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BOUNDEDNESS AND CONNECTEDNESS COMPONENTS FOR LOCALLY CONVEX CONES

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Abstract. We introduce topologies on locally convex cones which are in general coarser than the given topologies and take into account the presence of unbounded elements. Using these topologies, we investigate relations between the connectedness and the boundedness components of a locally convex cone.

1. Introduction

Endowed with suitable topologies, vector spaces yield rich and well-studied structures. Locally convex topological vector spaces in particular permit an extensive duality theory whose study gives valuable insight into the spaces themselves. Some important mathematical settings, however, while close to the structure of vector spaces do not allow subtraction of their elements or multiplication by negative scalars. Examples are certain classes of functions that may take infinite values or are characterized through inequalities rather than equalities. They arise naturally in integration and in potential theory. Likewise, families of convex subsets of vector spaces which are of interest in various contexts, do not form vector spaces. If the cancellation law fails, domains of this type may not even be embedded into larger vector spaces in order to apply results and techniques from classical functional analysis. They merit the investigation of a more general structure. The theory of locally convex cones, as developed in [**2**], deals with ordered cones that may not even be embeddable into vector spaces. A topological structure is introduced using order theoretical concepts. In Section 2 of this paper we shall review some of the main concepts and globally refer to [**2**] for details and proofs. A brief survey of the subject may also be found in [**3**]. In Section 4 we introduce the relative topologies of a locally convex cone and define and investigate different types of boundedness and connectedness components in Sections 5 and 6.

2. Locally Convex Cones

A *cone* is a set $\mathcal{P}$ endowed with an addition $(a,b)\mapsto a+b$ and a scalar multiplication $(\alpha,a)\mapsto \alpha a$ for real numbers $\alpha\geq 0$. The addition is supposed to be associative and commutative, and there is a neutral element $0\in\mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a)=(\alpha\beta)a$, $(\alpha+\beta)a=\alpha a+\beta a$, $\alpha(a+b)=\alpha a+\alpha b$, $1a=a$ and $0a=0$ for all $a,b\in\mathcal{P}$ and $\alpha,\beta\geq 0$. The *cancellation law*, stating that $a+c=b+c$ implies $a=b$, however, is

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not required in general. It holds if and only if the cone $\mathcal{P}$ can be embedded into a real vector space.

An *ordered cone* $\mathcal{P}$ carries a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. Equality on $\mathcal{P}$ is obviously such an order. Note that anti-symmetry is not required for the relation $\leq$.

The theory of locally convex cones as developed in [**2**] uses order theoretical concepts to introduce a quasiuniform topological structure on an ordered cone. In a first approach, the resulting topological neighborhoods themselves will be considered to be elements of the cone.

In this vein, a *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, that is a subset of positive elements which is directed downward, closed for addition and multiplication by strictly positive scalars. The elements v of $\mathcal{V}$ define *upper* resp˙*lower neighborhoods* for the elements of $\mathcal{P}$ by

$$v(a) = \{ b \in \mathcal{P} \mid b \leq a + v \} \quad \text{resp.} \quad (a)v = \{ b \in \mathcal{P} \mid a \leq b + v \},$$

creating the *upper* resp˙*lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology*. All elements of $\mathcal{P}$ are supposed to be *bounded below*, that is for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \lambda v$ for some $\lambda \geq 0$. Finally, a *locally convex cone*$(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone not necessarily containing the abstract neighborhood system $\mathcal{V}$. Every locally convex ordered topological vector space is a locally convex cone in this sense, as it may be canonically embedded into a full locally convex cone (see Example 2.1(c) below, or Example I.2.7 in [**2**]). It is shown in Chapter I.5.2 of [**2**] how a convex quasiuniform structure on a cone can be used to construct a full locally convex cone which contains the given one as a subcone and induces the given uniform structure. This yields a second, equivalent approach to locally convex cones.

Example 2.1. (a) In the extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we consider the usual order and algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. Endowed with the neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a+\varepsilon]$ are the upper and the intervals $[a-\varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with $+\infty$ as an isolated point.

(b) For the subcone $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} \mid a \geq 0\}$ of $\overline{\mathbb{R}}$ we may also consider the singleton neighborhood system $\mathcal{V} = \{0\}$. The elements of $\overline{\mathbb{R}}_+$ are obviously bounded below even with respect to the neighborhood $v = 0$, hence $\overline{\mathbb{R}}_+$ is a full locally convex cone. For $a \in \overline{\mathbb{R}}$ the intervals $(-\infty, a]$ and $[a, +\infty]$ are the only upper and lower neighborhoods, respectively. The symmetric topology is the discrete topology on $\overline{\mathbb{R}}_+$.

(c) The following is a principal model for the theory of locally convex cones: Let $(E, \leq)$ be a locally convex ordered topological vector space. Recall that equality is an order relation, hence this example will also cover locally convex spaces without any given order structure. Let $\mathcal{P} = \mathit{Conv}(E)$ be the cone of all non-empty convex subsets of E, endowed with the usual addition and multiplication of sets by non-negative scalars, that is $A+B = \{a+b \mid a \in A \quad \text{and} \quad b \in B\}$ and $\alpha A = \{\alpha a \mid a \in A\}$ for $A, B \in Conv(E)$ and $\alpha \geq 0$. For $A \in \mathcal{P}$ we denote by

$$\downarrow\! A = \{e \in E \mid e \leq a \text{ for some } a \in A\},$$

that is the decreasing subset of E generated by A. This set is of course again convex, hence an element of $\mathcal{P}$. We define the order on $\mathcal{P}$ by

$$A \leq B \qquad \text{if} \qquad A \subset \downarrow\! B.$$

The requirements for an ordered cone are easily checked. The neighborhood system in $\mathcal{P}$ is given by a basis $\mathcal{V} \subset \mathcal{P}$ of closed absolutely convex neighborhoods of the origin in E. We observe that for every $A \in \mathcal{P}$ and $V \in \mathcal{V}$ there is $\rho > 0$ such that $\rho V \cap A \neq \varnothing$. This yields $0 \in A + \rho V$. Therefore $\{0\} \leq A + \rho V$, and every element $A \in \mathcal{P}$ is indeed bounded below. Thus $(\mathcal{P}, \mathcal{V})$ is a full locally convex cone. Via the embedding of its elements onto singleton subsets, the space E itself may be considered as a subcone of $\mathcal{P}$. This embedding preserves the order structure of E, and on its image in $\mathcal{P}$, the three (upper, lower and symmetric) topologies of $\mathcal{P}$ coincide with the given topology on E. Thus E is a locally convex cone, but not a full cone. Other subcones of $\mathcal{P}$ that merit further investigation are those of all closed, closed and bounded, or compact convex sets in $\mathcal{P}$, respectively. Note that closed and bounded convex sets satisfy the cancellation law. Details on those and further related examples may be found in [**2**], I.1.7, I.2.7 and I.2.8.

This example may be further generalized if we replace the vector space E by a locally convex cone.

(d) Let $\mathcal{F}(X, \mathcal{P})$ be the cone of all $\mathcal{P}$-valued functions on a set X, where $(\mathcal{P}, \mathcal{V})$ is a locally convex cone. If we identify the elements $v \in \mathcal{V}$ with the constant functions $x \mapsto v$ for all $x \in X$, then $\mathcal{V}$ is a subset of $\mathcal{P}$. A function $f \in \mathcal{F}(X, \mathcal{P})$ is uniformly bounded below, if for every $v \in \mathcal{V}$ there is $\rho \geq 0$ such that $0 \leq f(x) + \rho v$ for all $x \in X$. These functions form a full locally convex cone $\big(\mathcal{F}_b(X, \mathcal{P}), \mathcal{V}\big)$. Alternatively, a more general neighborhood system $\mathcal{V}_{\mathcal{Y}}$ for $\mathcal{F}(X, \mathcal{P})$ may be created using a suitable family $\mathcal{Y}$ of subsets Y of X and the neighborhoods v_Y for $v \in \mathcal{V}$ and $Y \in \mathcal{Y}$, defined for functions $f, g \in \mathcal{F}(X, \mathcal{P})$ as $f \leq g + v_Y$ if $f(x) \leq g(x) + v$ for all $x \in Y$. In this case we consider the subcone $\mathcal{F}_{b_{\mathcal{Y}}}(X, \mathcal{P})$ of all functions in $\mathcal{F}(X, \mathcal{P})$ that are uniformly bounded below on the sets in $\mathcal{Y}$. Together with the neighborhood system $\mathcal{V}_{\mathcal{Y}}$, it forms a locally convex cone. $\big(\mathcal{F}_{b_{\mathcal{Y}}}(X, \mathcal{P}), \mathcal{V}_{\mathcal{Y}}\big)$ carries the topology of uniform convergence on the sets in $\mathcal{Y}$.

2.1. Linear operators.

For cones $\mathcal{P}$ and $\mathcal{Q}$ a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a+b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ holds for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $\mathcal{P}$ and $\mathcal{Q}$ are ordered, then T is called *monotone,* if $a \leq b$ implies $T(a) \leq T(b)$. If both $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $w \in \mathcal{W}$ one can find $v \in \mathcal{V}$ such that $T(a) \leq T(b) + w$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. Uniform continuity is not just continuity. It is immediate from the definition that it implies and combines continuity for the operator $T : \mathcal{P} \to \mathcal{Q}$ with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$, respectively.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathcal{V})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all *polars* v° of neighborhoods $v \in \mathcal{V}$, where $\mu \in v^\circ$ means that $\mu(a) \le \mu(b) + 1$, whenever $a \le b + v$ for $a, b \in \mathcal{P}$. Continuity implies that a linear functional μ is monotone, and for a full cone $\mathcal{P}$ it requires just that $\mu(v) \le 1$ holds for some $v \in \mathcal{V}$ in addition. We endow $\mathcal{P}^*$ with the topology $w(\mathcal{P}^*, \mathcal{P})$ of pointwise convergence on the elements of $\mathcal{P}$, considered as functions on $\mathcal{P}^*$ with values in $\overline{\mathbb{R}}$ with its usual compact Hausdorff topology. As in locally convex topological vector spaces, the polar v° of a neighborhood $v \in \mathcal{V}$ is seen to be $w(\mathcal{P}^*, \mathcal{P})$-compact and convex ([**2**], Theorem II.2.4).

Example 2.2. Revisiting the preceding Examples 2.1, we observe that the dual cone $\overline{\mathbb{R}}^*$ of $\overline{\mathbb{R}}$ (see 2.1(a)) consists of all positive reals (via the usual multiplication), and the singular functional $\overline{0}$ such that $\overline{0}(a) = 0$ for all $a \in \mathbb{R}$ and $\overline{0}(+\infty) = +\infty$. Likewise, in 2.1(b), the continuous linear functionals on $\overline{\mathbb{R}}_+$, endowed with the neighborhood system $\mathcal{V} = \{0\}$, are the positive reals together with $\overline{0}$, but further include the element $+\infty$, acting as $+\infty(0) = 0$ and $+\infty(a) = +\infty$ for all $0 \ne a \in \overline{\mathbb{R}}_+$. This functional is obviously contained in the polar of the neighborhood $0 \in \mathcal{V}$. In 2.1 (c) and (d) on the other hand, due to the generality of the settings, a complete description for the respective dual cones is not immediately available. We may, however, identify some of their elements: In 2.1(c), let μ be a continuous monotone linear function on the locally convex ordered topological vector space $(E, \le)$. Then the mapping

$$A \mapsto \sup\{\mu(a) \mid a \in A\} : Conv(E) \to \overline{\mathbb{R}}$$

is seen to be an element of $Conv(E)^*$. In 2.1(d), for all $x \in X$, the point evaluations, that is, the mappings

$$f \mapsto f(x) : \mathcal{F}_b(X, \mathcal{P}) \to \overline{\mathbb{R}}$$

constitute continuous linear functionals on $\mathcal{F}_b(X, \mathcal{P})$.

Hahn-Banach type extension and separation theorems for linear functionals are most important for the development of a powerful duality theory for locally convex cones. We shall mention a few results from [**2**] and [**5**]. A *sublinear functional* on a cone $\mathcal{P}$ is a mapping $p : \mathcal{P} \to \overline{\mathbb{R}}$ such that $p(\alpha a) = \alpha p(a)$ and $p(a+b) \le p(a) + p(b)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \ge 0$. Likewise, an *extended superlinear functional* on $\mathcal{P}$ is a mapping $q : \mathcal{P} \to \underline{\overline{\mathbb{R}}} = \mathbb{R} \cup \{+\infty, -\infty\}$ such that $q(\alpha a) = \alpha q(a)$ and $q(a+b) \ge q(a) + q(b)$ hold for all $a, b \in \mathcal{P}$ and $\alpha \ge 0$. (We set $\alpha + (-\infty) = -\infty$ for all $\alpha \in \underline{\overline{\mathbb{R}}}$, $\alpha \cdot (-\infty) = -\infty$ for all $\alpha > 0$ and $0 \cdot (-\infty) = 0$ in this context.) We cite Theorem 3.1 from [**5**]:

Theorem 2.3. Sandwich Theorem *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, and let $v \in \mathcal{V}$. For a sublinear functional $p : \mathcal{P} \to \overline{\mathbb{R}}$ and an extended superlinear functional $q : \mathcal{P} \to \underline{\overline{\mathbb{R}}}$ there exists a linear functional $\mu \in v^\circ$ such that $q \le \mu \le p$ if and only if $q(a) \le p(b) + 1$ holds whenever $a \le b + v$ for $a, b \in \mathcal{P}$.*

This leads to a variety of extension and separation results, the most general ones being Theorems 4.1 and 4.4 in [**5**]. We shall only mention the following simplified version of 4.1 in [**5**] (Theorem II.2.9 in [**2**]).

Corollary 2.4. *Let $(\mathcal{N},\mathcal{V})$ be a subcone of the locally convex cone $(\mathcal{P},\mathcal{V})$. Every continuous linear functional on $\mathcal{N}$ can be extended to a continuous linear functional on $\mathcal{P}$; more precisely: For every $\mu \in v^\circ_{\mathcal{N}}$ there is $\tilde{\mu} \in v^\circ_{\mathcal{P}}$ such that $\tilde{\mu}$ coincides with μ on $\mathcal{N}$.*

3. Weak Local and Global Preorders

In addition to the given order $\leq$ on a locally convex cone, one also considers the *weak (global) preorder* $\preccurlyeq$ (see [**6**]) which is slightly weaker then the given order and defined for $a, b \in \mathcal{P}$ by

$$a \preccurlyeq b \qquad \text{if} \qquad a \leq \gamma b + \varepsilon v$$

for all $v \in \mathcal{V}$ and $\varepsilon > 0$ with some $1 \leq \gamma \leq 1+\varepsilon$. This order represents a closure of the given order with respect to the linear and topological structures of $\mathcal{P}$. It is obviously coarser than the given order, that is $a \leq b$ implies $a \preccurlyeq b$ for $a, b \in \mathcal{P}$. In the preceding Examples 2.1(a), (b) and (d), however, both orders coincide. In 2.1(c), on the other hand, we have $A \preccurlyeq B$ if $A \subset \overline{\downarrow B}$, the topological closure in E of the set $\downarrow B$. Note that $\overline{\downarrow B}$ is again a convex and decreasing subset of E.

The weak preorder on $\mathcal{P}$ is again compatible with the algebraic operations, as Lemma 4.1 below will imply. Theorem 3.1 in [**6**] states that the weak preorder on a locally convex cone $\mathcal{P}$ is entirely determined by its dual cone $\mathcal{P}^*$, that is $a \preccurlyeq b$ holds for $a, b \in \mathcal{P}$ if and only if $\mu(a) \leq \mu(b)$ for all $\mu \in \mathcal{P}^*$. The weak preorder may also be used in a full cone containing $\mathcal{P}$ and $\mathcal{V}$. Consequently, the respective relation involving the neighborhoods in $\mathcal{V}$ is defined for elements $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ as

$$a \preccurlyeq b + v \qquad \text{if} \qquad a \leq \gamma(b+v) + \varepsilon u$$

for all $u \in \mathcal{V}$ and $\varepsilon > 0$ with some $1 \leq \gamma \leq 1+\varepsilon$. Endowed with the weak preorder $(\mathcal{P},\mathcal{V})$ forms again a locally convex cone. For details we refer to [**6**]. Theorem 3.2 in [**6**] states that for $a, b \in \mathcal{P}$ and a neighborhood $v \in \mathcal{V}$, we have $a \preccurlyeq b + v$ if and only if $\mu(a) \leq \mu(b) + 1$ holds for all $\mu \in v^\circ$. The neighborhoods with respect to the weak preorder in $\mathcal{P}$ are therefore entirely determined by their polars.

Given a neighborhood $v \in \mathcal{V}$ the *weak local preorder* (also see [**6**]) $\preccurlyeq_v$ on $\mathcal{P}$ is the weak (global) preorder with respect to the neighborhood subsystem $\mathcal{V}_v = \{\alpha v \mid \alpha > 0\}$. That is, for $a, b \in \mathcal{P}$ we have

$$a \preccurlyeq_v b \qquad \text{if} \qquad a \leq \gamma b + \varepsilon v$$

for all $\varepsilon > 0$ with some $1 \leq \gamma \leq 1+\varepsilon$. Theorem 3.1 in [**6**] states that $a \preccurlyeq_v b$ if and only if $\mu(a) \leq \mu(b)$ holds for all $\mu \in v^\circ$.

Moreover, it is evident that for a linear operator T between locally convex cones $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$, continuity with respect to the given orders implies continuity and monotonicity with respect to the respective weak preorders on $\mathcal{P}$ and $\mathcal{Q}$, that is $a \preccurlyeq b + v$ implies $T(a) \preccurlyeq T(b) + w$ and $a \preccurlyeq b$ implies $T(a) \preccurlyeq T(b)$.

The weak preorder may be used to establish a representation for a locally convex cone $(\mathcal{P},\mathcal{V})$ as a cone of continuous $\overline{\mathbb{R}}$-valued functions on some topological space and as a cone of convex subsets of some locally convex topological vector space, respectively. We cite Theorem 4.1 from [**6**]:

Theorem 3.1. *Every locally convex cone $(\mathcal{P},\mathcal{V})$ may be embedded into*

(i) *a locally convex cone of continuous $\overline{\mathbb{R}}$-valued functions on some topological space X, endowed with the pointwise order and operations and the topology of uniform convergence on a family of compact subsets of X.*

(ii) *a locally convex cone of convex subsets of a locally convex topological vector space, endowed with the usual addition and multiplication by scalars, the set inclusion as order and the neighborhoods inherited from the vector space.*

These embeddings are linear and preserve the weak global preorder and the neighborhoods of $(\mathcal{P}, \mathcal{V})$.

4. Boundedness and the Relative Topologies

While all elements of a locally convex cone are bounded below by definition, they need not to be bounded above. Given a neighborhood $v \in \mathcal{V}$, an element a of a locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *v-bounded (above)* (see [**2**], I.2.3) if there is $\lambda \geq 0$ such that $a \leq \lambda v$. The subset $\mathcal{B}_v \subset \mathcal{P}$ of all v-bounded elements is a subcone and even a face of $\mathcal{P}$. Correspondingly, by $\mathcal{B} = \bigcap_{v \in \mathcal{V}} \mathcal{B}_v$ we denote the subcone (and face) of all *bounded (above)* elements of $\mathcal{P}$ (see also Proposition 5.1 below). All invertible elements of $\mathcal{P}$ are bounded. Continuous linear functionals take only finite values on bounded elements. Similarly, given $v \in \mathcal{V}$, a subset A of $\mathcal{P}$ is said to be *v-bounded above* (or *v-bounded below*) if there is $\lambda \geq 0$ such that $a \leq \lambda v$ (or $0 \leq a + \lambda v$) holds for all $a \in A$; and finally, the set $A \subset \mathcal{P}$ is called *bounded above* (or *bounded below*) if it is v-bounded above (or v-bounded below) for all neighborhoods $v \in \mathcal{V}$.

The presence of unbounded elements constitutes a significant difference between locally convex cones and locally convex topological vector spaces. It tends to make matters more interesting, but also considerably more complicated. If, for example, the element $a \in \mathcal{P}$ is not bounded, then the mapping $\alpha \mapsto \alpha a : [0, +\infty) \to \mathcal{P}$, is discontinuous if we consider the usual topology of $\mathbb{R}_+$ and any of the given (upper, lower or symmetric) topologies on $\mathcal{P}$, which therefore appear to be rather restrictive. We shall therefore introduce slightly coarser neighborhoods on $\mathcal{P}$ which take unbounded elements suitably into account. Given a neighborhood $v \in \mathcal{V}$ and $\varepsilon > 0$, we define the corresponding *upper* and *lower relative neighborhoods* $v_\varepsilon(a)$ and $(a)v_\varepsilon$ for an element $a \in \mathcal{P}$ by

$$\begin{array}{rcll} v_\varepsilon(a) & = & \{\, b \in \mathcal{P} \mid b \leq \gamma a + \varepsilon v & \text{for some} \quad 1 \leq \gamma \leq 1+\varepsilon \,\} \\ (a)v_\varepsilon & = & \{\, b \in \mathcal{P} \mid a \leq \gamma b + \varepsilon v & \text{for some} \quad 1 \leq \gamma \leq 1+\varepsilon \,\}. \end{array}$$

Their intersection $v^s_\varepsilon(a) = v_\varepsilon(a) \cap (a)v_\varepsilon$ is the corresponding *symmetric relative neighborhood.* These are of course convex subsets of $\mathcal{P}$. Note that for a positive element $a \in \mathcal{P}$ the above expressions somewhat simplify. Since $\gamma a \leq (1+\varepsilon)a$, we have

$$v_\varepsilon(a) = \{\, b \in \mathcal{P} \mid b \leq (1+\varepsilon)a + \varepsilon v\}$$

and

$$(a)v_\varepsilon = \{\, b \in \mathcal{P} \mid a \leq (1+\varepsilon)b + \varepsilon v\}$$

in this case.

Lemma 4.1. *Let $a, b, c, a_i, b_i \in \mathcal{P}$, $v \in \mathcal{V}$, $\lambda \geq 0$ and $\varepsilon, \delta > 0$.*

(a) *If* $a \in v_\varepsilon(b)$ *and* $b \in v_\delta(c)$, *then* $a \in v_{(\varepsilon+\delta+\varepsilon\delta)}(c)$.

(b) *If* $a \in v_\varepsilon(b)$ *and* $0 \le b + \lambda v$, *then* $a \le (1+\varepsilon)b + \varepsilon(1+\lambda)v$.

(c) *If* $a \in v_\varepsilon(b)$ *and* $0 \le a + \lambda v$, *then* $a \le (1+\varepsilon)b + \varepsilon(1+\lambda+\varepsilon)v$ *and* $0 \le b + (\lambda+\varepsilon)v$.

(d) *If* $a_i \in v_\varepsilon(b_i)$ *and if* $0 \le b_i + \lambda v$ *for* $i = 1, \dots, n$, *then* $(a_1 + \ldots + a_n) \in v_{\varepsilon n(1+\lambda)}(b_1 + \ldots + b_n)$.

Proof. For (a), let $a \in v_\varepsilon(b)$ and $b \in v_\delta(c)$, that is $a \le \gamma b + \varepsilon v$ and $b \le \lambda c + \delta v$ for some $1 \le \gamma \le 1+\varepsilon$ and $1 \le \lambda \le 1+\delta$. Then $a \le \gamma\lambda c + (\gamma\delta + \varepsilon)v$. As $\gamma\delta + \varepsilon \le (1+\varepsilon)\delta + \varepsilon = \varepsilon + \delta + \varepsilon\delta$ and $1 \le \gamma\lambda \le (1+\varepsilon)(1+\delta) = 1 + \varepsilon + \delta + \varepsilon\delta$, we have $a \in v_{(\varepsilon+\delta+\varepsilon\delta)}(c)$. For (b), let $a \in v_\varepsilon(b)$, that is $a \le \gamma b + \varepsilon v$ for some $1 \le \gamma \le 1+\varepsilon$. If $0 \le b + \lambda v$, then

$$a \le \gamma b + \varepsilon v + (1+\varepsilon-\gamma)(b+\lambda v) \le (1+\varepsilon)b + (\varepsilon+\varepsilon\lambda)v.$$

For (c), let $a \in v_\varepsilon(b)$ and $\lambda \ge 0$ such that $0 \le a + \lambda v$. Then $a \le \gamma b + \varepsilon v$ with some $1 \le \gamma \le 1+\varepsilon$, hence $0 \le \gamma b + (\varepsilon+\lambda)v$, and indeed $0 \le b + \frac{\varepsilon+\lambda}{\gamma}v \le b + (\varepsilon+\lambda)v$. Part (b) yields $a \le (1+\varepsilon)b + \varepsilon(1+\lambda+\varepsilon)v$. For (d), let $a_i \in v_\varepsilon(b_i)$ and $0 \le b_i + \lambda v$. Then $a_i \le (1+\varepsilon)b_i + \varepsilon(1+\lambda)v$ by part (b). This yields $a_1 + \ldots + a_n \le (1+\varepsilon)(b_1 + \ldots + b_n) + n\varepsilon(1+\lambda)v$, hence our claim. □

4.1(a) implies in particular that $v_\varepsilon(a) \subset v_{3\varepsilon}(c)$ whenever $a \in v_\varepsilon(b)$ and $b \in v_\varepsilon(c)$ for $a, b, c \in \mathcal{P}$ and $0 < \varepsilon \le 1$. 4.1(d) shows compatibility of these neighborhoods with the addition. Compatibility with the multiplication by positive scalars is obvious. Similar statements as in Lemma 4.1 hold for the lower and for the symmetric relative neighborhoods.

For elements $a, b \in \mathcal{P}$ the weak local and global preorders on $\mathcal{P}$ as defined in Section 3 may be recovered as

$$a \preccurlyeq_v b \qquad \text{if} \qquad a \in v_\varepsilon(b)$$

for all $\varepsilon > 0$ and $v \in \mathcal{V}$, and

$$a \preccurlyeq b \qquad \text{if} \qquad a \in v_\varepsilon(b)$$

for all $v \in \mathcal{V}$ and $\varepsilon > 0$. Lemma 4.1(d) implies that these orders are compatible with the algebraic operations in $\mathcal{P}$.

For varying $v \in \mathcal{V}$ and $\varepsilon > 0$ the neighborhoods $v_\varepsilon(\cdot)$, $(\cdot)v_\varepsilon$ and $v^s_\varepsilon(\cdot)$ create the *upper, lower* and *symmetric relative topologies* on $\mathcal{P}$, respectively. These topologies are obviously coarser than the corresponding given topologies, but coincide locally with the latter on bounded elements of $\mathcal{P}$, as for $a \le \lambda v$ we have

$$(\varepsilon v)(a) \subset v_\varepsilon(a) \subset (\rho v)(a) \qquad \text{and} \qquad (a)(\varepsilon v) \subset (a)v_\varepsilon(a) \subset (a)(\rho v).$$

with $\rho = (1+\lambda)\varepsilon$. However, while the relative neighborhoods form convex subsets of $\mathcal{P}$, they do not create a locally convex cone topology. Indeed, the sets $\{(a,b) \mid a \in v_\varepsilon(b)\}$ are not necessarily convex in $\mathcal{P}^2$, hence do not establish a convex semiuniform structure on $\mathcal{P}$.

It is worthwhile to notice that a continuous linear operator T between two locally convex cones $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ is also continuous if we endow both $\mathcal{P}$ and $\mathcal{Q}$ with either their respective upper, lower or symmetric relative topologies. Indeed, given $w \in \mathcal{W}$, there is $v \in \mathcal{V}$ such that $a \le b + v$ implies $T(a) \le T(b) + w$ for elements

$a, b \in \mathcal{P}$. Thus $a \in v_\varepsilon(b)$, that is $a \le \gamma b + \varepsilon v$ with some $1 \le \gamma \le 1 + \varepsilon$, implies $T(a) \le \gamma T(b) + \varepsilon w$, hence $T(a) \in w_\varepsilon\big(T(b)\big)$. A similar argument shows continuity with respect to either the lower or symmetric relative topologies of $\mathcal{P}$ and $\mathcal{Q}$.

We shall also use the *(upper, lower, symmetric) relative v-topologies* on $\mathcal{P}$, generated by the relative neighborhoods for a fixed $v \in \mathcal{V}$. The symmetric relative v-topology, in particular, is induced by the pseudometric

$$d_v(a, b) = \inf\big\{1, \sqrt{\varepsilon} \mid a \in v_\varepsilon^s(b)\big\}.$$

The properties of a pseudometric (see Section 2.1 in [**8**]) are readily checked for this expression: We obviously have $d_v(a, b) \ge 0$, $d_v(a, a) = 0$ and $d_v(a, b) = d_v(b, a)$ for $a, b \in \mathcal{P}$. The triangular inequality, namely $d_v(a, c) \le d_v(a, b) + d_v(b, c)$ for $a, b, c \in \mathcal{P}$, holds trivially true if either $d_v(a, b) = 1$ or $d_v(b, c) = 1$. Otherwise, if $d_v(a, b) < \varepsilon < 1$ and $d_v(b, c) < \delta < 1$, then $a \in v_{\varepsilon^2}^s(b)$ and $b \in v_{\delta^2}^s(c)$ implies by Lemma 4.1(a) that $a \in v_\rho^s(c)$, where $\rho = \varepsilon^2 + \delta^2 + \varepsilon^2\delta^2 \le (\varepsilon + \delta)^2$. Thus $d_v(a, c) \le \varepsilon + \delta$, hence the triangular inequality holds.

The (upper, lower, symmetric) relative topologies on $\mathcal{P}$ are the common refinements of all (upper, lower, symmetric) relative v-topologies.

A linear functional μ on $\mathcal{P}$, contained in the polar v° of a neighborhood $v \in \mathcal{V}$ is also seen to be a continuous mapping between $\mathcal{P}$ and $\overline{\mathbb{R}}$, if we endow $\mathcal{P}$ with either the upper, lower or symmetric relative v-topology and, correspondingly, $\overline{\mathbb{R}}$ with its upper, lower or symmetric topology, as described in Example 1.1(a). Indeed, for elements $a, b \in \mathcal{P}$ and $\varepsilon > 0$, in case that $\mu(b) = +\infty$, we have $\mu(a) \le \mu(b) + \varepsilon$ for all $a \in \mathcal{P}$. In case that $\mu(b) < +\infty$, we choose $\delta > 0$ such that $\delta\big(|\mu(b)| + 1\big) \le \varepsilon$. Then $a \in v_\delta(b)$, that is $a \le \gamma b + \delta v$ with some $1 \le \gamma \le 1 + \varepsilon$, implies $\mu(a) \le \gamma\mu(b) + \delta \le \mu(b) + (\gamma - 1)|\mu(b)| + \delta \le \mu(b) + \varepsilon$. This shows continuity with respect to the upper topologies. Continuity with respect to either the lower or symmetric relative v-topologies may be shown analogously. In this context, note that for $\overline{\mathbb{R}}$ the given and the relative topologies coincide.

5. Boundedness Components

For an element $a \in \mathcal{P}$ we define the *upper* and *lower boundedness components* of a as

$$\mathcal{B}(a) = \bigcap_{v \in \mathcal{V}} \bigcup_{\varepsilon > 0} v_\varepsilon(a) \qquad \text{and} \qquad (a)\mathcal{B} = \bigcap_{v \in \mathcal{V}} \bigcup_{\varepsilon > 0} (a)v_\varepsilon,$$

respectively. The elements of $\mathcal{B}(a)$ are called *bounded (above) relative to a*. By the definition of a locally convex cone we have $0 \in \mathcal{B}(a)$ for all $a \in \mathcal{P}$, and $\mathcal{B}(0) = \mathcal{B}$ consists of all bounded elements of $\mathcal{P}$. We shall first list a few basic properties of the upper boundedness components.

Proposition 5.1. *Let $a, b, c \in \mathcal{P}$. Then*

(a) *$b \in \mathcal{B}(a)$ if and only if for every $v \in \mathcal{V}$ there are $\alpha, \beta \ge 0$ such that $b \le \alpha a + \beta v$.*

(b) *$\mathcal{B}(a)$ is a subcone of $\mathcal{P}$, and $\mathcal{B} \subset \mathcal{B}(a)$.*

(c) *$\mathcal{B}(a)$ is a face in $\mathcal{P}$, that is $b + c \in \mathcal{B}(a)$ implies both $b, c \in \mathcal{B}(a)$.*

(d) *$\mathcal{B}(\alpha a) = \mathcal{B}(a)$ for $\alpha > 0$, and $\mathcal{B}(a) + \mathcal{B}(b) \subset \mathcal{B}(a + b)$.*

(e) *If $b \in \mathcal{B}(a)$, then $\mathcal{B}(b) \subset \mathcal{B}(a)$.*

(f) *$b \in \mathcal{B}(a)$ if and only if for all $\mu \in \mathcal{P}^*$, $\mu(a) < +\infty$ implies $\mu(b) < +\infty$.*

(g) $\mathcal{B}(a)$ *is closed in* $\mathcal{P}$ *with respect to the lower relative topology of* $\mathcal{P}$.

Proof. For part (a), let $b \in \mathcal{B}(a)$. For every $v \in \mathcal{V}$ there is $\varepsilon > 0$ such that $b \in v_\varepsilon(a)$, that is $b \le \alpha a + \beta v$ for some $\alpha, \beta \ge 0$. If, on the other hand, for $v \in \mathcal{V}$ we have $b \le \alpha a + \beta v$ for $\alpha, \beta \ge 0$, we choose $\lambda \ge 0$ such that $0 \le a + \lambda v$. Then $b \le (\alpha + 1)a + (\beta + \lambda)v$, hence $b \in v_\varepsilon(a)$ for $\varepsilon > \max\{\alpha, \beta + \lambda\}$. If this holds true for all $v \in \mathcal{V}$, then we have $b \in \mathcal{B}(a)$.

Part (b) is obvious from (a), since $b \le \alpha a + \beta v$ and $c \le \gamma a + \delta v$ for $v \in \mathcal{V}$ and $\alpha, \beta, \gamma, \delta \ge 0$ implies that $b + c \le (\alpha + \gamma)a + (\beta + \delta)v$ and $\lambda b \le \lambda\alpha a + \lambda\beta v$ for $\lambda \ge 0$.

For (c), let $b + c \in \mathcal{B}(a)$, that is, given $v \in \mathcal{V}$, we have $b + c \le \alpha a + \beta v$ for some $\alpha, \beta \ge 0$. Because all elements of a locally convex cone are bounded below, there is $\lambda \ge 0$ such that $0 \le c + \lambda v$. Thus $b \le b + c + \lambda v \le \alpha a + (\beta + \lambda)v$. Hence $b \in \mathcal{B}(a)$. Similarly, one verifies that $c \in \mathcal{B}(a)$.

The first statement of (d) is obvious. For the second statement, let $c \in \mathcal{B}(a)$, $d \in \mathcal{B}(b)$ and $v \in \mathcal{V}$. Then $c \le \alpha a + \beta v$ and $d \le \gamma b + \delta v$ for some $\alpha, \beta, \gamma, \delta \ge 0$. Let $\lambda \ge 0$ such that both $0 \le a + \lambda v$ and $0 \le b + \lambda v$. In case that $\alpha \le \gamma$, this yields $c \le \alpha a + \beta v + (\gamma - \alpha)(a + \lambda v) = \gamma a + \rho v$, where $\rho = \beta + (\gamma - \alpha)\lambda$. Thus $c + d \le \gamma(a + b) + (\rho + \delta)v$. In case that $\alpha > \gamma$, a similar argument leads to $c + d \le \alpha(a + b) + (\rho' + \beta)v$, where $\rho' = \delta + (\alpha - \gamma)\lambda$. This verifies $c + d \in \mathcal{B}(a + b)$.

For (e), let $b \in \mathcal{B}(a)$, $c \in \mathcal{B}(b)$ and $v \in \mathcal{V}$. Then $b \le \alpha a + \beta v$ and $c \le \gamma b + \delta v$ for some $\alpha, \beta, \gamma, \delta \ge 0$. Thus $c \le \alpha\gamma a + (\beta\gamma + \delta)v$, hence $c \in \mathcal{B}(a)$.

For part (f), let $b \in \mathcal{B}(a)$ and $\mu \in \mathcal{P}^*$ such that $\mu(a) < +\infty$. There is $v \in \mathcal{V}$ such that $\mu \in v^\circ$ and $\alpha, \beta \ge 0$ such that $b \le \alpha a + \beta v$. This implies $\mu(b) \le \alpha\mu(a) + \beta < +\infty$. For the converse, given $v \in \mathcal{V}$, we define a functional μ_v on $\mathcal{P}$ setting $\mu_v(b) = 0$ for all $b \in \mathcal{P}$ such that $b \le \alpha a + \beta v$ for some $\alpha, \beta \ge 0$, and $\mu_v(b) = +\infty$, else. It is straightforward to check that μ_v is linear. Indeed, if $\mu_v(b) = \mu_v(c) = 0$, that is $b \le \alpha a + \beta v$ and $c \le \gamma + \delta v$ for $\alpha, \beta, \gamma, \delta \ge 0$, then $b + c \le (\alpha + \gamma)a + (\beta + \delta)v$, hence $\mu_v(b + c) = 0$. If, on the other hand, $\mu_v(b + c) = 0$, that is $b + c \le \alpha a + \beta v$ for some $\alpha, \beta \ge 0$, we choose $\lambda \ge 0$ such that $0 \le c + \lambda v$ and have $b \le b + c + \lambda v \le \alpha a + (\beta + \lambda)v$. This shows $\mu_v(b) = 0$. Similarly, one verifies that $\mu_v(c) = 0$. Moreover, we realize that μ_v is an element of v°, as for $b \le c + v$, we have $\mu_v(b) = 0$ whenever $\mu_v(c) = 0$ hence $\mu_v(b) \le \mu_v(c) + 1$ holds in any case. Using this, we proceed with our argument: If $b \notin \mathcal{B}(a)$, then there is $v \in \mathcal{V}$ such that $b \not\le \alpha a + \beta v$ for all choices of $\alpha, \beta \ge 0$, hence $\mu_v(b) = +\infty$, while $\mu_v(a) = 0$.

Finally, for part (g), we remarked before that a linear functional $\mu \in \mathcal{P}^*$ is a continuous mapping from $\mathcal{P}$ into $\overline{\mathbb{R}}$ if we endow $\mathcal{P}$ with either its upper, lower or symmetric relative topology, and $\overline{\mathbb{R}}$ with either its given upper, lower or symmetric topology, respectively. We shall use this observation for the functionals μ_v, for $v \in \mathcal{V}$, that we constructed in our argument for part (f). Because $\mathbb{R}$ is a closed subset of $\overline{\mathbb{R}}$ in the lower topology of $\overline{\mathbb{R}}$ (see Example 1.1(a)), its inverse image $\mu_v^{-1}(\mathbb{R})$ under μ_v is closed in the lower relative topology of $\mathcal{P}$. We have $\mathcal{B}(a) = \bigcap_{v \in \mathcal{V}} \mu_v^{-1}(\mathbb{R})$ by part (e). Thus $\mathcal{B}(a)$ is indeed closed in the lower relative topology of $\mathcal{P}$. □

We now come to the corresponding properties of the lower boundedness components:

Proposition 5.2. *Let $a, b, c \in \mathcal{P}$. Then*

(a) $b \in (a)\mathcal{B}$ *if and only if* $a \in \mathcal{B}(b)$.

(b) $b \in (a)\mathcal{B}$ *if and only if for every* $v \in \mathcal{V}$ *there are* $\alpha, \beta > 0$ *such that* $\alpha a \leq b + \beta v$.

(c) *If* $b \in (a)\mathcal{B}$ *and* $c \in \mathcal{P}$, *then* $\beta b + c \in (a)B$ *for all* $\beta > 0$.

(d) *If* $\mathcal{B}(a) = \mathcal{B}(b)$, *then* $(a)\mathcal{B} = (b)\mathcal{B}$.

(e) $b \in \mathcal{B}(a) \cap (a)\mathcal{B}$ *if and only if* $\mathcal{B}(b) = \mathcal{B}(a)$.

(f) $b \in (a)\mathcal{B}$ *if and only if for all* $\mu \in \mathcal{P}^*$, $\mu(a) = +\infty$ *implies* $\mu(b) = +\infty$.

(g) $(a)\mathcal{B}$ *is closed in* $\mathcal{P}$ *with respect to the upper relative topology of* $\mathcal{P}$.

Proof. For part (a), we observe that $b \in (a)\mathcal{B}$ if and only if for every $v \in \mathcal{V}$ there is $\varepsilon > 0$ such that $b \in (a)v_\varepsilon$, that is $a \in v_\varepsilon(b)$. The latter means that $a \in \mathcal{B}(b)$.

For part (b), suppose that for every $v \in \mathcal{V}$ there are $\alpha, \beta > 0$ such that $\alpha a \leq b + \beta v$. Then $a \leq (1/\alpha)b + (\beta/\alpha)v$, hence $a \in \mathcal{B}(b)$ by 5.1(a), and $b \in (a)\mathcal{B}$ by part (a). For the converse, let $b \in (a)\mathcal{B}$, that is $a \in v_\varepsilon(b)$ for every $v \in \mathcal{V}$ with some $\varepsilon > 0$. This yields $a \leq \gamma b + \varepsilon v$ for some $1 \leq \gamma \leq 1 + \varepsilon$, hence $(1/\gamma)a \leq b + (\varepsilon/\gamma)v$, as claimed.

For part (c), let $b \in (a)\mathcal{B}$, that is $a \in \mathcal{B}(b)$, let $c \in \mathcal{P}$ and $\beta > 0$. 5.1 (d) shows that $a \in \mathcal{B}(\beta b)$, hence $a \in \mathcal{B}(\beta b) + \mathcal{B}(c) \subset \mathcal{B}(\beta b + c)$. Thus $\beta b + c \in (a)\mathcal{B}$.

For (d), suppose that $\mathcal{B}(a) = \mathcal{B}(b)$ and let $c \in (a)\mathcal{B}$. Then $a \in \mathcal{B}(c)$, hence $\mathcal{B}(b) = \mathcal{B}(a) \subset \mathcal{B}(c)$ by 5.1(e). Thus $b \in \mathcal{B}(c)$, hence $c \in (b)\mathcal{B}$. Similarly, one shows that $(b)\mathcal{B} \subset (a)\mathcal{B}$.

For part (e), let $b \in \mathcal{B}(a) \cap (a)\mathcal{B}$. Then $b \in \mathcal{B}(a)$ and $a \in \mathcal{B}(b)$ implies that $\mathcal{B}(a) = \mathcal{B}(b)$ by 5.1(e). For the converse, suppose that $\mathcal{B}(a) = \mathcal{B}(b)$. Then $b \in \mathcal{B}(a)$ and $a \in \mathcal{B}(b)$, hence $b \in \mathcal{B}(a) \cap (a)\mathcal{B}$.

For part (f), let $b \in (a)\mathcal{B}$ and let $\mu \in \mathcal{P}^*$ such that $\mu(a) = +\infty$. As $a \in \mathcal{B}(b)$, this implies $\mu(b) = +\infty$ by 5.1(f). Conversely, if $b \notin (a)\mathcal{B}$, that is $a \notin \mathcal{B}(b)$, then by 5.1(f) there is $\mu \in \mathcal{P}^*$ such that $\mu(a) = +\infty$ and $\mu(b) < +\infty$.

For part (g) we recall that the singleton set $\{+\infty\}$ is closed in the upper topology of $\overline{\mathbb{R}}$, hence its inverse image under any $\mu \in \mathcal{P}^*$ is closed with respect to the upper relative topology of $\mathcal{P}$. Following part (e), $(a)\mathcal{B}$ is the intersection of the inverse images for all $\mu \in \mathcal{P}^*$ such that $\mu(a) = +\infty$, hence $(a)\mathcal{B}$ is indeed closed for the upper relative topology. □

The sets

$$\mathcal{B}^s(a) = \mathcal{B}(a) \cap (a)\mathcal{B}$$

are called the *symmetric boundedness components* of $\mathcal{P}$.

Proposition 5.3. *The symmetric boundedness components are closed for addition and multiplication by strictly positive scalars. They satisfy a version of the cancellation law, that is* $a + c \preccurlyeq b + c$ *for elements* a, b *and* c *of the same boundedness component implies that* $a \preccurlyeq b$.

Proof. The first part of our statement follows from 5.1(b) and 5.2(c). For the second part, suppose that $a, b, c \in \mathcal{P}$ are bounded relative to each other and that $a + c \preccurlyeq b + c$. Given $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq c + \lambda v$. Thus $a + (c + \lambda v) \preccurlyeq b + (c + \lambda v)$. As we observed before, $(\mathcal{P}, \mathcal{V})$ endowed with the weak preorder $\preccurlyeq$ forms again a locally convex cone. Following Lemma 1.4.2 in [**2**], if applied to this order and the positive element $(a + \lambda v)$ of a full cone containing $\mathcal{P}$, the above implies $a \preccurlyeq b + \varepsilon(c + \lambda v)$ for all $\varepsilon > 0$. By our assumption, there are $\alpha, \beta \geq 0$ such that $c \leq \alpha b + \beta v$. Now combining the above yields

$$a \preccurlyeq b + \varepsilon\big(\alpha b + (\beta + \lambda)v\big) = (1 + \varepsilon\alpha)b + \varepsilon(\beta + \lambda)v$$

for all $\varepsilon > 0$. This shows $a \preccurlyeq_v b$ by our definition of the weak local preorder in Section 3. Finally, because $a \preccurlyeq_v b$ holds for all $v \in \mathcal{V}$ we infer that $a \preccurlyeq b$. □

Proposition 5.4. *The symmetric boundedness components furnish a partition of $\mathcal{P}$ into disjoint convex subsets that are closed and connected in the symmetric relative topology.*

Proof. Let $a \in \mathcal{P}$. By Propositions 5.1(g) and 5.2(g), the sets $\mathcal{B}(a)$ and $(a)\mathcal{B}$, hence $\mathcal{B}^s(a)$, are indeed closed in the symmetric relative topology, which is finer than both the upper and the lower relative topologies of $\mathcal{P}$. Both sets $\mathcal{B}(a)$ and $(a)\mathcal{B}$ are convex by 5.1(b) and 5.2(c), respectively. Thus $\mathcal{B}^s(a)$ is also convex. For any $b \in \mathcal{B}^s(a)$ we have $\mathcal{B}(b) = \mathcal{B}(a)$ by 5.2(e), and also $(b)\mathcal{B} = (a)\mathcal{B}$ by 5.2(d). This shows $\mathcal{B}^s(b) = \mathcal{B}^s(a)$. Any two symmetric boundedness components of $\mathcal{P}$ therefore either coincide or are disjoint. For connectedness, let $b, c \in \mathcal{B}^s(a)$. We shall verify that the mapping

$$t \mapsto f(t) = t\, b + (1 - t)c \,:\, [0\,, 1] \to \mathcal{B}^s(a)$$

is continuous with respect to the symmetric relative topology of $\mathcal{P}$. For this let $v \in \mathcal{V}$ and $\varepsilon > 0$. There are $\beta, \gamma, \lambda \geq 1$ such that $b \leq \gamma c + \lambda v$ and $c \leq \beta b + \lambda v$ as well as $0 \leq b + \lambda v$ and $0 \leq c + \lambda v$. Set $\delta = \varepsilon \min\{1/\beta, 1/\gamma, 1/(2\lambda)\}$. Then for $t_1, t_2 \in [0, 1]$ such that $t_1 \leq t_2 \leq t_1 + \delta$ we have

$$t_1 b \leq t_1 b + (t_2 - t_1)(b + \lambda v) \leq t_2 b + \delta\lambda v$$

and

$$\begin{aligned} c &= \frac{\beta(1-t_2)}{t_2 + \beta(1-t_2)}c + \frac{t_2}{t_2 + \beta(1-t_2)}c \\ &\leq \frac{\beta(1-t_2)}{t_2 + \beta(1-t_2)}c + \frac{t_2}{t_2 + \beta(1-t_2)}(\beta b + \lambda v) \\ &\leq \frac{\beta}{t_2 + \beta(1-t_2)}\Big((1-t_2)c + t_2 b\Big) + \frac{t_2}{t_2 + \beta(1-t_2)}\lambda v. \end{aligned}$$

As $t_2/\big(t_2 + \beta(1 - t_2)\big) \leq 1$ (recall that $\beta \geq 1$) and $(t_2 - t_1) \leq \delta$, the above yields

$$(t_2 - t_1)c \leq \frac{\beta(t_2 - t_1)}{t_2 + \beta(1-t_2)}\Big((1-t_2)c + t_2 b\Big) + \delta\lambda v.$$

Using this we calculate

$$\begin{aligned} f(t_1) &= t_1 b + (1-t_1)c = t_1 b + (1-t_2)c + (t_2-t_1)c \\ &\le (t_2 b + \delta\lambda v) + (1-t_2)c + \frac{\beta(t_2-t_1)}{t_2+\beta(1-t_2)}\Big((1-t_2)c + t_2 b\Big) + \delta\lambda v \\ &= \left(1 + \frac{\beta(t_2-t_1)}{t_2+\beta(1-t_2)}\right) f(t_2) + 2\delta\lambda v. \end{aligned}$$

Because both $\beta(t_2-t_1)/\big(t_2+\beta(1-t_2)\big) \le \beta\delta \le \varepsilon$ and $2\delta\lambda \le \varepsilon$, this shows $f(t_1) \in v_\varepsilon\big(f(t_2)\big)$. Similarly, one verifies that $f(t_2) \in v_\varepsilon\big(f(t_1)\big)$ holds in this case. Thus $|t_1 - t_2| \le \delta$ implies that $f(t_1) \in v^s_\varepsilon\big(f(t_2)\big)$. Summarizing, the continuity of the mapping $t \mapsto f(t)$ shows that every convex subset of $\mathcal{B}^s(a)$ is pathwise connected, hence connected in the symmetric relative topology of $\mathcal{P}$ (see Theorem 27.2 in [**8**]). □

We may also consider the local boundedness components of a locally convex cone $\mathcal{P}$ which arise if we endow $\mathcal{P}$ with the neighborhood subsystem $\mathcal{V}_v = \{\alpha v \mid \alpha > 0\}$ consisting of the multiples of a single neighborhood $v \in \mathcal{V}$. For an element $a \in \mathcal{P}$ and a neighborhood $v \in \mathcal{V}$, we define the *(local) upper, lower* and *symmetric v-boundedness components* of a as

$$\mathcal{B}_v(a) = \bigcup_{\varepsilon>0} v_\varepsilon(a), \quad (a)\mathcal{B}_v = \bigcup_{\varepsilon>0} (a)v_\varepsilon, \quad \text{and} \quad \mathcal{B}^s_v(a) = \mathcal{B}_v(a) \cap (a)\mathcal{B}_v,$$

respectively. The elements of $\mathcal{B}_v(a)$ are called *v-bounded (above) relative to a.* $\mathcal{B}_v(0) = \mathcal{B}_v$ consists of all v-bounded elements of $\mathcal{P}$. The global boundedness components may be recovered as

$$\mathcal{B}(a) = \bigcap_{v\in\mathcal{V}} \mathcal{B}_v(a), \qquad (a)\mathcal{B} = \bigcap_{v\in\mathcal{V}} (a)\mathcal{B}_v \qquad \text{and} \qquad \mathcal{B}^s(a) = \bigcap_{v\in\mathcal{V}} \mathcal{B}^s_v(a),$$

respectively. Obviously, the statements of Propositions 5.1, 5.2, 5.3 and 5.4 apply also to the local boundedness components, since we may replace the given neighborhood system $\mathcal{V}$ by the subsystem $\mathcal{V}_v$ and consider the locally convex cone $(\mathcal{P}, \mathcal{V}_v)$ for this purpose. The cancellation law in 5.3 holds with the weak local preorder $\preccurlyeq_v$ in this case. The dual cone $\mathcal{P}^*$ of $(\mathcal{P}, \mathcal{V}_v)$ consists only of the multiples of the functionals in v°, and the relative topologies of $\mathcal{P}$ are the relative v-topologies.

The main benefit in considering the local boundedness components as compared to the global ones, is the following: We shall proceed to verify that the disjoint partition of $\mathcal{P}$ into symmetric local boundedness components provides indeed a topological partition as well.

Proposition 5.5. *Let $a \in \mathcal{P}$ and $v \in \mathcal{V}$.*

(a) *$\mathcal{B}_v(a)$ is open in $\mathcal{P}$ with respect to the upper, closed with respect to the lower and both open and closed with respect to the symmetric relative v-topology of $\mathcal{P}$.*

(b) *$(a)\mathcal{B}_v$ is closed in $\mathcal{P}$ with respect to the upper, open with respect to the lower and both open and closed with respect to the symmetric relative v-topology of $\mathcal{P}$.*

Proof. Let $a \in \mathcal{P}$ and $v \in \mathcal{V}$ Proposition 5.1(g) states that $\mathcal{B}_v(a)$ is closed in the lower relative v-topology of $\mathcal{P}$. Let $b \in \mathcal{B}_v(a)$, that is $b \le \alpha b + \beta v$ for some $\alpha, \beta \ge 0$, and let $v_\varepsilon(b)$ be a lower neighborhood of b. Then for $c \in v_\varepsilon(b)$ we have $c \le \gamma b + \varepsilon v$ with some $1 \le \gamma \le 1+\varepsilon$, and therefore $c \le (\alpha\gamma)a + (\beta\gamma+\varepsilon)v$. This shows $c \in \mathcal{B}_v(a)$, hence $v_\varepsilon(b) \subset \mathcal{B}_v(a)$, and $\mathcal{B}_v(a)$ is seen to be open in the lower relative v-topology of $\mathcal{P}$. Moreover, because the symmetric relative v-topology is the common refinement of the upper and lower topologies, $\mathcal{B}_v(a)$ is indeed both open and closed in this topology. This completes part (a).

The argument for part (b) is similar: Proposition 5.2(g) states that $(a)\mathcal{B}_v$ is closed in the upper relative v-topology of $\mathcal{P}$. Let $b \in (a)\mathcal{B}_v$, that is $\alpha a \le b + \beta v$ for some $\alpha, \beta > 0$, and let $(b)v_\varepsilon$ be a lower neighborhood of b. Then for $c \in (b)v_\varepsilon$ we have $b \le \gamma c + \varepsilon v$ with some $1 \le \gamma \le 1+\varepsilon$, and therefore $\alpha a \le \gamma c + (\varepsilon+\delta)v$, hence $(\alpha/\gamma)a \le c + (\varepsilon+\delta)/\gamma v$. This shows $c \in (a)\mathcal{B}_v$, hence $(b)v_\varepsilon \subset (a)\mathcal{B}_v$, and $(a)\mathcal{B}_v$ is seen to be open in the lower relative v-topology of $\mathcal{P}$. Hence $(a)\mathcal{B}_v$ is both open and closed in the symmetric relative v-topology. □

Propositions 5.5 and 5.4 now yield a topological and algebraic partition of a locally convex cone into local boundedness components.

Proposition 5.6. *For every neighborhood $v \in \mathcal{V}$, the symmetric v-boundedness components furnish a partition of $\mathcal{P}$ into disjoint convex subsets that are open, closed and connected in the symmetric relative v-topology.*

A subset of $\mathcal{P}$ that is open or closed in any of the relative v-topologies is of course also open or closed in the corresponding (global) relative topology of $\mathcal{P}$. The same does however not hold for connectedness.

6. Connectedness Components

Topological vector spaces are connected and all of their elements are bounded. This does not hold for locally convex cones in general. However, Propositions 5.4 and 5.6 suggest relations between the boundedness and the connectedness components of a locally convex cone. Let us recall some of the relevant concepts from topology: The *quasi-component of a point x* in a topological space X is the intersection of all closed and open subsets of X which contain x. The quasi-components constitute a decomposition of X into pairwise disjoint and closed subsets (see VIII.26 in [**8**] or VI.1 in [**1**]). The *component of a point $x \in X$*, on the other hand is the largest connected subset of X which contains the point x. The components are subsets of the quasi-components and constitute a decomposition of X into pairwise disjoint, connected and closed subsets. A topological space is locally connected, if each of its points has a basis of connected neighborhoods. In locally connected spaces the quasi-components and components coincide and are both open and closed (see Corollary 27.10 in [**8**]).

Proposition 6.1. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone.*

(a) *In the symmetric relative topology of $\mathcal{P}$ the components, quasi-components and the symmetric boundedness components coincide.*

(b) *For every neighborhood $v \in \mathcal{V}$ and the symmetric relative v-topology, $\mathcal{P}$ is locally connected and the components, quasi-components and the symmetric v-boundedness components coincide.*

Proof. (a) For an element $a \in \mathcal{P}$ Proposition 5.4 implies that $\mathcal{B}^s(a)$ is contained in its (connectedness) component. On the other hand, $\mathcal{B}^s(a)$ is the intersection of the sets $\mathcal{B}^s_v(a)$ for all $v \in \mathcal{V}$, all of which are open and closed in the respective symmetric relative v-topologies, hence in the symmetric relative topology of $\mathcal{P}$ by Proposition 5.6. This shows that the quasi-component of a is contained in $\mathcal{B}^s(a)$. Hence these three components coincide.

For part (b) let $v \in \mathcal{V}$ and $a \in \mathcal{P}$. The v-boundedness component $\mathcal{B}^s_v(a)$ of a contains all the neighborhoods $v^s_\varepsilon(a)$ for $\varepsilon > 0$. Convexity then guarantees (see the corresponding argument in the proof of Proposition 5.4) that these neighborhoods are pathwise connected in the symmetric relative v-topology, hence $\mathcal{P}$ is locally connected. The components, quasi-components and the symmetric v-boundedness components of $\mathcal{P}$ coincide by part (a) if we endow $\mathcal{P}$ with the neighborhood subsystem $\mathcal{V}_v = \{\alpha v \mid \alpha > 0\}$. □

Proposition 6.2. *A locally convex cone $(\mathcal{P}, \mathcal{V})$ is locally connected in its symmetric relative topology if and only if every point $a \in \mathcal{P}$ has a basis of symmetric relative neighborhoods that are contained in $\mathcal{B}^s(a)$.*

Proof. Let $a \in \mathcal{P}$. The argument in the proof of Proposition 5.4 shows that every convex subset of $\mathcal{B}^s(a)$ is pathwise connected, hence connected in the symmetric relative topology. On the other hand, every connected subset of $\mathcal{P}$ containing the element a is a subset of $\mathcal{B}^s(a)$, the component of a by 6.1(a). Because the symmetric relative neighborhoods of a are convex, our claim follows. □

Example 6.3.

(a) Let $\mathcal{P} = \overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} \mid a \geq 0\}$, endowed with the neighborhood system $\mathcal{V} = \{0\}$ $\big($see Example 1.1(b)$\big)$. For the only neighborhood $v = 0 \in \mathcal{V}$ and $\varepsilon > 0$ the relative neighborhoods of an element $a \in \overline{\mathbb{R}}_+$ are

$$v_\varepsilon(a) = \big[0,\, (1+\varepsilon)a\big], \quad (a)v_\varepsilon = \big[\tfrac{a}{1+\varepsilon},\, +\infty\big] \quad \text{and} \quad v^s_\varepsilon = \big[\tfrac{a}{1+\varepsilon},\, (1+\varepsilon)a\big].$$

The symmetric relative topology therefore coincides with the Euclidean topology on $(0, +\infty)$, but renders $0 \in \mathcal{P}$ and $+\infty \in \mathcal{P}$ as isolated points. Recall from Example 2.1(b) that the symmetric given topology on $\overline{\mathbb{R}}_+$, in contrast, is the discrete topology. For the boundedness components of $\overline{\mathbb{R}}_+$ we have

$$\mathcal{B}(a) = [0, +\infty), \qquad (a)\mathcal{B} = (0, +\infty] \quad \text{and} \quad \mathcal{B}^s(a) = (0, +\infty)$$

for $a \in (0, +\infty)$,

$$\mathcal{B}(0) = \{0\}, \qquad (0)\mathcal{B} = [0, +\infty] \quad \text{and} \quad \mathcal{B}^s(0) = \{0\},$$

and

$$\mathcal{B}(+\infty) = [0, +\infty], \qquad (+\infty)\mathcal{B} = \{\infty\} \quad \text{and} \quad \mathcal{B}^s(+\infty) = \{\infty\}.$$

As claimed, the symmetric boundedness components furnish a partition of $\mathcal{P} = \overline{\mathbb{R}}_+$ into disjoint subsets that are both open and closed in the symmetric relative topology.

(b) Let X be a set and let $\mathcal{F}(X, \overline{\mathbb{R}})$ be the cone of all $\overline{\mathbb{R}}$-valued functions on X (see Example 2.1(d)), endowed with the pointwise operations and order. Let $\mathcal{Y}$ be a family of subsets of X and let $\mathcal{P}$ be the subcone of all functions in $\mathcal{F}(X, \overline{\mathbb{R}})$ that are uniformly bounded below on every set $Y \in \mathcal{Y}$. For every $Y \in \mathcal{Y}$ let

$v_Y \in \mathcal{P}$ be the function such that $v_Y(x) = 1$ for all $x \in Y$ and $v_Y(x) = +\infty$ else, and let $\mathcal{V}_\mathcal{Y}$ be the neighborhood system for $\mathcal{P}$ consisting of all strictly positive multiples and sums of such functions v_Y. In this way $(\mathcal{P}, \mathcal{V}_\mathcal{Y})$ is a full locally convex cone carrying the topology of uniform convergence on the sets in $\mathcal{Y}$. For a function $f \in \mathcal{P}$ and a neighborhood $v_Y \in \mathcal{V}_Y$, the v_Y-boundedness component $\mathcal{B}^s_{v_Y}(f)$ consists of all $g \in \mathcal{P}$ such that

$$\alpha f(x) - \beta \leq g(x) \leq \gamma f(x) + \delta$$

holds with some constants $\alpha, \beta, \gamma, \delta > 0$ for all $x \in Y$. Thus, obviously, $(v_Y)^s_\varepsilon(g) \subset \mathcal{B}^s_{v_Y}(f)$ for all $\varepsilon > 0$ whenever $g \in \mathcal{B}^s_{v_Y}(f)$. This observation confirms that the component $\mathcal{B}^s_{v_Y}(f)$ is both open and closed in the symmetric relative v_Y-topology, which is the topology of uniform convergence on Y. Yet the (global) boundedness component $B^s(f) = \bigcap_{Y \in \mathcal{Y}} \mathcal{B}^s_{v_Y}(f)$ is in general only closed in the symmetric relative topology, which is the topology of uniform convergence on all sets $Y \in \mathcal{Y}$. However, if the set X itself is contained in $\mathcal{Y}$, then the multiples of the neighborhood v_X form already a basis for $\mathcal{V}_\mathcal{Y}$, and the v_X-boundedness components coincide with the global ones. Following Proposition 6.2, $\mathcal{P}$ is locally connected in this case. Its boundedness components therefore coincide with the components and quasi-components in the symmetric relative topology (Proposition 6.1) and are both open and closed.

If, for another special case, $\mathcal{Y}$ consists of all finite subsets of X, then for $Y \in \mathcal{Y}$ the above condition yields that two functions $f, g \in \mathcal{P}$ are contained in the the same v_Y-boundedness component if and only if they take the value $+\infty$ at exactly the same points of Y. The symmetric relative v_Y-topology is the topology of pointwise convergence on the set Y in this case. Correspondingly, the global boundedness components consist of functions that take the value $+\infty$ at exactly the same points of X, and the symmetric relative topology is the topology of pointwise convergence on X. If X itself is an infinite set, then the global boundedness components are seen to be closed but not open in this topology.

Remark 6.4. It is interesting to notice that for a locally convex cone $(\mathcal{P}, \mathcal{V})$ the mapping

$$(\alpha, a) \mapsto \alpha a \ : \ [0, +\infty) \times \mathcal{P} \to \mathcal{P},$$

while generally not continuous with respect to the given topologies of $\mathbb{R}$ and $\mathcal{P}$, is indeed continuous if we consider the respective symmetric relative topologies of $\overline{\mathbb{R}}_+$ (see 6.3(a)) and $\mathcal{P}$ instead. For this, given $\alpha \in [0, +\infty)$ and $a \in \mathcal{P}$, for $v \in \mathcal{V}$ and $\varepsilon > 0$ let $\lambda \geq 2$ such that $0 \leq a + (\lambda - 2)v$. For $0 < \delta < \min\{1, \varepsilon/3, \varepsilon/(14\alpha\lambda)\}$ we consider the symmetric relative neighborhoods $u_\delta(\alpha) = \left[\frac{\alpha}{1+\delta},\ (1+\delta)\alpha\right]$ of α in $\overline{\mathbb{R}}_+$ and $v^s_\delta(a)$ of a in $\mathcal{P}$. For every $b \in v^s_\delta(a)$ we have $0 \leq b + \lambda v$ by 4.1(c) and $b \leq (1+\delta)a + \delta\lambda v$. Hence for $\beta \in u_\delta$ we estimate

$$\begin{aligned} \beta b + \left(\frac{\alpha\lambda}{1+\delta}\right) &\leq \beta(b + \lambda v) \leq (1+\delta)\alpha(b + \lambda v) \\ &\leq (1+\delta)\alpha\big((1+\delta)a + (\delta\lambda + \lambda)v\big) \\ &\leq (1+\delta)^2(\alpha a) + \alpha\lambda(1+\delta)^2 v. \end{aligned}$$

The cancellation law for the positive element v (see Lemma I.4.2 in [**2**]) together with $\alpha\lambda\left((1+\delta)^2 - 1/(1+\delta)\right) \le \alpha\lambda(7\delta) < \varepsilon$ yields

$$\beta a \le (1+\delta)^2(\alpha a) + \varepsilon v.$$

Because $1 \le (1+\delta)^2 \le 1 + 3\delta \le 1 + \varepsilon$, this demonstrates $\beta b \in v_\varepsilon(\alpha a)$. Similarly one verifies that $\alpha a \in (\beta b)v_\varepsilon$, hence $\beta b \in v_\varepsilon^s(\alpha a)$.

Similar and related notions of boundedness components in locally convex cones had previously been established in [**4**].

References

1. R. Engelking, *General Topology*, Monografie Matematyczne, Polish Scientific Publishers, Warszawa, 1977.
2. K. Keimel and W. Roth, *Ordered Cones and Approximation*, Lecture Notes in Mathematics, vol. 1517, Springer Verlag, Heidelberg-Berlin-New York, 1992.
3. W. Roth, *Locally convex cones as generalizations of locally convex vector spaces*, Far East Journal of Mathematical Sciences, Special Volume, Part II (1998), 215–245.
4. W. Roth, *A uniform boundedness theorem for locally convex cones*, Proceedings of the American Mathematical Society, **126** (7) (1998), 1973–1982.
5. W. Roth, *Hahn-Banach type theorems for locally convex cones*, Journal of the Australian Math. Soc. (Series A), **68** (1) (2000), 104–125.
6. W. Roth, *Separation properties for locally convex cones*, Journal of Convex Analysis, **9** (1) (2002), 301–307.
7. H.H. Schäfer, *Topological Vector Spaces*, Springer Verlag, Heidelberg-Berlin-New York, 1980.
8. S. Willard, *General Topology*, Addison-Wesley Publishing Company, Reading, Massachusetts, 1970.

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