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ON RELATIVE STAR-LINDELÖF SPACES

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Abstract. In this paper, we prove the following statements:

(1) There exist a Tychonoff space X and a subspace Y of X such that Y is strongly star-Lindelöf in X and $e(Y,X)$ is arbitrarily large, but X is not star-Lindelöf.

(2) There exist a Tychonoff space X and a subspace Y of X such that Y is star-Lindelöf in X, but Y is not strongly star-Lindelöf in X.

1. Introduction

By a space, we mean a topological space. Recall from [**1, 2, 6**] that a subspace Y of a space X is *Lindelöf* in X if for every open cover $\mathcal{U}$ of X, there exists a countable subfamily covering Y. A space X is *star-Lindelöf* (by different names, see [**3, 4, 7**]) if for every open cover $\mathcal{U}$ of X, there exists a countable subset F of X such that $St(F,\mathcal{U}) = X$, where $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \varnothing\}$. These definitions motivate us to introduce the following concepts.

Definition 1.1. A subspace Y of a space X is called *strongly star-Lindelöf* in X if for every open cover $\mathcal{U}$ of X, there exists a countable subset $F \subseteq Y$ such that $Y \subseteq St(F,\mathcal{U})$.

Definition 1.2. A subspace Y of a space X is called *star-Lindelöf* in X if for every open cover $\mathcal{U}$ of X, there exists a countable subset $F \subseteq X$ such that $Y \subseteq St(F,\mathcal{U})$.

From the above definitions, it is clear that if Y is strongly star-Lindelöf in X then Y is star-Lindelöf in X, but the converse does not hold (see below Example 2.3). Recall that the *extent* $e(X)$ of a space X is the smallest cardinal number τ such that the cardinality of every discrete closed subset in X has cardinality at most τ; Moreover, Arhangel'skii [**1**] defined the *extent* $e(Y,X)$ of Y in X is the smallest cardinal number τ such that the cardinality of every closed in X discrete subspace of Y is not greater than τ. It is well-known that the extent of a Lindelöf space is countable. Matveev [**8**] proved that the extent of a Tychonoff star-Lindelöf space can be arbitrarily large. Arhangel'skii [**1**] proved that if Y is Lindelöf in X, then $e(Y,X)$ is countable. It is natural for us to consider the following questions.

Question 1.3. Do there exist a Tychonoff space X and a subspace Y of X such that Y is strongly star-Lindelöf in X and $e(Y,X)$ is arbitrarily large, but X is not star-Lindelöf?

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Question 1.4. Do there exist a Tychonoff space X and a subspace Y of X such that Y is star-Lindelöf in X, but Y is not strongly star-Lindelöf in X?

The purpose of this paper is to answer positively the above questions by constructing two examples. Moreover, the cardinality of a set A is denoted by $|A|$. For a cardinal κ, κ^+ denotes the smallest cardinal greater than κ. Let ω denote the first infinite cardinal and $\mathfrak{c}$ the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each pair of ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$. Other terms and symbols that we do not define will be used as in [**5**].

2. Two Examples on Relative Star–Lindelöf Spaces

In this section, we construct two examples stated in the abstract. The first example uses Matveev's space. We now sketch the construction of Matveev's space Y defined in the proof of Theorem 1 of [**8**]. Let κ be an infinite cardinal and $D = \{0, 1\}$ be the two-point discrete space. For every $\alpha < \kappa$, let z_α be the point of D^κ defined by $z_\alpha(\alpha) = 1$ and $z_\alpha(\beta) = 0$ for $\beta \neq \alpha$. Put $Z = \{z_\alpha : \alpha < \kappa\}$. Let τ be a cardinal such that $cf(\tau) > \kappa$. Matveev's space M is defined to be the subspace

$$M \ = \ (D^\kappa \times \tau) \cup (Z \times \{\tau\})$$

of the product space $D^\kappa \times (\tau + 1)$. Then, M is a Tychonoff pseudocompact space with $e(Y) \geq \kappa$, since $Z \times \{\tau\}$ is a discrete closed set in Y. Matveev [**8**] proved that M is star-Lindelöf.

We need the following lemma:

Lemma 2.1 ([**9**], [**10**]). *For each family $\{V_\alpha : \alpha < \kappa\}$ of open sets in D^κ such that $z_\alpha \in V_\alpha$ for each $\alpha < \kappa$, there exists a countable set $S \subseteq D^\kappa$ such that $S \cap V_\alpha \neq \varnothing$ for each $\alpha < \kappa$ and $\mathrm{cl}_{D^\kappa} S \cap Z = \varnothing$.*

To construct the first example, we use the Alexandroff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is a set of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X. It is well-known that $A(X)$ is countably compact iff X is countably compact.

Example 2.2. For every regular uncountable cardinal κ, there exist a Tychonoff space X and a subspace Y of X such that Y is strongly star-Lindelöf in X and $e(Y, X) \geq \kappa$, but X is not star-Lindelöf.

Proof. Let κ be a regular uncountable cardinal and τ a cardinal such that $cf(\tau) \geq \kappa$. Let $X = A(M)$. Then, X is a Tychonoff space. Let

$$Y \ = \ A(M) \setminus \big((Z \times \{\tau\}) \times \{1\}\big).$$

Then, Y is a subspace of X. Let

$$Y' \ = \ (Z \times \{\tau\}) \times \{0\}.$$

Then, Y' is closed in X discrete subspace of Y and $|Y'| = \kappa$. Hence, $e(Y, X) \geq \kappa$.

First, we show that Y is strongly star-Lindelöf in X. Let $\mathcal{U}$ be an open cover of X. For every $\alpha < \kappa$, choose $U_\alpha \in \mathcal{U}$ such that $\langle\langle z_\alpha, \tau\rangle, 0\rangle \in U_\alpha$. Further, for every $\alpha < \kappa$ choose $\beta_\alpha < \tau$ and an open neighborhood V_α of z_α in D^κ such that

$$\big(((V_\alpha \cap Z) \times \{\tau\}) \times \{0\}\big) \cup (V_\alpha \times (\beta_\alpha, \tau) \times \{0,1\}\big) \subseteq U_\alpha.$$

By Lemma 2.1, there exists a countable set $S = \{s_i : i \in \omega\} \subseteq D^\kappa$ such that $S \cap V_\alpha \neq \varnothing$ for every $\alpha < \kappa$. Let $\beta^* = \sup\{\beta_\alpha : \alpha < \kappa\}$. Than, $\beta^* < \tau$, since $cf(\tau) > \kappa$. Let $S' = (S \times \{\beta^*\}) \times \{0\}$. Then, $Y' \subseteq St(S', \mathcal{U})$, since $U_\alpha \cap S' \neq \varnothing$ for each $\alpha < \kappa$. Observe that $A(D^\kappa \times \tau)$ is countably compact, since $D^\kappa \times \tau$ is countably compact and τ has the uncountable cofinality. Thus, we can find a finite set $F' \subseteq A(D^\kappa \times \tau)$ such that $A(D^\kappa \times \tau) \subseteq St(F', \mathcal{U})$. If we put $F = F' \cup S'$, then F is a countable subset of Y and $Y \subseteq St(F, \mathcal{U})$, which shows that Y is strongly star-Lindelöf in X.

Next, we show that X is not star-Lindelöf. Let

$$Y'' = (Z \times \{\tau\}) \times \{1\}.$$

Then, Y'' is clopen in X and every point of Y'' is isolated. Let us consider the open cover

$$\mathcal{U} = Y \cup \{\{\langle\langle z_\alpha, \tau\rangle, 1\rangle\} : \alpha < \kappa\}$$

of X. Then, for any countable subset F of X, there exists an $\alpha_0 < \kappa$ such that

$$\langle\langle z_{\alpha_0}, \tau\rangle, 1\rangle \notin F.$$

Hence, $\langle\langle z_{\alpha_0}, \tau\rangle, 1\rangle \notin St(F, \mathcal{U})$, since $\{\langle\langle z_{\alpha_0}, \tau\rangle, 1\rangle\}$ is the only element of $\mathcal{U}$ containing $\langle\langle z_{\alpha_0}, \tau\rangle, 1\rangle$, which completes the proof. □

For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 2.3. There exist a Tychonoff space X and a subspace Y of X such that Y is star-Lindelöf in X, but Y is not strongly star-Lindelöf in X.

Proof. Let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$. Define

$$S_1 = (\mathfrak{c} \times \omega) \cup \mathcal{R}.$$

We topologize S_1 as follows: $\mathfrak{c} \times \omega$ has the usual product topology and is an open subspace of S_1, and a basic neighbourhood of $r \in \mathcal{R}$ takes the form

$$G_{\beta,K}(r) = (\{\alpha : \beta < \alpha < \mathfrak{c}\} \times (r \setminus K)) \cup \{r\}$$

for $\beta < \mathfrak{c}$ and a finite subset K of ω. Then, the space S_1 is Tychonoff and $e(S_1) = \mathfrak{c}$, because $\mathcal{R}$ is discrete closed in S_1.

Define

$$S_2 = (\mathfrak{c}^+ \times \omega) \cup \mathcal{R}.$$

We topologize S_2 as follows: $\mathfrak{c}^+ \times \omega$ has the usual product topology and is an open subspace of S_2, and a basic neighbourhood of $r \in \mathcal{R}$ takes the form

$$G'_{\beta,K}(r) = (\{\alpha : \beta < \alpha < \mathfrak{c}^+\} \times (r \setminus K)) \cup \{r\}$$

for $\beta < \mathfrak{c}^+$ and a finite subset K of ω. Then, the space S_2 is Tychonoff and $e(S_2) = \mathfrak{c}$, because $\mathcal{R}$ is discrete closed in S_2.

Assume $S_1 \cap S_2 = \varnothing$. Let $\varphi : \mathcal{R} \to \mathcal{R}$ be the identity mapping. Let X be the quotient space obtained from the discrete sum $S_1 \oplus S_2$ by identifying r and $\varphi(r)$ for each $r \in \mathcal{R}$. Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map. Let $Y = \pi(S_1)$.

First, we show that Y is star-Lindelöf in X. To this end, let $\mathcal{U}$ be an open cover of X. For each $n \in \omega$, since $\pi(\mathfrak{c} \times \{n\})$ is countably compact, there exists a finite subset $F_n \subseteq \pi(\mathfrak{c} \times \{n\})$ such that

$$\pi(\mathfrak{c} \times \{n\}) \subseteq St(F_n, \mathcal{U}).$$

Let $F' = \bigcup\{F_n : n \in \omega\}$. Then,

$$\pi(\mathfrak{c} \times \omega) \subseteq St(F', \mathcal{U}).$$

For each $r \in \mathcal{R}$, take $U_r \in \mathcal{U}$ with $\pi(r) \in U_r$, and fix $\alpha_r < \mathfrak{c}^+$ and $n_r \in r$ such that

$$\pi(\{\langle \alpha, n_r \rangle : \alpha_r < \alpha < \mathfrak{c}^+\}) \subseteq U_r.$$

For each $n \in \omega$, let $\mathcal{R}_n = \{r \in \mathcal{R} : n_r = n\}$ and choose $\beta_n \in \mathfrak{c}^+$ with $\beta_n > \sup\{\alpha_r : r \in \mathcal{R}_n\}$. Then,

$$\pi(\mathcal{R}_n) \subseteq St(\pi(\langle \beta_n, n \rangle), \mathcal{U}).$$

Thus, if we put $F'' = \pi(\{\langle \beta_n, n \rangle : n \in \omega\})$, then $\pi(\mathcal{R}) \subseteq St(F'', \mathcal{U})$. Let $F = F' \cup F''$. Then, F is a countable subset of X such that $Y \subseteq St(F, \mathcal{U})$, which shows that Y is star-Lindelöf in X.

Next, to show that Y is not strongly star-Lindelöf in X. Since $|\mathcal{R}| = \mathfrak{c}$, we can enumerate $\mathcal{R}$ as $\{r_\alpha : \alpha < \mathfrak{c}\}$. Let us consider the open cover

$$\mathcal{U} = \{\pi(G_{\alpha,\varnothing}(r_\alpha) \cup \big(G'_{\alpha,\varnothing}(r_\alpha)\big) : \alpha < \mathfrak{c}\} \cup \{\pi(\mathfrak{c} \times \omega)\} \cup \{\pi(\mathfrak{c}^+ \times \omega)\} \neq \varnothing$$

of X. It remains to show that $Y \setminus St(F, \mathcal{U}) \neq \varnothing$ for every countable subset F of Y. To show this, let F be a countable set of Y. Since $F \cap \pi(\mathcal{R})$ is countable, there exists an $\alpha' < \mathfrak{c}$ such that

$$F \cap \pi(\{r_\alpha : \alpha > \alpha'\}) = \varnothing.$$

On the other hand, $F \cap \pi(\mathfrak{c} \times \omega)$ is countable, so there exists $\alpha'' < \mathfrak{c}$ such that

$$F \cap \pi((\alpha'', \mathfrak{c}) \times \omega) = \varnothing.$$

If we pick $\alpha_0 > \max\{\alpha', \alpha''\}$, then

$$\pi(r_{\alpha_0}) \notin St(F, \mathcal{U}),$$

since $\pi(G_{\alpha_0,\varnothing}\big(r_{\alpha_0}) \cup G'_{\alpha_0,\varnothing}(r_{\alpha_0})\big)$ is the only element of $\mathcal{U}$ containing $\pi(r_{\alpha_0})$, which completes the proof. □

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References

1. A.V. Arhangel'skiĭ, *A generic theorem in the theory of cardinal invariants of topological spaces*, Comment. Math. Univ. Carolin. **36** (1995), 303–325.
2. A.V. Arhangel'skiĭ and M.M. Genedi Khamdi, *The Origin of the Theory of Relative Topological Properties*, General Topology, Space and Mapping, pp. 3–48, Moskov. Gos. Univ., Moscow, 1989
3. E.K. van Douwen, G.M. Reed, A. W. Roscoe and I. J. Tree, *Star covering properties*, Topology Appl. **39** (1991), 71–103.

4. M.M. Dai, *A class of topological spaces which contains all Lindelöf spaces and all separable spaces*, (Chinese), Chinese, Ann. Math . Ser. A, **4** (1983), 571–575.
5. R. Engelking *General Topology, Revised and Completed Edition*, Heldermann Verlag, Berlin, 1989.
6. Lj.D. Kočinac, *Some relative topological properties*, Mat. Vesnik. **44** (1992), 33–44.
7. M.V. Matveev, *A Survey on Star Covering Properties*, Topology, Atlas, preprint No. 330, 1998.
8. M.V. Matveev, *How weak is weak extent?*, Topology, Appl. **119** (2002), 229–232.
9. M.V. Matveev, *On spaces in countable web*, preprint.
10. Y. K. Song, *Spaces with large extent and large star-Lindelöf number*, Houston, J. Math. **29** (2003), 347–354.

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