INVERTIBILITY OF MULTIPLICATION MODULES

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Abstract. All rings are commutative with identity and all modules are unital. We give several properties of invertible submodules of multiplication modules generalizing those of invertible ideals. We give characterizations of faithful multiplication Dedekind modules and derive a number of their properties.

1. Introduction

Let $R$ be a commutative ring with identity and $M$ an $R$-module. $M$ is a multiplication module if every submodule of $M$ has the form $IM$ for some ideal $I$ of $R$. Equivalently, $N = [N : M]M$, [5]. A submodule $K$ of $M$ is multiplication if $N \cap K = [N : K]K$ for all submodules $N$ of $M$, [17, Lemma 1.3]. Anderson [8] defined $\theta(M) = \sum_{m \in M} [Rm : M]$ and showed the usefulness of this ideal in studying multiplication modules. He proved for example that if $M$ is multiplication then $M = \theta(M)M$, and a finitely generated module $M$ is multiplication if and only if $\theta(M) = R$, [8, Proposition 1 and Theorem 1]. It is shown, [3, Corollary 1.2], that $M$ is multiplication if and only if $Rm = \theta(M)m$ for each $m \in M$, equivalently $R = \theta(M) + \text{ann}(m)$ for each $m \in M$. Several characterizations of multiplication module are given in [7, Theorem 2.1]. Finitely generated faithful multiplication modules are cancellation, [19, Corollary to Theorem 9]. It follows that if $M$ is a finitely generated faithful multiplication module then $I[N : M] = [IN : M]$ for all submodules $N$ of $M$ and all ideals $I$ of $R$. Multiplication modules have recently received considerable attention, see for example [4], [7]-[9], [13] and [19].

Let $N$ be a submodule of $M$ and $I$ an ideal of $R$. The residual submodule of $N$ by $I$ is $[N : M I] = \{x \in M : xI \subseteq N\}$. If $M$ is a multiplication module then $[N : M I] = [N : IM]M$, [2, Lemma 2.1]. If $M$ is a faithful multiplication module then $[0 : M I] = (\text{ann}I)M$. It follows from [2, Proposition 2.6] that if $N$ is a finitely generated submodule of a faithful multiplication module $M$ and $I$ is a finitely generated faithful multiplication ideal of $R$ such that $N \subseteq IM$, then $[N : M I]$ is a finitely generated submodule of $M$.

Let $M$ be an $R$-module. An ideal $I$ of $R$ is $M$-cancellation, ($M$-weak cancellation) if for all submodules $N, K$ of $M$, $IN = IK$ implies that $N = K$, $(N + [0 : M I] = \theta(M)M,$
2. Invertible Submodules of Multiplication Modules

In this section we consider invertible submodules of multiplication modules. We start with a result generalizing some facts about invertible ideals.

**Proposition 2.1.** Let $R$ be a ring and $M$ a multiplication $R$-module. Let $N$ be a submodule of $M$ and $I$ an ideal of $R$.

1. If $M$ is finitely generated faithful then $N$ is invertible in $M$ if and only if $[N : M]$ is an invertible ideal of $R$. 

$I = K + [0 : M I])$.

If $M$ is a faithful multiplication $R$-module then every finitely generated (faithful) multiplication ideal of $R$ is $M$-weak cancellation ($M$-cancellation), [2, Corollary 1.3].

Let $R$ be a commutative ring with identity. Let $S$ be the set of non-zero divisors of $R$ and $R_S$ the total quotient ring of $R$. For a non-zero ideal $I$ of $R$, let

$$I^{-1} = \{x \in R_S : xI \subseteq R\}.$$ 

$I$ is an invertible ideal of $R$ if $II^{-1} = R$. Let $M$ be an $R$-module and

$$T = \{t \in S : \text{for all } m \in M, tm = 0 \text{ implies } m = 0\}.$$ 

$T$ is a multiplicatively closed subset of $S$, and if $M$ is torsion free then $T = S$. In particular, $T = S$ if $M$ is a faithful multiplication module, see [9, Lemma 4.1]. Also $T = S$ if $M$ is an ideal of $R$.

Let $N$ be a non-zero submodule of $M$, and let

$$N^{-1} = \{x \in R_T : xN \subseteq M\}.$$ 

$N^{-1}$ is an $R$-submodule of $RT$, $R \subseteq N^{-1}$, and $NN^{-1} \subseteq M$. Following [15], $N$ is invertible in $M$ if $NN^{-1} = M$. Clearly $M$ is invertible in $M$, and it is proved in [15, Remark 3.8] that $R$ is an integral domain if and only if every non-zero cyclic submodule of faithful multiplication $R$-module $M$ is invertible in $M$.

Naoum and Al-Alwan, [15], introduced invertibility of submodules generalizing the concept for ideals and gave several properties and examples of such submodules. They also introduced Dedekind and Prüfer modules: an $R$-module $M$ is Dedekind (resp. Prüfer) if every non-zero (non-zero finitely generated) submodule of $M$ is invertible. They proved, [15, Theorems 3.4, 3.5 and 3.6], that a faithful multiplication $R$-module $M$ is Dedekind (Prüfer) if and only if $R$ is a Dedekind (Prüfer) domain. It is shown, [15, Note 1.6], that if $Rm$ is an invertible submodule of $M$ then $\text{ann}(Rm) = \text{ann}M$. More generally, if $N$ is an invertible submodule of $M$ then $\text{ann}N = \text{ann}M$. For if $r \in \text{ann}N$, then $rN = 0$, and hence $rM = rNN^{-1} = 0$. Then $\text{ann}N \subseteq \text{ann}M$. The other inclusion is always true. In this note we consider invertible submodules of multiplication modules, several properties of which are given in Proposition 2.1. We give several characterizations of faithful multiplication Prüfer and Dedekind modules, see Theorems 2.3 and 3.4 respectively. Among other properties of faithful multiplication Dedekind modules, we prove in Corollaries 3.8 and 3.9 that every submodule $N$ of a faithful multiplication Dedekind module $M$ may be generated by two elements, and every submodule of $M/N$ is cyclic.

All rings are commutative with identity and all modules are unital. For the basic concepts used, we refer the reader to [10], [11], [12], [14], [15], and [18].
(2) If \( M \) is finitely generated faithful and \( I \) is invertible then \( N \) is invertible in \( IM \) if and only if \([N : M] \) is invertible in \( M \).

(3) If \( M \) is finitely generated faithful and \( N \) is invertible in \( M \) then \( N \) is finitely generated faithful and multiplication.

(4) If \( M \) is finitely generated faithful and \( IN \) is invertible in \( M \) then \( N \) is invertible in \( M \) and \( I \) is an invertible ideal.

(5) Suppose \( M \) is non-torsion. If \( N \) is invertible in \( M \) then \( N \) is multiplication and contains a non-torsion element. The converse is true if \( R \) is a Bezout ring.

(6) If \( R \) is an integral domain, \( M \) is faithful and \( N \) is finitely generated faithful multiplication then \( N \) is invertible in \( M \).

(7) If \( R \) is an integral domain, \( M \) is faithful and \( N \) is finitely generated then \( N \) is invertible in \( M \) if and only if \( N \) is locally cyclic.

**Proof.** (1) [15, Remark 3.2 and Lemma 3.3].

(2) By [2, Lemma 2.1], \([N : M] I = [N : IM] M\). By [15, Remark 3.2], \( IM \) is invertible and hence by (1), \([N : IM] \) is an invertible ideal of \( R \). The result follows again by [15, Remark 3.2].

(3) If \( N \) is invertible in \( M \) then by (1), \([N : M] \) is an invertible ideal of \( R \), and hence is finitely generated faithful multiplication. The result follows by [9, Corollary 1.4].

(4) If \( IN \) is invertible in \( M \) then \((IN)^{-1}IN = M \). It is easy to verify that \((IN)^{-1}I \subseteq N^{-1}\). Hence

\[
M = (IN)^{-1}IN \subseteq N^{-1}N \subseteq M,
\]

so that \( M = N^{-1}N \) and \( N \) is invertible in \( M \). To show that \( I \) is invertible, we have by (3) that \( N \) is finitely generated faithful multiplication, and by (1), \( I = [IN : N] \) is invertible.

(5) \( M \) non-torsion means that there exists \( m \in M \) with \( \text{ann}(m) = 0 \). Hence \( M \) is faithful. By [3, Corollary 1.2], \( R = \theta(M) + \text{ann}(m) \), and hence \( \theta(M) = R \), so that \( M \) is finitely generated, [8, Proposition 1]. Suppose that \( N \) is invertible in \( M \). By (1), \([N : M] \) is an invertible ideal of \( R \), and hence \([N : M] \) contains a non-zero divisor of \( R \), say \( r \). Now \( rm \in rM \subseteq N \). Let \( s \in R \) with \( srm = 0 \). Then \( sr = 0 \), and hence \( s = 0 \) and \( rm \) is a non-torsion element of \( N \). By (3), \( N \) is multiplication. Conversely, suppose that \( N \) is a non-torsion multiplication submodule of \( M \), and \( R \) is a Bezout ring. \([N : M] \) is a multiplication ideal of \( R \), [1, Proposition 2.2] and [13, Lemma 1.4]. Let \( n \in N \) be non-torsion. \([Rn : M] \) is a finitely generated multiplication ideal of \( R \), [1, Proposition 2.2], [13, Lemma 1.4]. Hence it is principal since \( R \) is Bezout. Let \([Rn : M] = Ra \) for some \( a \in R \). Then \( Rn = [Rn : M]M = aM \), so that \( n \in aM \), and hence there exists \( 0 \neq m \in M \) such that \( n = am \). If \( s \in R \) with \( sa = 0 \) then \( sn = 0 \), and hence \( s = 0 \), so \( a \) is a non-zero divisor of \( R \). Next, \( a \in [Rn : M] \subseteq [N : M] \), so that \([N : M] \) is a multiplication ideal and contains a non-zero divisor. Therefore \([N : M] \) is invertible and hence \( N \) is invertible.
(6) As \( R \) is an integral domain, \( M \) is non-torsion and hence is finitely generated, \([4, p. 572]\). For \( 0 \neq n \in N \), \( Rn \) is an invertible submodule of \( M \) and \( Rn = JN \) for some ideal \( J \) of \( R \). By (4), \( N \) is invertible in \( M \).

(7) Again, \( M \) is finitely generated. If \( N \) is invertible, then \( N \) is multiplication by (3), and hence \( N \) is locally cyclic, \([5, Proposition 4]\). Conversely, let \( N \) be a finitely generated locally cyclic submodule of \( M \). Then \( N \) is multiplication, \([5, Proposition 5]\), and by (5) \( N \) is invertible.

An \( R \)-module \( M \) is called a Bezout module if for all \( m, n \in M \), \( Rm + Rn \) is cyclic, equivalently every finitely generated submodule of \( M \) is cyclic, see \([20, p. 33]\). In this note we call an \( R \)-module \( M \) a valuation module if for all \( m, n \in M \), either \( Rm \subseteq Rn \) or \( Rn \subseteq Rm \). Equivalently, for all submodules \( N, K \) of \( M \), either \( N \subseteq K \) or \( K \subseteq N \). Also we call an \( R \)-module \( M \) a greatest common divisor (GCD) module if the intersection of every two cyclic submodules of \( M \) is cyclic. It is clear that every valuation module is a Bezout module. Moreover, every Bezout module is a GCD module. For let \( m, n \) be non-zero elements of \( M \). Then \( Rm + Rn \) is a cyclic (and hence multiplication) submodule of \( M \). It follows by \([4, Corollary 2.4], [17, Corollary 3.4], and [19, Proposition 12]\), that \( Rm \cap Rn \) is a 2-generated submodule of \( M \), (in fact, \( Rm \cap Rn = Rxn + Rym \) for some \( x \in [Rm : Rn] \) and \( y \in [Rn : Rm] \), where \( x + y = 1 \)). As \( M \) is Bezout, \( Rm \cap Rn \) is cyclic, and hence \( M \) is a GCD module.

**Proposition 2.2.** Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module.

(1) \( M \) is a Bezout module if and only if \( R \) is a Bezout domain.

(2) \( M \) is a valuation module if and only if \( R \) is a valuation domain.

(3) \( M \) is a GCD module if and only if \( R \) is a GCD domain.

(4) \( M \) is a local module if and only if \( R \) is a local domain.

**Proof.** First we show that \( M \) is finitely generated. Let \( m \) be a non-zero element of \( M \). Since \( R \) is an integral domain, \( M \) is a \( D_1 \)-module, i.e. a module in which every non-zero cyclic submodule is invertible, \([15]\). It follows by \([15, Note 1.6]\) that \( \text{ann}(m) = \text{ann}M = 0 \). As \( M \) is a multiplication module, it follows by \([3, Corollary 1.2]\) that \( R = \theta(M) + \text{ann}(m) = \theta(M) \), and by \([8, Theorem 1]\), \( M \) is finitely generated. Alternatively, since \( M \) is multiplication, we obtain by \([8, Proposition 1]\) that \( M = \theta(M)M \). As \( M \) is faithful, \( \theta(M) \neq 0 \). Hence there exists \( m \in M \) such that \([Rm : M] \neq 0 \). Hence \( rM \subseteq Rm \) for some non-zero element \( r \in R \). It follows that \( \text{ann}(m) \subseteq \text{ann}(rM) = 0 \), and \( R = \theta(M) \) which gives that \( M \) is finitely generated.

(1) If \( R \) is a Bezout domain then \( M \) is a \( D_1 \)-module, \([15, p. 402]\). Let \( m, n \in M \). As \( M \) is multiplication, \( Rn = IM \) for some ideal \( I \) of \( R \). By Proposition 2.1(4), \( I \) is an invertible ideal of \( R \), hence finitely generated and therefore principal, say \( I = Ri \) with \( i \in I \). Then \( Rn = iM \), and hence \( im = rn \) for some \( r \in R \). Now \( Ri + Rr \) is a principal ideal of \( R \), and hence for some \( k \in M \),

\[
Rk = Rm + Rrm = Rrn + Rrm = Rr(Rn + Rm).
\]

As \( Rr \) is an \( M \)-cancellation ideal of \( R \), \([2, Corollary 1.3]\), we infer that \( Rn + Rm = [Rk : M Rr] \). Since \( Rk \subseteq rM \), it follows by \([2, Proposition 2.6 (3)]\) that \([Rk : M Rr] \)
is a cyclic module and hence $M$ is Bezout. Conversely, suppose that $M$ is a Bezout module and let $r, s \in R$. If $0 \neq m \in M$, then $Rrm + Rsm = (Rr + Rs)m$ is cyclic, say $(Rr + Rs)m = Rn$ with $n \in M$. As $M$ is a $D_1$-module, $Rn$ is invertible, and by Proposition 2.1(3), $\text{ann}(m) = 0$. Hence $Rm$ is a cancellation module, [19, Corollary to Theorem 9], so that $Rr + Rs = [Rn : Rm]$. Since $Rn \subseteq Rm$, it follows by [19, Proposition 13(iii)] that $[Rn : Rm]$ is a principal ideal, and $R$ is a Bezout domain.

(2) Let $M$ be a valuation module and let $a, b \in R$. For all $0 \neq m \in M$ we have $am, bm \in M$, and hence $Ram \subseteq Rbm$ or $Rbm \subseteq Ram$. Since $Rm$ is cancellation, it follows that $Ra \subseteq Rb$ or $Rb \subseteq Ra$. Conversely, let $R$ be a valuation domain and $m, n \in M$. Then $Rn = IM$ and $Rm = JM$ for some ideals $I, J$ of $R$. Since either $I \subseteq J$ or $J \subseteq I$, we get $Rn \subseteq Rm$ or $Rm \subseteq Rn$ and $M$ is valuation.

(3) Suppose that $R$ is a GCD domain and let $m, n \in M$. Then $Rn = IM$ for some ideal $I$ of $R$. $Rm$ is an invertible submodule of $M$ and hence it is faithful multiplication by Proposition 2.1. By [1, Proposition 2.2] and [13, Lemma 1.4], $I$ is a finitely generated multiplication ideal of $R$, hence is invertible. It follows by [11, Ex. 15, p. 42] that $I$ is principal, say $I = Ri$ with $i \in I$. Then $Rn = iM$, and hence there exists $r \in R$ such that $im = rn$. It follows by [9, Theorem 1.6], [1, Theorem 2.1] and [2, Proposition 2.7] that

$$r(Rn \cap Rm) = Rrn \cap Rrm = Rrm \cap Rrn = (Ri \cap Rr)m.$$  

Since $Ri \cap Rr$ is principal, we infer that $r(Rn \cap Rm) = Rk$ with $k \in M$. By [2, Proposition 2.6], $Rn \cap Rm = [Rk : M Rr]$ is a cyclic submodule of $M$, hence $M$ is a GCD-module. Conversely, suppose that $M$ is a GCD-module and let $r, s \in R$, $m \in M$. Since $R$ is an integral domain, $M$ is a $D_1$-module. Hence $Rm$ is invertible and by Proposition 2.1, $Rm$ is faithful and multiplication. By [9, Theorem 1.6] and [1, Theorem 2.1], $Rrm \cap Rsm = (Rr \cap Rs)m$ is a cyclic submodule of $M$, say $(Rr \cap Rs)m = Rk$ with $k \in M$. It follows by [19, Proposition 13] that $Rr \cap Rs = [Rk : Rm]$ is a principal ideal of $R$, and hence $R$ is a GCD domain.

(4) Suppose that $(R, P)$ is a local ring (not necessarily an integral domain) and $M$ is a non-zero multiplication $R$-module (not necessarily faithful). By [9, Theorem 2.5], $PM$ is the unique maximal submodule of $M$. Conversely, suppose that $M$ is a local module and $Q$ is the maximal submodule of $M$. Then $Q = PM$ for $P$ a maximal ideal of $R$, [9, Theorem 2.5], and $P$ is unique since $M$, being finitely generated faithful multiplication, is cancellation.

We use the following terminology in the next result. A submodule $N$ of $M$ is meet principal if $IN \cap N' = (I \cap [N' : N])N$ for all submodules $N'$ of $M$ and all ideals $I$ of $R$, and join principal if $[IN + N'] : N = I + [N' : N]$ for all submodules $N'$ of $M$ and all ideals $I$ of $R$. Following [6, p. 14], $N$ is a principal submodule of $M$ if $N$ is both meet and join principal.

An $R$-module $M$ is a Pr"ufer module if every non-zero finitely generated submodule of $M$ is invertible. If $M$ is a faithful multiplication module over an integral domain $R$ then $M$ is a Pr"ufer module if and only if $R$ is a Pr"ufer domain, [15]. In the next theorem we give several equivalent conditions for a faithful multiplication module over an integral domain to be a Pr"ufer module. Compare with [12, Theorem 6.6].

**Theorem 2.3.** Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module. The following conditions are equivalent.
(1) $M$ is a Prüfer module.
(2) $[N : K] + [K : N] = R$ for all finitely generated submodules $N, K$ of $M$.
(3) $[R^n : Rk] + [Rk : R^n] = R$ for all $n, k \in M$.
(4) Every non-zero 2-generated submodule of $M$ is invertible.
(5) $M_P$ is a valuation module for every prime ideal $P$ of $R$.
(6) $N_P$ is a cyclic submodule of $M_P$ for every finitely generated submodule $N$ of $M$ and every prime ideal $P$ of $R$.
(7) Every finitely generated submodule of $M$ is principal in $M$.
(9) $[(\sum_{i=1}^n N_i) : K] = \sum_{i=1}^n [N_i : K]$ for every finite collection of submodules $N_i (1 \leq l \leq n)$ of $M$ and every finitely generated submodule $K$ of $M$.
(10) $[K : \bigcap_{i=1}^n N_i] = \sum_{i=1}^n [K : N_i]$ for every finite collection of finitely generated submodules $N_i (1 \leq l \leq n)$ of $M$ and every submodule $K$ of $M$.
(11) $K \cap \sum_{i=1}^n N_i = \sum_{i=1}^n (K \cap N_i)$ for every finite collection of submodules $N_i (1 \leq l \leq n)$ of $M$ and every submodule $K$ of $M$.
(12) $K + \bigcap_{i=1}^n N_i = \bigcap_{i=1}^n (K + N_i)$ for every finite collection of submodules $N_i (1 \leq l \leq n)$ of $M$ and every submodule $K$ of $M$.

Proof. (1) $\Rightarrow$ (2) Let $M$ be a Prüfer module. Then $M$ is a $D_1$-module, and hence $R$ is an integral domain. This implies that $M$ is finitely generated. Let $N, K$ be finitely generated submodules of $M$. If either $N = 0$ or $K = 0$ the equality is obviously true. If $N \neq 0$ and $K \neq 0$ then $N + K$ is a non-zero finitely generated submodule of $M$, and is therefore invertible, and by Proposition 1.1, $N + K$ is multiplication. The result follows by [19, Corollary 3 to Theorem 1] and [17, Lemma 3.3].

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) $Rn + Rk$ is a finitely generated multiplication submodule of $M$, [17, Lemma 3.3] and [19, Corollary 3 to Theorem 1]. Hence, $[(Rn + Rk) : M]$ is a finitely generated multiplication ideal of $R$. Since $R$ is an integral domain, $[(Rn + Rk) : M]$ is invertible and by [15, Remark 3.2], $Rn + Rk = [(Rn + Rk) : M]M$ is an invertible submodule of $M$.

(4) $\Rightarrow$ (3) By Proposition 2.1(3), $Rn + Rk$ is multiplication, and the result follows by [17, Lemma 3.3] and [19, Corollary 3 to Theorem 1].

(3) $\Rightarrow$ (5) Let $P$ be a prime ideal of $R$. We must show that if $\frac{a}{x}, \frac{b}{y} \in M_P$ then either $R_P(\frac{a}{x}) \subseteq R_P(\frac{y}{x})$ or $R_P(\frac{b}{y}) \subseteq R_P(\frac{y}{x})$. Equivalently, $(Rk)_P \subseteq (Rn)_P$ or $(Rn)_P \subseteq (Rk)_P$. It follows from (3) that $[(Rn)_P : (Rk)_P] + [(Rk)_P : (Rn)_P] = R_P$. Hence, $[(Rn)_P : (Rk)_P] = R_P$ or $[(Rk)_P : (Rn)_P] = R_P$. This implies that $(Rn)_P \subseteq (Rk)_P$ or $(Rk)_P \subseteq (Rn)_P$, and $M_P$ is a valuation $R_P$-module.


(5) \(\implies\) (6) If \(N\) is finitely generated then \(NP\) is a finitely generated submodule of \(MP\), hence it is cyclic.

(6) \(\implies\) (7) If \(N\) is finitely generated and \(NP\) is cyclic for each prime ideal \(P\) of \(R\), then each \(NP\) is a principal submodule of \(MP\), \([6, \text{p. } 14]\), so that \(NP\) is meet- and join-principal in \(MP\). Therefore for all ideals \(I\) of \(R\) and all submodules \(N'\) of \(M\), the equations \(IN \cap N' = (I \cap [N' : N])N\) and \([(IN + N') : N] = I + [N' : N]\) are true locally and hence globally.

(7) \(\implies\) (1) Let \(N\) be a non-zero finitely generated submodule of \(M\). Since \(N\) is a principal submodule of \(M\), it is join-principal and hence, taking \(I = R\) in the definition, we have for all submodules \(N'\) of \(M\), \(N \cap N' = [N' : N]N\), and hence \(N\) is multiplication. The result follows by Proposition 2.1(5).

(8) \(\implies\) (9) Let \(N_i (1 \leq i \leq n)\) be a finite collection of submodules of \(M\) and \(K\) any finitely generated submodule of \(M\). Since \([N_1 : N] + [N_j : N] = R\) is true locally, we infer that the equation \(\sum_{i=1}^{n} N_i : K = \sum_{i=1}^{n} [N_i : K]\) is true locally and hence globally, \([4, \text{Corollary 1.2}]\).

(9) \(\implies\) (2) Let \(N_1, N_2\) be finitely generated submodules of \(M\). Then
\[
R = [N_1 + N_2 : N_1 + N_2] = [N_1 : N_1 + N_2] + [N_2 : N_1 + N_2] = [N_1 : N_2] + [N_2 : N_1].
\]

(2) \(\implies\) (10) See \([4, \text{Corollary 1.2}]\).

(10) \(\implies\) (2) For any finitely generated submodules \(N_1, N_2\) of \(M\),
\[
R = [(N_1 \cap N_2) : (N_1 \cap N_2)] = [(N_1 \cap N_2) : N_1] + [(N_1 \cap N_2) : N_2] = [N_2 : N_1] + [N_1 : N_2].
\]

(8) \(\implies\) (11) Follows by \([4, \text{Corollary 1.2}]\).

(11) \(\implies\) (12) By induction it suffices to assume \(n = 2\).
\[
(K + N_1) \cap (K + N_2) = (K \cap K) + (K \cap N_1) + (K \cap N_2) + (N_1 \cap N_2) = K + (N_1 \cap N_2).
\]

(12) \(\implies\) (11) Again it suffices to assume \(n = 2\).
\[
(N_1 \cap K) + (N_2 \cap K) = (N_1 + N_2) \cap (N_1 + K) \cap (N_2 + K) \cap (K + K) = (N_1 + N_2) \cap K.
\]

(11) \(\implies\) (2) From \(K \cap (N_1 + N_2) = (K \cap N_1) + (K \cap N_2)\) it follows by \([19, \text{Proposition 4}]\) that \([N_1 : N_2] + [N_2 : N_1] = R\).

Corollary 2.4. Let \(R\) be an integral domain and \(M\) a faithful multiplication \(R\)-module.

(1) \(M\) is a valuation module if and only if \(M\) is a Bezout local module.

(2) \(M\) is a Bezout module if and only if \(M\) is a Prüfer GCD module.

Proof. (1) By Proposition 2.2, \(M\) is valuation if and only if \(R\) is valuation if and only if \(R\) is a local Bezout domain, \([11, \text{Theorem 63}]\), if and only if \(M\) is a local Bezout module.

(2) By Proposition 2.2, \([15, \text{Theorem 3.6}]\) and \([11, \text{Ex. 15, p. 42}]\), \(M\) is a Bezout module if and only if \(R\) is a Bezout domain if and only if \(R\) is a Prüfer GCD domain if and only if \(M\) is a Prüfer GCD module.
It is useful to mention that the “only if” part of the Corollary 2.4(2) above can be proved as follows: Suppose that \( M \) is a faithful multiplication Bezout module. If \( N \) is a non-zero finitely generated submodule of \( M \), then \( N \) is cyclic. Since \( R \) is an integral domain, \( M \) is a \( D_1 \)-module and \( N \) is invertible, so that \( M \) is Prüfer. By the remark made before Proposition 2.2, \( M \) is a GCD module.

3. Dedekind Modules

Following [15], an \( R \)-module \( M \) is a Dedekind module if every non-zero submodule of \( M \) is invertible. A faithful multiplication module \( M \) over an integral domain \( R \) is Dedekind if and only if \( R \) is a Dedekind domain, [15, Theorems 3.4, 3.5]. In this section we give several more characterizations of faithful multiplication Dedekind modules. First we give three lemmas.

**Lemma 3.1.** Let \( R \) be a ring and \( M \) a faithful multiplication Dedekind module. Then every prime submodule of \( M \) is maximal.

**Proof.** Since \( M \) is Dedekind, \( M \) is a \( D_1 \)-module. Hence \( R \) is an integral domain, and therefore \( M \) is finitely generated. Suppose that \( P \) is a prime submodule of \( M \) which is not maximal. By [9, Theorem 2.5], there is a maximal submodule \( Q \) such that \( P \subsetneq Q \). Since \( Q \) is invertible in \( M \), it is multiplication by Proposition 2.1, and hence \( P = [P : Q]Q \). As \( M \) is finitely generated faithful multiplication, \( [P : M] = [P : Q][Q : M] \). Since \( P \) is a prime submodule of \( M \), \( [P : M] \) is a prime ideal of \( R \), and hence either \( [P : M] = [Q : M] \) or \( [P : M] = [P : Q] \). If \( [P : M] = [Q : M] \) then \( P = [P : M]M = [Q : M]M = Q \), a contradiction. If \( [P : M] = [P : Q] \) then \( P = [P : Q]M \), and by Proposition 2.1, \( [P : Q] \) is an invertible ideal of \( R \). As \( [P : Q]Q = P = [P : Q]M \), we infer that \( Q = M \), a contradiction. \( \square \)

**Lemma 3.2.** Let \( M \) be a faithful multiplication module over an integral domain \( R \). If \( P \) is maximal among all non-multiplication submodules of \( M \) then \( P \) is a prime submodule of \( M \).

**Proof.** Suppose not, and let \( a \in R, m \in M \) with \( am \in P, m \notin P, a \notin [P : M] \). Now \( P \subsetneq P + Rm \). As \( a \in [P : P + Rm] \) and \( a \notin [P : M] \), it follows that \( [P : M] \subseteq [P : P + Rm] \). But \( M \) is a cancellation module since it is finitely generated faithful multiplication. Thus \( P \subsetneq [P : P + Rm]M \), and therefore \( [P : P + Rm]M \) is a multiplication submodule of \( M \), and by [1, Proposition 2.2] and [13, Lemma 1.4], \( [P : P + Rm] \) is a multiplication ideal of \( R \). It follows by [9, Corollary 1.4] that \( P = [P : P + Rm](P + Rm) \) is a multiplication submodule of \( M \), a contradiction. \( \square \)

By Proposition 2.1, invertible submodules of a faithful multiplication module over an integral domain \( R \) are multiplication, so it follows from Lemma 3.2 that if \( M \) is a faithful multiplication module over an integral domain \( R \) and \( P \) is maximal among non-invertible submodules of \( M \), then \( P \) is a prime submodule of \( M \).

**Lemma 3.3.** Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module. If every non-zero prime submodule of \( M \) is invertible then \( M \) is a Dedekind module.
Theorem 3.4. Let $N$ be a CSM. Then $N$ is a CSM if and only if $M$ is a faithfully flat module of $M$. Let $\lambda \in \Lambda$ be a chain of elements of $C$. Then $N = \bigcup_{\lambda \in \Lambda} N_\lambda$ is a non-zero submodule of $M$. If $N$ is invertible in $M$ then by Proposition 2.1, $\bigcup_{\lambda \in \Lambda} [N_\lambda : M] = [N : M]$ is an invertible ideal of $R$, and hence there exist $a_i \in [N : M]$ and $x_i \in [N : M]^{-1}$ such that $1 = \sum_{i=1}^n a_i x_i$. Since $[N_\lambda : M](\lambda \in \Lambda)$ is a chain of ideals of $R$, there exists $\lambda_0 \in \Lambda$ such that $a_i \in [N_{\lambda_0} : M]$ for all $1 \leq i \leq n$. Moreover, $x_i \in [N : M]^{-1} \subseteq [N_{\lambda_0} : M]^{-1}$. Hence,

$$1 = \sum_{i=1}^n a_i x_i \in [N_{\lambda_0} : M][N_{\lambda_0} : M]^{-1},$$

so that $[N_{\lambda_0} : M][N_{\lambda_0} : M]^{-1} = R$, and $[N_{\lambda_0} : M]$ is an invertible ideal of $R$. By Proposition 2.1, $N_{\lambda_0}$ is therefore an invertible submodule of $M$, a contradiction. By Zorn’s lemma, if $C$ is non-empty, then it has a maximal element, which by Lemma 3.2 is a prime submodule of $M$, contradicting the choice of $C$. \hfill \Box

We will call an $R$-module $M$ a **cyclic submodule module** (CSM), if every submodule of $M$ is cyclic. In particular, $M$ is a cyclic module. Obviously a CSM is a Bezout module. A faithful multiplication module $M$ over an integral domain $R$ is a CSM if and only if $M$ is a PID. For let $M = Rm$, and let $I$ be an ideal of $R$. Then $Im$ is a cyclic submodule of $M$, say $Im = Rk$ with $k \in M$. Hence by [19, Proposition 13], $I = [Rk : Rm]$ is a principal ideal of $R$, and $R$ is a PID. Conversely, let $R$ be a PID and $M$ a multiplication (not necessarily faithful) $R$-module. By [6, Theorem 5.4(1)], $M$ is cyclic. Let $N$ be a submodule of $M$. Then $[N : M]$ is a principal ideal of $R$ and hence $N = [N : M]M$ is a cyclic submodule of $M$, and $M$ is a CSM.

**Theorem 3.4.** Let $R$ be a ring and $M$ a faithful multiplication $R$-module. The following are equivalent.

1. $M$ is a Dedekind module.
2. $R$ is an integral domain and every prime submodule of $M$ is invertible.
3. $M$ is a Noetherian Pr"ufer module.
4. $R$ is an integral domain, $M$ is Noetherian and $M_P$ is a CSM for each prime ideal $P$ of $R$.
5. $R$ is an integral domain, $M$ is Noetherian and every maximal submodule of $M$ is cancellation.
6. $R$ is an integral domain and every non-zero proper submodule $N$ of $M$ has the form $P_1^{r_1}P_2^{r_2} \cdots P_n^{r_n}M$ where $P_1, P_2, \cdots, P_n$ are comaximal prime ideals of $R$, and $r_1, r_2, \cdots, r_n$ are positive integers. Moreover, the prime ideals $P_1, P_2, \cdots, P_n$ are uniquely determined by $N$.

**Proof.**

1. $\implies$ 2. Obvious.
2. $\implies$ 1. Lemma 3.3.
1. $\implies$ 3. Clearly $M$ is Pr"ufer. Let $N$ be a non-zero submodule of $M$. Then $N$ is invertible. As $M$ is Dedekind and hence a $D_1$-module, $R$ is an integral
domain and therefore $M$ is finitely generated. It follows by Proposition 2.1 that $N$ is finitely generated, and hence $M$ is Noetherian.

(3) $\implies$ (4) Since $M$ is Prüfer, $M$ is a $D_1$-module and hence $R$ is an integral domain. Moreover, by Theorem 2.3, $M_P$ is a valuation module for each prime ideal $P$ of $R$. As $M$ is Noetherian, so too is $M_P$, and hence $M_P$ is a CSM.

(4) $\implies$ (1) Let $0 \neq N$ be a submodule of $M$. Let $P$ be a prime ideal of $R$. Since $M$ is Noetherian, $M$ and $N$ are finitely generated. Since $M_P$ is a CSM, $N_P$ is cyclic. By Proposition 2.1, $N$ is invertible.

(5) $\implies$ (1) Since $M$ is a faithful Noetherian module, $R$ is a Noetherian domain, [14, Theorem 3.5]. Let $Q$ be a maximal submodule of $M$. By [9, Theorem 2.5] there exists a maximal ideal $P$ of $R$ such that $Q = PM$. Since $M$ is finitely generated faithful multiplication, we infer again from [9, Theorem 2.5] that $P$ is unique and $Q \hookrightarrow P$ is a bijection between maximal submodules of $M$ and maximal ideals of $R$. For any ideals $I, J$ of $R$, if $IP = JP$ then $IQ = JQ$, and hence $I = J$. Thus every maximal ideal of $R$ is cancellation. It follows by [10, Ex. 12, p. 456] and [11, Ex. 12, p. 73] that $R$ is a Dedekind domain, and by [15, Theorem 3.4], $M$ is Dedekind.

(1) $\implies$ (5) Obvious.

(3) $\implies$ (6) Let $C$ be the set of all non-zero proper submodules of $M$ which are not representable in the form $IM$ with $I$ a product of powers of comaximal prime ideals of $R$. Suppose that $C \neq \emptyset$. Since $M$ is Noetherian, it follows by [18, p. 89] that $C$ has a maximal element, say $N$. By [9, Lemma 2.10] $N$ is a prime submodule of $M$, and by Lemma 3.1, $N$ is not a maximal submodule of $M$. It follows by [9, Theorem 2.5] that there exists a maximal submodule $Q$ of $M$ such that $N \subset Q$. As every non-zero submodule of $M$ is invertible, $Q$ is invertible and by Proposition 2.1, $Q$ is multiplication. Hence $N = [N : Q]Q$, and therefore $[N : M] = [N : Q][Q : M]$. It follows that $[N : M] \not\subset [N : Q]$. In fact, if $[N : M] = [N : Q]$ then $[N : Q] = [N : Q][Q : M]$, [19, Theorem 10(i)], and as $[N : Q]$ is an invertible ideal of $R$, $Q = M$, a contradiction. Since $M$ is finitely generated faithful multiplication, $N \not\subset [N : Q]M$, and hence $[N : Q]M \notin C$, and $Q \notin C$. Suppose that $Q = JM$ and $[N : Q]M = KM$, where $J, K$ are products of powers of comaximal prime ideals of $R$. Since $M$ is finitely generated faithful multiplication, $M$ is a cancellation module, and hence $[N : Q] = K$. This implies that $N = [N : Q]QJKM$, contradicting the choice of $N$, Hence, $C = \emptyset$. For uniqueness, suppose that $IM = JM$, where $I, J$ are products of powers of the comaximal prime ideals $\{P_1, \cdots, P_s\}$ and $\{Q_1, \cdots, Q_t\}$ respectively. If, say, $P_1 \notin \{Q_1, \cdots, Q_t\}$, then we consider two cases. First, if no $Q_j$ is comparable with $P_1$ then $P_1 + Q_j = R$ for all $j$. As $IM = JM$, we infer that for some $r \geq 0$, $(P_1^r)_{P_1} = M_{P_1}$, and by Nakayama’s lemma, $M_{P_1} = 0_{P_1}$, and this contradicts that $\text{ann}(M)_{P_1} = \text{ann}(M_{P_1}) = 0_{P_1}$. Second, if some $Q_j$ is comparable with $P_1$, say $P_1 \subset Q_1$, then since $R$ is a Dedekind domain, $R$ is of Krull dimension at most 1 so that any other $Q_j$ which is comparable with $P_1$ properly contains $P_1$. Localization of $IM = JM$ at $P_1$ then gives the same contradiction as the first case. Similarly if $Q_1 \subset P_1$.

(6) $\implies$ (1) Let $I$ be a non-zero ideal of $R$. Then $IM$ is a non-zero submodule of $M$ since $M$ is faithful. Hence $IM = JM$, where $J$ is a product of powers of comaximal prime ideals of $R$. Since $R$ is an integral domain, $M$ is finitely generated
and hence it is cancellation, [19, Corollary to Theorem 9]. It follows that \( I = J \), and hence \( R \) is a Dedekind domain. By [15, Theorem 3.4], \( M \) is Dedekind. 

The following five corollaries generalize some facts about Dedekind domains.

We say that a submodule \( P \) of an \( R \)-module \( M \) is indecomposable if for all ideals \( I \) of \( R \) and submodules \( N \) of \( M \), \( P = IN \) implies that either \( P = N \) or \( P = IM \). Prime submodules are indecomposable. In fact if \( P = IN \) and \( P \neq N \) and \( P \neq IM \), then there exist \( n \in N, i \in I, m \in M \) with \( n \notin P \) and \( im \notin P \). Then \( in \in IN = P \) but \( n \notin P \) and \( i \notin [P : M] \), so that \( P \) is not a prime submodule of \( M \).

**Corollary 3.5.** Let \( R \) be a ring and \( M \) a faithful multiplication Dedekind \( R \)-module. Let \( P \) be an indecomposable submodule of \( M \).

1. If \( P = IN \) for some ideal \( I \) of \( R \) and submodule \( N \) of \( M \), then either \( I = R \) or \( N = M \).

2. \( P \) is a prime submodule of \( M \).

**Proof.** (1) is clear since \( R \) is a Dedekind domain, [15, Theorem 3.5] and \( N \) is a finitely generated faithful multiplication submodule of \( M \) by Proposition 2.1.

(2) \( R \) is an integral domain by Theorem 3.4(2). Suppose that \( P = 0 \). Then \( P = 0M \), so that by [9, Corollary 2.11], \( P \) is a prime submodule of \( M \). Suppose that \( P 

It is well known that an integral domain \( R \) is a PID if and only if \( R \) is a Dedekind GCD domain. We extend this result to modules as follows.

**Corollary 3.6.** Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module. Then \( M \) is a CSM if and only if \( M \) is a Dedekind GCD module.

**Proof.** Suppose that \( M \) is a Dedekind GCD module. It follows by Corollary 2.4 that \( M \) is a Bezout module. Moreover, \( M \) is Noetherian, hence \( M \) is a CSM. Conversely, since \( R \) is an integral domain, \( M \) is a \( D_1 \)-module and hence it is Dedekind. Obviously a CSM is a GCD module.

Alternatively, suppose that \( M \) is a faithful multiplication module. By the remark made before Theorem 3.4, Proposition 1.2 and [15, Theorem 3.5], \( M \) is a Dedekind GCD module if and only if \( R \) is a Dedekind GCD domain if and only if \( R \) is a PID if and only if \( M \) is a CSM. 

Before proceeding further, we need the following lemma.

**Lemma 3.7.** Let \( R \) be a ring and \( A, B \) non-zero submodules of a faithful multiplication Dedekind \( R \)-module \( M \). Then there exists \( a \in A \) such that \( M = B + aA^{-1} \).

**Proof.** First of all, for any \( a \in A \), \( aA^{-1} \) is a submodule of \( M \) since \( aA^{-1} \subseteq AA^{-1} = M \). Suppose that \( B = P^r M \) where \( P \) is a non-zero prime ideal of \( R \) and \( r \) is a positive integer. \( A \) is cancellation, hence \( PA \subseteq A \), so let \( a \in A \) with \( a \notin PA \). Then \( Ra \notin PA \), and hence \( aA^{-1} \not\subseteq PM \). \( PM \) is a prime submodule of \( M \), [9, Lemma 2.10], and by Lemma 2.2 \( PM \) is maximal. Hence \( aA^{-1} + PM = M \), and by [17, Lemma 3.3] and [19, Corollary 3 of Theorem 8 and Corollary of Theorem 9],
Proof. This is immediate from the previous corollary since \( \bigoplus \) multiplication, we infer that \( R \). Then every submodule \( aA \) this finally gives that \( a \). Let \( 0 \) be chosen arbitrarily in \( N \). Putting \( mN \) \( M \). \( aA \) are comaximal prime ideals of \( R \), multiplication. By \( mN \) \( Rm \). Finally, \( M \) is finitely generated faithful multiplication, so \( [mN^{-1} : M] = [Rm : N] \), and hence \( Rm = [Rm : N]N \). Hence, \( mN^{-1} = [Rm : N]N^{-1} = [Rm : N]M \). Finally, \( M \) is finitely generated faithful multiplication, so \( [mN^{-1} : M] = [Rm : N] \), and hence \( Rm = [Rm : N]N = [mN^{-1} : M]N \). □

Corollary 3.8. Let \( R \) be a ring and \( M \) a faithful multiplication Dedekind \( R \)-module. Then every submodule \( N \) of \( M \) may be generated by two elements, one of which may be chosen arbitrarily in \( N\backslash\{0\} \).

Proof. Any submodule of \( M/N \) has the form \( K/N \) with \( N \subseteq K \subseteq M \), so \( K \neq 0 \) is multiplication. By [15, Theorem 3.5] \( R \) is a Dedekind domain and hence \( R \) has Krull dimension 1, so that \( R/[N : K] \) has Krull dimension \( \leq 1 \). Since \( R \) is Noetherian, \( R/[N : K] \) is a Noetherian ring and hence Artinian by [12, Theorem 7.4]. \( K/N \) is multiplication since \( K \) is. Using [9, p.761] and the fact that \( R/\text{ann}(K/N) = R/[N : K] \), we infer that \( K/N \) is a faithful multiplication \( R/[N : K] \)-module. It follows from [16, Theorem 1.5] that \( K/N \) is an Artinian \( R/[N : K] \)-module, and by [9, Corollary 2.9], we get that \( K/N \) is a cyclic \( R/[N : K] \)-module and hence a cyclic \( R \)-module. □

Corollary 3.9. Let \( R \) be a ring and \( N \) a non-zero submodule of a faithful multiplication Dedekind \( R \)-module \( M \). Then \( M/N \) is a CSM.

Proof. Let \( 0 \neq n \in N \). Putting \( B = Rn \) and \( A = N \) in the above lemma, we get that \( M = Rn + mN^{-1} \) for some \( m \in N \). Since \( M \) is finitely generated faithful multiplication, we infer that \( R = [Rn : M] + [mN^{-1} : M] \), and hence \( N = [Rn : M]N + [mN^{-1} : M]N \subseteq [Rn : M]M + [mN^{-1} : M]N = Rn + [mN^{-1} : M]N \). To complete the proof, we show that \( [mN^{-1} : M]N = Rm \). As \( m \in N \) and \( N \) is multiplication, \( Rm = [Rm : N]N \). Hence, \( mN^{-1} = [Rm : N]N^{-1} = [Rm : N]M \). Finally, \( M \) is finitely generated faithful multiplication, so \( [mN^{-1} : M] = [Rm : N] \), and hence \( Rm = [Rm : N]N = [mN^{-1} : M]N \). □

Corollary 3.10. Let \( R \) be a ring and \( N \) a non-zero submodule of a faithful multiplication Dedekind \( R \)-module \( M \). If \( N = P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n} M \) as in Theorem 3.4, then \( \bigoplus_{i=1}^n M/P_i^{r_i} M \) is a CSM.

Proof. This is immediate from the previous corollary since \( M/N \cong \bigoplus_{i=1}^n M/P_i^{r_i} M \). □

References


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