N$ are comaximal submodules of $T(K + N)M$. It follows that $T(K + N)[K : M]$ and $T(K + N)[N : M]$ are comaximal ideals of $T(K + N)$. So for all positive integers $n$, we have
\[ (T(K + N)[K : M])^n + (T(K + N)[N : M])^n = T(K + N). \]
and hence
\[ T(K + N)[K : M]^n + T(K + N)[N : M]^n = T(K + N). \]
This gives that
\[ T(K + N)[K : M]^{n-1}K + T(K + N)[N : M]^{n-1}N = T(K + N)M, \]
as required.

(2) We have $T([K : M]H) \subseteq T([K : M][N : M]H)$ and $T([N : M]H) \subseteq T([K : M][N : M]H)$. It follows that
Let $z \in T([K : M][N : M]H)$. Then for some positive integer $n$,
Hence $z[K : M]^{n+1}[N : M]^{n+1}[H : M]^n H \subseteq M$. Since $K$ and $N$ are comaximal submodules of $M$, $[K : M]$ and $[N : M]$ are maximal ideals of $R$ and hence
\[ R = [K : M]^{n+1} + [N : M]^{n+1} \]. There exist \( a \in [K : M]^{n+1} \) and \( b \in [N : M]^{n+1} \) such that \( 1 = a + b \), and hence \( za [N : M]^{n+1} [H : M]^{n} H \subseteq M \). This gives that \( za [N : M] H : M]^{n} H \subseteq M \). Similarly, \( zb [[K : M] H : M]^{n} H \subseteq M \). Hence \( z = za + zb \in T ([N : M] H) + T ([K : M] H) \), and this shows that
\[
\]

(3) Let \( z \in T (K) T (N) \). Then \( z = \sum_{i=1}^{k} x_{i} y_{i} \) such that \( x_{i} \in T (K) \) and \( y_{i} \in T (N) \). It follows that \( x_{i} [K : M]^{n} K \subseteq M \) and \( y_{i} [N : M]^{n} N \subseteq M \) for some positive integers \( m \) and \( n \). Take \( m = \max \{ m_{1}, m_{2}, \ldots, m_{k} \} \) and \( n = \max \{ n_{1}, n_{2}, \ldots, n_{k} \} \). It follows that \( x_{i} [K : M]^{m+1} \subseteq R \) and hence \( x_{i} [K : M]^{(m+1)(n+1)} \subseteq R \). Similarly \( y_{i} [N : M]^{(m+1)(n+1)-1} N \subseteq M \). It follows that \( x_{i} y_{i} [K : M]^{(m+1)(n+1)-1} [N : M]^{(m+1)(n+1)-1} N \subseteq M \) and hence
\[
z [K : M]^{(m+1)(n+1)-1} [N : M]^{(m+1)(n+1)-1} N \subseteq M.
\]
So \( z \in T ([K : M] N) \) and hence \( T (K) T (N) \subseteq T ([K : M] N) \). Next, let \( w \in T ([K : M] N) \). There exists a positive integer \( m \) such that
\[
w [K : M]^{m} [K : M] N \subseteq M,
\]
and hence
\[
w [K : M]^{m+1} [N : M]^{m} N \subseteq M.
\]
It follows that \( w [K : M]^{m+1} \subseteq T (N) \). Since \( K \) is an invertible submodule of \( M, [K : M] \) and hence \( [K : M]^{m+1} \) is an invertible ideal of \( R \). Hence \( w \in [K : M]^{-(m+1)} T (N) \subseteq T ([K : M]) T (N) = T (K) T (N) \). This finally shows that \( T ([K : M] N) = T (K) T (N) \).

(4) Let \( x \in T ([H : M] N) \). Then \( x [H : M]^{m} [H : M] N \subseteq M \) for some positive integer \( n \). Hence \( x [H : M]^{m+1} [N : M]^{n} N \subseteq M \) and this shows that
\[
x [H : M]^{m+1} \subseteq T (N) \subseteq T (K) \].
\]
As \( H \) is a finitely generated submodule of \( M, [H : M] \) is a finitely generated ideal of \( R, [15, \text{Theorem 10}] \). Hence \( [H : M]^{m+1} \) is finitely generated. Let \( [H : M]^{m+1} = \sum_{i=1}^{t} R h_{i} \). Then for all \( i \), \( x h_{i} \in T (K) \) and hence \( x h_{i} [K : M]^{m} K \subseteq M \). Take \( m = \max \{ m_{1}, m_{2}, \ldots, m_{t} \} \). We have \( x h_{i} [K : M]^{m} K \subseteq M \), and hence \( x [H : M]^{m+1} [K : M]^{m} K \subseteq M \). It follows that \( x [H : M]^{m+1} [K : M]^{m+1} \subseteq R \). Therefore \( x \in T ([H : M]^{m+1} [K : M]^{m+1}) \). But
\[
T ([H : M]^{m+1} [K : M]^{m+1}) = T ([H : M] [K : M]) = T ([H : M] K).
\]
Thus \( x \in T ([H : M] K) \), and hence \( T ([H : M] N) \subseteq T ([H : M] K) \).

(5) Obviously, \( T (H) \subseteq T ([H : M]^{n} H) \). For the reverse inclusion, let \( x \in T ([H : M]^{n} H) \). Then \( x [H : M]^{m} H : M]^{m} H \subseteq M \) for some positive integer \( m \). Hence \( x [H : M]^{m(n+1)+n} H \subseteq M \), and \( x \in T (H) \). So \( T ([H : M]^{n} H) \subseteq T (H) \) and the equality holds. This finishes the proof of the theorem.

**Proposition 3.** Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module. Let \( K \) and \( N \) be submodules of \( M \). If
\[
T (K + N) (K + N) = T (K + N) M,
\]
then

**Proof.** By Theorem 1 (2), \( T(K) + T(N) \subseteq T([K : M]N) \). Let \( z \in T([K : M]N) \). Then \( z[[K : M]N : M]N \subseteq M \) for some positive integer \( n \). Hence \( z[K : M]^{n+1}[N : M]N \subseteq M \), and hence \( z[K : M]^{n+1}[N : M]^{n+1} \subseteq R \). Since \( T(K + N)K + T(K + N)N = T(K + N)M, T(K + N)K \) and \( T(K + N)N \) are comaximal submodules of \( T(K + N)M \). Hence \( T(K + N)[M : M] \) and \( T(K + N)[N : M] \) are comaximal ideals of \( T(K + N) \). This gives that
\[ [K : M]^{n+1}T(K + N) + [N : M]^{n+1}T(K + N) = T(K + N). \]

Write \( 1 = a + b \), where \( a \in [K : M]^{n+1}T(K + N) \subseteq [K : M]^{n+1}T(N) \) and \( b \in [N : M]^{n+1}T(K + N) \subseteq [N : M]^{n+1}T(K) \). Write \( z = za + zb \). We show that
\[ za \in T(N) \text{ and } zb \in T(K). \]

Since \( a \in [K : M]^{n+1}T(N) \), \( a = \sum_{i=1}^{r} a_i y_i \), where \( a_i \in [K : M]^{n+1} \) and \( y_i \in T(N) \). As \( y_i \in T(N) \), \( y_i [N : M]^{m}N \subseteq M \) for some positive integers \( m_i \). Let \( m = \max\{m_1, m_2, ..., m_r\} \). Then \( y_i [N : M]^{m}N \subseteq M \), for all \( i = 1, ..., r \). Hence
\[
\begin{align*}
az [N : M]^{m+n+1}N &= \sum_{i=1}^{r} a_i y_i [N : M]^{m+n+1}N \\
&\subseteq z[K : M]^{n+1}[N : M]^{n+1}([y_i [N : M]^{m}N] \\
&\subseteq RM = M.
\end{align*}
\]

This gives that
\[ az [N : M]^{m+n+1}N \subseteq M, \]
and hence \( az \in T(N) \). Similarly, \( bz \in T(K) \) and hence \( T([K : M]N) \subseteq T(K) + T(N) \). \( \square \)

The next result shows that a faithful multiplication Prufer module over an integral domain satisfies \( T_2 \)-property.

**Proposition 4.** Let \( R \) be an integral domain and \( M \) a faithful multiplication Prufer \( R \)-module. Then \( M \) satisfies \( T_2 \)-property.

**Proof.** Let \( N \) be a finitely generated submodule of \( M \). Then \( N \) is invertible in \( M \). Since \( T(N)N \subseteq T(N)M \), we have
\[ T(N)M = T(N)NN^{-1} \subseteq T(N)N^{-1}M, \]
and hence \( T(N) \subseteq T(N)N^{-1} = T(N)[N : M]^{-1} \subseteq T(N) \), [1, Lemma 1]. So \( T(N) = T(N)N^{-1} \) and hence \( T(N)N = T(N)N^{-1}N \), that is \( T(N)N = T(N)M \). Now, let \( K \) and \( N \) be finitely generated submodules of \( M \). Then \( K + N \) is invertible and hence \( T(K + N)(K + N) = T(K + N)M \). It follows by Proposition 3 that \( T([K : M]N) = T(K) + T(N) \) and \( M \) satisfies \( T_2 \)-property. \( \square \)

We remark that if \( R \) is an integral domain and \( M \) a faithful multiplication \( R \)-module and \( K \) and \( N \) are submodules of \( M \) such that \( K + N \) is invertible then \( M \) satisfies \( T_2 \)-property. In particular, every faithful multiplication Dedekind module over an integral domain satisfies \( T_1 \)-property.
Corollary 5. Let \( N \) be an invertible submodule of a faithful multiplication module \( M \) over an integral domain \( R \). Let \( P \) be a prime submodule of \( M \) then \( N \subseteq P \) if and only if \( T(N)P = T(N)M \).

Proof. Let \( N \subseteq P \). Since \( N \) is invertible, we have from the proof of Proposition 4 that

\[
T(N)M = T(N)N \subseteq T(N)P \subseteq T(N)M,
\]

so that \( T(N)P = T(N)M \). Conversely, let \( T(N)P = T(N)M \). Since \( M \) is finitely generated faithful multiplication (hence cancellation), we infer that \([P : M]T(N) = T(N) = T([N : M])\). Hence \( 1 = \sum_{i=1}^{n} p_is_i \), where \( p_i \in [P : M] \) and \( s_i \in T(N) \). By the definition of \( T(N) \), there exists a positive integer \( k \) such that \( s_i [N : M]^k N \subseteq M \), and hence \( s_i [N : M]^{k+1} \subseteq R \) for each \( i = 1, ..., k \). Thus, for each \( y \in [N : M]^{k+1} \), \( y = \sum_{i=1}^{n} p_i(s_iy) \in [P : M] \). Hence \([N : M]^{k+1} \subseteq [P : M] \).

Since \( P \) is a prime submodule of \( M \), \([P : M] \) is a prime ideal of \( R \). Hence \([N : M] \subseteq [P : M] \) and hence \( N = [N : M] M \subseteq [P : M] M = P \). \( \square \)

Let \( R \) be an integral domain and \( \text{spec}(R) \) the set of all prime ideals of \( R \). Recall that \( \text{Spec}(R) \) is Noetherian if and only if the ascending chain condition on radical ideals of \( R \) is satisfied, [8]. This fact can be generalized to multiplication modules. Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module. Let \( \text{Spec}(M) \) denote the set of all prime submodules of \( M \). Then \( \text{Spec}(M) \) is called Noetherian if the ascending chain condition on radical submodules of \( M \) holds.

Before we give our next result we need the following lemma.

Lemma 6. Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module.

1. An ideal \( I \) of \( R \) is radical if and only if \( IM \) is a radical submodule of \( M \).
2. \( N \) is a radical submodule of \( M \) if and only if \([N : M] \) is a radical ideal of \( R \).
3. \( \text{Spec}(R) \) is Noetherian if and only if \( \text{Spec}(M) \) is Noetherian.
4. \( \text{Spec}(M) \) is Noetherian if and only if for all submodules \( N \), \( \text{rad}N = \text{rad}N_0 \) for some finitely generated submodule \( N_0 \subseteq N \).
5. \( \text{Spec}(M) \) is Noetherian if and only if for all submodules \( N \) of \( M \), \( N \) has only finitely many minimal prime submodules.

Proof. (1) If \( I \) is radical, \( I = \sqrt{I} \) and hence \( IM = \sqrt{IM} = \sqrt{[IM : M]M} = \text{rad}(IM), [12, Proposition 2] \) and [7, Theorem 2.12]. Conversely, if \( IM \) is radical, then \( \text{rad}(IM) = IM \). Hence \( \sqrt{IM} = IM \). Since \( M \) is cancellation, \( \sqrt{I} = I \), and hence \( I \) is radical.

(2) If \( N \) is radical, \( N = \text{rad}N = \sqrt{[N : M]M}, [7, Theorem 2.12] \). Hence \([N : M]M = \sqrt{[N : M]M} \). As \( M \) is cancellation, \([N : M] = \sqrt{[N : M]} \), and \([N : M] \) is radical. The converse is obvious now.

(3) Suppose \( \text{Spec}(R) \) is Noetherian. Let \( N_1 \subseteq N_2 \subseteq ... \) be ascending chain of radical submodules of \( M \). Then \([N_1 : M] \subseteq [N_2 : M] \subseteq ... \) is ascending chain of radical ideals of \( R \). There exists \( k \in \mathbb{Z}^+ \) such that \([N_k : M] = [N_{k+1} : M] = ... \) and hence \( N_k = N_{k+1} = ... \) So \( \text{Spec}(M) \) is Noetherian. The converse is clear by using (1).
Let $N$ be a submodule of $M$. Hence $[N : M]$ is an ideal of $R$. By (3) Spec$(R)$ is Noetherian. It follows by [8, Theorem 3.1.11] that there exists a finitely generated ideal $I_0 \subseteq [N : M]$ such that $\sqrt{[N : M]} = \sqrt{I_0}$. Hence $\text{rad}N = \sqrt{[N : M]M} = \sqrt{I_0M} = \text{rad}(I_0M)$, [7, Theorem 2.12] and [12, Proposition 2]. Since $I_0 \subseteq [N : M]$ is finitely generated, $I_0M \subseteq N$ is finitely generated and the result follows. Conversely, let $\text{rad}N = \text{rad}N_0$ for some finitely generated submodule $N$ of $M$. Then $\sqrt{[N : M]} = \sqrt{[N_0 : M]}$ where $[N_0 : M]$ is a finitely generated ideal of $R$. By [8, Theorem 3.1.11] Spec$(R)$ is Noetherian and by (3) Spec$(M)$ is Noetherian.

(5) Let $N$ be a submodule of $M$. By (3), Spec$(R)$ is Noetherian and by [8, Theorem 3.1.11], $[N : M]$ has only finitely many minimal prime ideals of $R$. It follows by [7, Corollary 2.11] that $N$ has only finitely many minimal prime submodules of $M$. For the converse, suppose $N$ has only finitely many minimal prime submodules $P$ of $M$. Then $[N : M]$ has only finitely many minimal prime ideals $[P : M]$ of $R$, [7, Corollary 2.10]. It follows by [8, Theorem 3.1.11] that Spec$(R)$ is Noetherian and by (3), Spec$(M)$ is Noetherian.

**Proposition 7.** Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module. If $M$ is a Prufer module with Noetherian spectrum, then the transform formula holds for $M$.

**Proof.** Let $N$ be a submodule of $M$. By Lemma 6, $N$ has only finitely many minimal prime submodules of $M$ and hence $[N : M]$ has only finitely many prime ideals of $R$. It follows by [8, Theorem 3.2.5] that $T([N : M]) = T\left(\sqrt{[N : M]}\right)$, and hence $T(N) = T(\sqrt{[N : M]M}) = T(\text{rad}N)$. Let $K$ and $N$ be submodules of $M$. By Lemma 6, there exist finitely generated submodules $K_0$ and $N_0$ such that $\text{rad}K = \text{rad}K_0$ and $\text{rad}N = \text{rad}N_0$. It follows by [3, Proposition 16], [8, Theorem 3.2.5] and Proposition 4 that,

$$T([K : M]N) = T(\text{rad}[K : M]N) = T(\text{rad}K \cap \text{rad}N) = T(\text{rad}K_0 \cap \text{rad}N_0) = T([K_0 : M]N_0) = T(K_0) + T(N_0) = T(K) + T(N),$$

and hence $M$ satisfies the transform formula. □

We add other sufficient conditions for the formula $T([K : M]N) = T(K) + T(N)$ to be satisfied.

**Theorem 8.** Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module. Let $K$ and $N$ be submodules of $M$.

(1) Let $K + N$ be an invertible submodule of $M$. Then for any submodule $H$ of $M$,


(2) If $K$ is invertible and $T(K) + T(N)$ is a subring of $Q(R)$, then $T([K : M]N) = T(K) + T(N)$.

(3) Let $K$ be finitely generated. If $T(K) + T(N) = T(H)$ for some finitely generated submodule $H$ of $M$, then $T([K : M]N) = T(K) + T(N)$. 


Therefore, and hence $T$ rules of module. Let $T_N$ similarly, $N = [N_1 : M](K + N)$. So $K + N = ([K_1 : M] + [N_1 : M])(K + N)$, and this gives that

$$M = (K + N)(K + N)^{-1} = ([K_1 : M] + [N_1 : M])(K + N)(K + N)^{-1}$$

$$= ([K_1 : M] + [N_1 : M])M = K_1 + N_1,$$

that is $K_1$ and $N_1$ are comaximal submodules of $M$. Since $K = [K_1 : M](K + N)$, we infer that $[H : M]K = [K_1 : M][H : M](K + N)$. Similarly, $[H : M]N = [N_1 : M][H : M](K + N)$. It follows by Theorem 2(2) that


(2) Since $T(K) + T(N) = T([K_1 : M]) + T([N_1 : M])$ is a ring, we have that $T(K)T(N) = T([K_1 : M])T([N_1 : M])T([K_1 : M]) + T([N_1 : M]) = T(K) + T(N)$.

By Theorem 2, $T(K)T(N) = T([K_1 : M]N)$. Hence $T([K_1 : M]N) = T(K) + T(N)$, see Theorem 1(2).

(3) Since $T(K) \subseteq T(H)$, we get from Theorem 2(4) that


Since $[K_1 : M]H \subseteq H$

$$T(H) \subseteq T([K_1 : M]H) = T([H : M]K),$$

and hence $T(H) = T([K_1 : M]H)$. Next $T(N) \subseteq T(H) = T([K_1 : M]H)$ and $[K_1 : M]H$ is finitely generated. Again we use Theorem 2(4) to get

$$T([K_1 : M]N) \subseteq T([K_1 : M][H : M]N) \subseteq T([K_1 : M][H : M]^2)$$

$$= T([K_1 : M][H : M]) = T([H : M]) = T(H).$$

Therefore,

$$T(H) = T(K) + T(N) \subseteq T([K_1 : M]N) \subseteq T(H),$$

which implies that $T([K_1 : M]N) = T(K) + T(N)$. \qed

As a consequence of Theorem 8 we give the following corollary.

**Corollary 9.** Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module. Let $L$ be the set of all submodule transforms of finitely generated submodules of $M$. The following conditions are equivalent.

(1) $M$ is a $T_2$-module
(2) $(L, +, \cap)$ is a distributive lattice.
(3) $(L, +, \cap)$ is a lattice.
Proof. (1) $\implies$ (2) By Theorem (1), $T(K) \cap T(N) = T(K + N)$ and since $M$ is a $T_2$-module, $T(K) + T(N) = T([K : M] N)$. This is true for all finitely generated submodules $K$ and $N$ of $M$. Hence $(L, +, \cap)$ is a lattice. To show that $L$ is distributive, let $T(K)$, $T(N)$ and $T(H)$ be in $L$. Obviously, $(T(K) \cap T(H)) + (T(N) \cap T(H)) \subseteq (T(K) + T(N)) \cap T(H)$. To prove the reverse inclusion it is equivalent to show that $T([K : M] N + H) \subseteq T([K + H] : M)(N + H)$. But this follows from the fact that

$$[(K + H) : M](N + H) \subseteq [K : M](N + H).$$

(2) $\implies$ (3) Obvious.

(3) $\implies$ (1) Let $T(K)$, $T(N) \in L$. Since $L$ is closed under addition, there exists a finitely generated submodule $H$ of $M$ such that $T(K) + T(N) = T(H)$. Theorem 8 (3) shows that $M$ is a $T_2$-module.

We make two observations on Corollary 9. Let $\mathcal{L} = \{T(N) : N$ is a submodule of $M\}$. Then $(\mathcal{L}, +, \cap)$ is a distributive lattice if and only if $M$ satisfies $T_1$-property. Secondly, $T([K : N]M) = T(K \cap N)$ for submodules $K$ and $N$ of $M$ leads to an interesting connection between the submodules of $M$ and the submodule transforms of $M$, when $M$ is a $T_1$-module. Let $(S, +, \cap)$ be the lattice of all submodules of $M$. The map $(S, +, \cap) \to (\mathcal{L}, +, \cap)$ defined by $N \to T(N)$ is an order reversing lattice homomorphism which interchange “addition” and “intersection”.

Before we give the next results, we need a lemma.

Lemma 10. Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module. Let $N$ be a finitely generated submodule of $M$. Then for every multiplicatively closed subset $S$ of $R$, $T(N_S) = T(N)_S$.

Proof. Let $x \in T(N)_S = T(N) R_S$. Then $x = \sum_{i=1}^r y_i z_i$ where $y_i \in T(N)$ and $z_i \in R_S$. Choose a positive integer $m$ such that $y_i [N : M]^m N \subseteq M$. It follows that $y_i z_i [N : M]^m N \subseteq R_S M = MS$ for all $i$. Hence $x [N : M]^m N \subseteq MS$ and hence $x \in T(N_S)$. Conversely, let $N = \sum_{i=1}^k R_{n_i}$ and let $x \in T(N_S) = T \left( \sum_{i=1}^k (R_{n_i})_S \right) = \bigcap_{i=1}^k T(R_{n_i})_S$. Pick a positive integer $m$ such that $x [R_{n_i} : M]^m R_{n_i} \subseteq MS$ for each $i$. Hence $xs[R_{n_i} : M]^m n_i \subseteq M$ for each $i$, and hence $xs[R_{n_i} : M]^{m+1} \subseteq R$. It follows that $xs \in \bigcap_{i=1}^k T ([R_{n_i} : M]^m) = \bigcap_{i=1}^k T ([R_{n_i} : M])$. Since $N$ is finitely generated and $M$ finitely generated faithful multiplication $R$-module, we infer from [14, Proposition 1] that $[N : M] = \left[ \bigcap_{i=1}^k (R_{n_i}) : M \right] = \sum_{i=1}^k [R_{n_i} : M]$. It follows by [8, Theorem 3.2.3] that $T([N : M]) = \bigcap_{i=1}^k T([R_{n_i} : M])$ and hence $xs \in T([N : M]) = T(N)$, so $x \in T(N)_S$ and hence $T(N_S) \subseteq T(N)_S$. □

Now, we give the following property of faithful multiplication modules.
Theorem 11. Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module. Then $R$ is a $T_1$-domain (resp. $T_2$-domain) if and only if $M$ is a $T_1$-module (resp. $T_2$-module). If $R$ is a $T_3$-domain, then $M$ is a $T_3$-module.

Proof. Let $R$ be a $T_1$-domain. Let $K$ and $N$ be submodules of $M$. Then

$$T([K : M][N : M]) = T([K : M]) + T([N : M]),$$

and hence


and hence $M$ is a $T_1$-module. Conversely, let $M$ be a $T_1$-module and let $I, J$ be ideals of $R$. Then

$$T(IM : M)[JM] = T(IM) + T(JM),$$

and hence $T(IJM) = T(IM) + T(JM)$. This gives that

$$T(II) = T(IM : M) = T([IM : M]) + T([JM : M]) = T(I) + T(J),$$

and $R$ is a $T_1$-domain. The proof that $R$ is a $T_2$-domain if and only if $M$ is a $T_2$-module follows by the same proof above and the fact that $N$ is finitely generated if and only if $[N : M]$ is finitely generated and an ideal $I$ of $R$ is finitely generated if and only if $IM$ is a finitely generated submodule of $M$, [15 Theorem 10] and [4, Proposition 2.2]. Next, suppose $R$ is a $T_3$-domain. Let $K = Rk$ and $N = Rn$ be submodules of $M$. Then $[K : M]$ and $[N : M]$ are finitely generated multiplication ideals of $R$. Let $P$ be a maximal ideal of $R$. Then $[K : M]_P$ and $[N : M]_P$ are principal ideals of $R_P$. Since $R$ is a $T_3$-domain, $R_P$ is a $T_3$-domain, [8, Theorem 4.5.13]. Hence by [8, Lemma 4.4.3] we get that


and hence

$$\bigcap_P T([K : M][N : M])_P = \bigcap_P T(([K : M]) + T([N : M]))_P.$$ 

This implies that

$$T([K : M][N : M]) = T([K : M]) + T([N : M]),$$

and hence

$$T([K : M]N) = T(K) + T(N),$$

so $M$ is a $T_3$-module. This finishes the proof of the Theorem.

As a consequence of the above theorem, we give the following three corollaries.

Corollary 12. Let $R$ be an integral domain and $M$ a faithful multiplication Prufer module. If $K$ and $N$ are submodules of $M$ such that either $K$ or $N$ is finitely generated then $T([K : M]N) = T(K) + T(N)$. 

Proof. Let \( K \) be finitely generated. Since \( M \) is Prufer, \( K \) is invertible. Hence \([K : M]\) is invertible, \cite[Proposition 2.1]{2} and \cite[Lemma 3.3]{13}. Now, as \( M \) is Prufer, \( R \) is a Prufer domain, \cite[Theorem 3.6]{13} and \cite[Theorem 2.3]{2}. It follows by \cite[Theorem 4.5.9]{8} that

\[
T([K : M] [N : M]) = T([K : M]) + T([N : M]),
\]

and hence

\[
T([K : M] N) = T(K) + T(N),
\]

as required. \(\square\)

Corollary 13. Let \( R \) be an integral domain and \( M \) a faithful multiplication Noetherian \( R \)-module. The following conditions are equivalent:

(1) \( \dim M \leq 1 \).

(2) \( M \) is a \( T_1 \)-module.

(3) \( M \) is a \( T_3 \)-module.

Proof. It follows by \cite[Theorem 2.10 and Corollary 2.11]{7} that \( \dim R \leq 1 \) if and only if \( \dim M \leq 1 \). Also \( R \) is Noetherian, since \( M \) is multiplication Noetherian, \cite[p.767]{7}. The result follows by Theorem 11 and \cite[Theorem 4.5.20]{8}. \(\square\)

Corollary 14. Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module. If \( M \) is a Noetherian integrally closed module, then the following statements are equivalent:

(1) \( M \) is a Dedekind module.

(2) \( M \) is a \( T_1 \)-module.

(3) \( M \) is a \( T_3 \)-module.

Proof. By \cite[p.767]{7}, \( R \) is a Noetherian domain. Also by \cite[Lemma 23]{1}, \( R \) is integrally closed. Now the result follows by \cite[Corollary 4.5.21]{8}. \(\square\)

We close our work by the following property of faithful multiplication modules. It shows that if \( M \) is a faithful multiplication module over an integral domain, then \( M \) has the transform property if and only if its localization has.

Proposition 15. Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module. Then \( M \) is a \( T_2 \)-module (resp. \( T_3 \)-module) if and only if \( M_P \) is a \( T_2 \)-module (resp. \( T_3 \)-module) for every maximal ideal \( P \) of \( R \).

Proof. We prove the result for the \( T_2 \)-hypothesis. Let \( K, N \) be finitely generated submodule of \( M \). Hence \([K : M]\) and \([N : M]\) are finitely generated ideals of \( R \), \cite[Theorem 10]{15} and \cite[Proposition 2.2]{4}. Assume that for each maximal ideal \( P \) of \( R \), \( M_P \) is a \( T_2 \)-module (and hence \( R_P \) is a \( T_2 \) domain by Theroem 11). It follows by \cite[Lemma 4.4.3]{8} and Lemma 10 that

\[
T([K : M] N) = T([K : M] [N : M]) = \bigcap_P T([K : M] [N : M])_P
\]

\[
= \bigcap_P T([K : N]_P [N : M]_P) = \bigcap_P T([K : M]_P + T[N : M]_P)
\]

\[
= \bigcap_P T([K : M] + [N : M])_P = T([K : M]) + T([N : M])
\]

\[
= T(K) + T(N).
\]

The proof of the converse easily follows by Lemma 10. \(\square\)
Finally, we can show that the converse of the last statement of Theorem 11 is true. That is if $M$ is a $T_3$-module then $R$ is a $T_3$-domain. For let $a, b \in R$ and $P$ a maximal ideal of $R$. Then $(aM)_P$ and $(bM)_P$ are cyclic submodules of $M_P$. Since $M$ is a $T_3$-module, by Proposition 15, $M_P$ is a $T_3$-module. It follows that

$$T(abM)_P = T(abM_P) = T(aM_P) + T(bM_P) = T((aM) + T(bM))_P.$$  

This gives that $T(ab) = T(a)+T(b)$, and hence $T(Rab) = T(Ra) + T(Rb)$. Hence $R$ is a $T_3$-domain.

References


Majid M. Ali
Department of Mathematics and Statistics
Sultan Qaboos University
Muscat, Oman
mali@squ.edu.om