A FIXED POINT THEOREM IN THE MENER
PROBABILISTIC METRIC SPACE

ABDOLEHMAN RAZANI

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Abstract. In this article, a fixed point theorem in the Menger probabilistic
metric space is proved. In addition, the existence of a periodic point in this
space is shown. Finally, two questions would arise.

1. Introduction

The notion of a probabilistic metric space was introduced by Menger [4] in 1942.
Fixed point theory in this space is studied by many authors, for instance [2], [5],
[7] and so on. It is necessary to mention that fixed point theorems are main tools
for mathematicians to study the problem of existence of a solution for a system
of differential equations in probabilistic metric space, for instance see [2]. The
propose of this paper is to present a fixed point theorem in Menger probabilistic
metric space. Due to do this and for the sake of convenience, first, we recall some
definitions and notations in [1], [2], [3], [6] and [7].

Definition 1.1. A mapping $F : R \rightarrow R^+$ is called a distribution function
if it is nondecreasing and left-continuous with $\inf_{t \in R} F(t) = 0$ and
$\sup_{t \in R} F(t) = 1$.

Let $D^+$ be the set of all distribution functions $F$ such that $F(0) = 0$ ($F$ is a
nondecreasing, left-continuous mapping from $R$ into $[0, 1]$ such that $\sup_{x \in R} F(x) = 1$).

Definition 1.2. A probabilistic metric space (briefly, PM-space) is an order pair
$(S, F)$ where $S$ is a nonempty set and $F : S \times S \rightarrow D^+$ ($F(p, q)$ is denoted by $F_{p,q}$
for every $(p, q) \in S \times S$) satisfies the following conditions:

1. $F_{u,v}(x) = 1$ for all $x > 0$ if and only if $u = v$ ($u, v \in S$).
2. $F_{u,v}(x) = F_{v,u}(x)$ for all $u, v \in S$ and $x \in R$.
3. If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$, then $F_{u,w}(x + y) = 1$ for $u, v, w \in S$ and
$x, y \in R$.

If only 1 and 2 hold, the ordered pair $(S, F)$ is said to be a probabilistic semimetric
space.

Definition 1.3. A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a triangular norm
(abbreviated, a $t$-norm) if the following conditions are satisfied:

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Example 1.4. The following are the four basic t-norms:
(I) The minimum t-norm, $T_M$, is defined by

$$T_M(x, y) = \min(x, y),$$

(II) The product t-norm, $T_P$, is defined by

$$T_P(x, y) = x \cdot y,$$

(III) The Lukasiewicz t-norm, $T_L$, is defined by

$$T_L(x, y) = \max(x + y - 1, 0),$$

(IV) The weakest t-norm, the drastic product, $T_D$, is defined by

$$T_D(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

As regards the pointwise ordering, we have the inequalities

$$T_D < T_L < T_P < T_M.$$

Definition 1.5. A Menger probabilistic metric space (briefly, Menger PM-space) (see [6]) is a triple $(S, F, T)$, where $(S, F)$ is a probabilistic metric space, $T$ is a triangular norm (abbreviated t-norm) and the following inequality holds:

$$F_{u,v}(x + y) \geq T(F_{u,w}(x), F_{w,v}(y)), \quad (1)$$

for all $u, v, w \in S$ and every $x > 0, y > 0$.

Schweizer, Sklar and Thorp [7] proved that if $(S, F, T)$ is a Menger PM-space with $\sup_{0 < t < 1} T(t, t) = 1$, then $(S, F, T)$ is a Hausdorff topological space in the topology $\tau$ induced by the family of $(\varepsilon, \lambda)$-neighborhoods

$$\{U_p(\varepsilon, \lambda) : p \in S, \varepsilon > 0, \lambda > 0\},$$

where

$$U_p(\varepsilon, \lambda) = \{u \in S : F_{u,p}(\varepsilon) > 1 - \lambda\}.$$

Definition 1.6. Let $(S, F, T)$ be a Menger PM-space with $\sup_{0 < t < 1} T(t, t) = 1$.

1. A sequence $\{u_n\}$ in $S$ is said to be $\tau$-convergent to $u \in S$ (we write $u_n \to u$) if for any given $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{u_n, u}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.

2. A sequence $\{u_n\}$ in $S$ is called a $\tau$-Cauchy sequence if for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{u_n, u_m}(\varepsilon) > 1 - \lambda$, whenever $n, m \geq N$.

3. A Menger PM-space $(S, F, T)$ is said to be $\tau$-complete if each $\tau$-Cauchy sequence in $S$ is $\tau$-convergent to some point in $S$. 
The rest of the paper is organized as follows: In Section 2, a fixed point theorem is given. Finally, the existence of a periodic point is proved in Section 3.

2. Fixed Point Under Contractive Map

In this section, the definition of contractive map is rewritten and an iterative theorem is proved. This theorem shows that if we have the existence of a convergent subsequence of an iterate sequence (of a contractive map) then we can prove the existence of a fixed point. In order to do this, we recall the following definition:

**Definition 2.1.** Let \((S, F, T)\) be a Menger PM-space. We say the mapping \(f : S \to S\) is contractive, if for all \(u \neq v \in S\),

\[
    F_{f(u), f(v)}(x) \geq F_{u, v}(x),
\]

for all every \(x > 0\), but \(F_{f(u), f(v)} \neq F_{u, v}\).

**Lemma 2.2.** Let \((S, F, T)\) be a Menger PM-space, where \(T\) is a continuous \(t\)–norm. Let \((u_n, v_n, x_n)\) be a sequence in \(S \times S \times \mathbb{R}^+\) and \((u_n, v_n, x_n) \to (u, v, x_0)\), where \(u, v \in S\) and \(x_0 > 0\). If \(F_{u, v}\) is continuous at \(x_0\), then \(F_{u_n, v_n}(x_n) \to F_{u, v}(x_0)\).

**Proof.** Note that

\[
    \liminf_{n} F_{u_n, v_n}(x_n) \geq \liminf_{n} T(F_{u_n, u}(\delta), F_{u, v}(x_n - 2\delta)), F_{v, v}(\delta))
\]

\[
= T \left( T(1, \liminf_{n} F_{u, v}(x_n - 2\delta)), 1 \right)
\]

\[
= \liminf_{n} F_{u, v}(x_n - 2\delta)
\]

\[
\geq F_{u, v}(x_0 - 3\delta)\quad \text{since } x_n > x_0 - \delta \text{ eventually,}
\]

and so \(\liminf_{n} F_{u_n, v_n}(x_n) \geq F_{u, v}(x_0^-)\) since \(\delta > 0\) is arbitrary. By the same argument, \(\limsup_{n} F_{u_n, v_n}(x_n) \leq F_{u, v}(x_0^+)\). Since \(F_{u, v}\) is continuous at \(x_0\), then

\[
    F_{u_n, v_n}(x_n) \to F_{u, v}(x_0).
\]

Now we have the main theorem as follows:

**Theorem 2.3.** Let \((S, F, T)\) be a Menger PM-space, where \(T\) is a continuous \(t\)–norm. If \(A\) is a contractive mapping of \(S\) into itself such that there exists a point \(u \in S\) whose sequence of iterates \((A^n(u))\) contains a convergent subsequence \((A^{n_i}(u))\); then \(\xi = \lim_{i \to \infty} A^{n_i}(u) \in S\) is a unique fixed point.

**Proof.** Suppose \(A(\xi) \neq \xi\) and consider the sequence \((A^{n_i+1}(u))\) which, it can easily be verified, converges to \(A(\xi)\).

Choose \(x\) to be such that

\[
    F_{A(\xi), A^2(\xi)}(x) > F_{\xi, A(\xi)}(x),
\]

with \(F_{A(\xi), A^2(\xi)}\) and \(F_{\xi, A(\xi)}\) are both continuous at \(x\). This is possible, because there must be some value \(x_1 > 0\) with

\[
F_{A(\xi), A^2(\xi)}(x_1) > F_{\xi, A(\xi)}(x_1),
\]
and since $F_{A(\xi), A^2(\xi)}$ and $F_{\xi, A(\xi)}$ are both left continuous at $x_1$, we have
\[ F_{A(\xi), A^2(\xi)}(x) > F_{\xi, A(\xi)}(x) \]
for all $x$ in some interval. Each function has at most countably many discontinuities, so a point can be found in this interval at which both are continuous.

Now, construct neighborhoods $B_1, B_2$ of $\xi, A(\xi)$, respectively, such that for some $\varepsilon > 0$ and all $u \in B_1, v \in B_2$ we have
\[ F_{A(u), A(v)}(x) > F_{u, v}(x) + \varepsilon. \]
For large enough $i$, we have
\[ F_{A^{n_i+1}(u), A^{n_i+2}(u)}(x) > F_{A^{n_i+1}(u), A^{n_i+2}(u)}(x) + \varepsilon. \]
Applying $A$ a few more times, gives
\[ F_{A^{n_i+1}(u), A^{n_i+1+i}(u)}(x) > F_{A^{n_i+1}(u), A^{n_i+1+i}(u)}(x) + \varepsilon, \]
and hence
\[ F_{A^{n_i+1}(u), A^{n_j+i}(u)}(x) > F_{A^{n_i+1}(u), A^{n_j+i}(u)}(x) + (j-i)\varepsilon \]
for sufficiently large $i$ and $j$. Letting $j \to \infty$ gives the contradiction.

In order to prove the uniqueness of $\xi$, suppose there is a $\eta \neq \xi$ with $A(\eta) = \eta$, then it follows that
\[ F_{\xi, \eta}(x) = F_{A(\xi), A(\eta)}(x) < F_{\xi, \eta}(x), \]
which is contradiction. This proves the uniqueness and, thus, accomplishes the proof of this theorem. \qed

Theorem 2.3 implies some information on the convergence of a sequence of iterates.

**Remark 2.4.** Let all assumptions of Theorem 2.3 hold. If $(A^n(u)), u \in S$, contains a convergent subsequence $(A^{n_i}(u))$, then $\lim_{n \to \infty} A^n(u)$ exists and coincides with the fixed point $\xi$.

**Proof.** We have $\lim_{i \to \infty} A^{n_i}(u) = \xi$. Given $1 > \delta > 0$ there exists, then, a positive integers $N_0$ such that $i > N_0$ implies $F_{\xi, A^{n_i}(u)}(t) > 1 - \delta$. If $m = n_i + l$ ($n_i$ fixed, $l$ variable) is any positive integer $> n_i$ then
\[ F_{\xi, A^{n_i}(u)}(x) = F_{A^l(\xi), A^n(\xi)(u)}(x) > F_{\xi, A^{n_i}(u)}(x) > 1 - \delta, \]
which proves the above assertion. \qed

### 3. Periodic Points

In this section, first, we define a periodic point or an eventually fixed point. Then we prove the existence of a periodic point in the Menger PM-space. Finally, two questions would arise.

**Definition 3.1.** Let $(S, F, T)$ be a Menger PM-space, and $f$ is a self-mapping of $S$. Then $\xi$ is a periodic point or an eventually fixed point, if there exists a positive integer $k$ such that $f^k(\xi) = \xi$.
Definition 3.2. Let \((S, F, T)\) be a Menger PM-space, a mapping \(f : S \to S\) is called locally contractive if and only if:

\[
\forall u \in S \exists 0 < \lambda < 1 \forall p, q \in \{v \in S : F_{n+1}(x) > 1 - \lambda\} \quad F_{p,q}(x) \leq F_{f(p), f(q)}(x) \tag{3}
\]

for all \(x > 0\).

Definition 3.3. Let \((S, F, T)\) be a Menger PM-space. A mapping \(f : S \to S\) is called \(\lambda\)-uniformly locally contractive if and only if:

\[
\text{there exists a positive integer } N \text{ such that for all } x > 0, \quad \text{Definition } 3.3.
\]

Theorem 3.4. Let \((S, F, T)\) be a Menger PM-space, with a continuous \(t\)-norm \(T\) defined as \(T(a, b) = T_M(a, b)\) for \(a, b \in [0, 1]\). Suppose \(f\) is a \(\lambda\)-uniformly locally contractive self-map of \(S\) such that

there exists a point \(u \in S\) whose sequence of iterates \((f^n(u))\) contains a convergent subsequence \((f^{n_i}(u))\),

then \(\xi = \lim_{i \to \infty} f^{n_i}(u)\) is a periodic point of \(f\).

Proof. By the condition (4), there exists a positive integer \(N_1\) such that for \(i > N_1\) implies

\[
F_{f^{n_i}(u), \xi}(x) > 1 - \lambda \quad \text{for all } 0 < \lambda < 1 \quad \text{and } x > 0. \tag{5}
\]

Also \(f\) is \(\lambda\)-uniformly locally contractive, thus the last inequality implies

\[
F_{f^{n_{i+1}}(u), f(\xi)}(x) \geq F_{f^{n_i}(u), \xi}(x)
\]

and so \(F_{f^{n_{i+1}}(u), f(\xi)}(x) > 1 - \lambda\). After \(n_{i+1} - n_i\) iterations we obtain:

\[
F_{f^{n_{i+1}}(u), f^{n_{i+1} - n_i}(\xi)}(x) > 1 - \lambda.
\]

Note that

\[
F_{f^{n_{i+1} - n_i}(\xi)}(x) \geq T(F_{\xi, f^{n_{i+1} - n_i}(\xi)}(x_0), F_{f^{n_{i+1}}(u), f^{n_{i+1} - n_i}(\xi)}(x_i))
\]

\[
> T((1 - \lambda), (1 - \lambda)), \tag{6}
\]

where \(x = x_0 + x_1\), and also the last equality is hold because \(F_{f^{n_{i+1} - n_i}(\xi)}(x_0) > 1 - \lambda\) by (5) and \(F_{f^{n_{i+1}}(u), f^{n_{i+1} - n_i}(\xi)}(x_1) > 1 - \lambda\) by the same argument as above for \(x_1\) instead of \(x_0\). Now, due to the definition of \(T_M\), i.e. \(T_M(a, b) = \min\{a, b\}\), we obtain:

\[
F_{f, f^{n_{i+1} - n_i}(\xi)}(x) > 1 - \lambda. \tag{7}
\]

Suppose that \(\eta = f^{n_{i+1} - n_i}(\xi) \neq \xi\). If we call \(A = f^{n_{i+1} - n_i}\), then by the same proof of Theorem 2.3, we obtain a contradiction. Hence, putting \(k = n_{i+1} - n_i\), we have

\[
F_{\xi, f^{n_{i+1} - n_i}(\xi)}(x) > 1 - \lambda.
\]

Corollary 3.5. If, in Theorem 3.4, \(F_{\xi, f(\xi)}(x) > 1 - \lambda\), then \(k = 1\). Indeed

\[
F_{\xi, f^{\xi}(\xi)}(x) = F_{\xi, f(\xi)}(x),
\]

hence, \(f(\xi) \neq \xi\) contradicts (4).

Question 1. It is natural to ask whether Theorem 3.4 would remain true if \(\lambda\)-uniformly locally contractive self-map is substituted by locally contractive self-map.

Question 2. It is natural to ask whether Theorem 3.4 would remain true if \(T\) is defined in general case.
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References