NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 44 (2014), 45–60

EXCESS TOPOLOGIES IN METRIC SPACES

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(Received 6 September, 2013)

Abstract. Given a family $\mathscr{S}$ of nonempty subsets of a metric space $\langle X, d\rangle$ containing the singletons, we consider topologies on the nonempty subsets of X generated by families of excess functionals of the from $\{e_d(S, \cdot) : S \in \mathscr{S}\}$. Such topologies can be broken into lower and upper halves: the lower (resp. upper) half is the weakest topology on the nonempty subsets such that each excess functional in the family is upper (resp. lower) semicontinuous. Remarkably, one such topology can be stronger than another while its lower half can fail to be. We also study excess topologies generated by families of the form $\{e_d(\cdot, S) : S \in \mathscr{S}\}$; the results we give exhibit a decided lack of symmetry with those for the former class of excess topologies. This paper can be viewed as a sequel to a recent paper by the authors on gap topologies [**3**]. While the methodology and point-of-view are similar, the subject matter here is considerably more subtle.

1. Introduction

Let $\langle X, d\rangle$ be a metric space (assumed to always contain at least two points), and let $\mathscr{P}_0(X)$ (resp. $\mathscr{C}_0(X)$) denote the family of its nonempty subsets (resp. nonempty closed subsets). For $x \in X$ and $\varepsilon > 0$ we denote the open ball with center x and radius ε by $B_d(x, \varepsilon)$, and the ε-enlargement $\cup_{a\in A} B_d(a, \varepsilon)$ of $A \in \mathscr{P}_0(X)$ by $B_d(A, \varepsilon)$. For $x \in X$ and $A \in \mathscr{P}_0(X)$, we put $d(x, A) := \inf\{d(x, a) : a \in A\}$ and for nonempty subsets A and B, we denote the *gap* between them by

$$D_d(A, B) := \inf\{d(a, b) : a \in A, b \in B\},$$

while the *excess of A over B* is given by

$$e_d(A, B) := \sup\{d(a, B) : a \in A\}.$$

Gap is symmetric in A and B while excess is not; additionally, excess can assume values of infinity, e.g., if A is unbounded and B is bounded.

Much of the literature on topologies on spaces of subsets of X has restricted attention to topologies on $\mathscr{C}_0(X)$, where distance functionals were seen long ago to play a fundamental role. One reason for this focus is that for any nonempty subset A, $d(\cdot, A) = d(\cdot, \overline{A})$. The weak topology on $\mathscr{P}_0(X)$ (or on $\mathscr{C}_0(X)$) determined by $\{d(x, \cdot) : x \in X\}$ is called the *d-Wijsman topology* [**2, 4, 9, 11, 15, 20**]. We denote this topology by τ_{W_d}. Evidently, a net of nonempty sets $\langle A_j\rangle$ converges in the Wijsman topology to $A \in \mathscr{P}_0(X)$ provided $\langle d(\cdot, A_j)\rangle$ converges pointwise to

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2010 *Mathematics Subject Classification* Primary 54B20.
Key words and phrases: excess functional, excess topology, Wijsman topology, dual Wijsman topology, Hausdorff distance, gap functional, quasi-uniformity, strict inclusion, bornology.

$d(\cdot, A)$. Thus, we can view $\mathscr{P}_0(X)$ as sitting in $C(X, \mathbb{R})$ equipped with the topology of pointwise convergence. Stronger topologies result if we equip the function space with stronger topologies of uniform convergence:

(1) If we equip $C(X, \mathbb{R})$ with the topology of uniform convergence, then under the identification $A \leftrightarrow d(\cdot, A)$, $\mathscr{P}_0(X)$ is equipped with the pseudometrizable topology of *d-Hausdorff distance* [**2, 13**];

(2) If we equip $C(X, \mathbb{R})$ with the topology of uniform convergence on bounded sets, then under the identification $A \leftrightarrow d(\cdot, A)$, we obtain the pseudometrizable *Attouch-Wets topology*, also known as the *bounded Hausdorff topology* [**1, 2, 17**].

Nothing stronger than the Wijsman topology is obtained if we equip the function space with the topology of uniform convergence on compact subsets because the family of distance functions is equicontinuous, in fact, equi-Lipschitzian.

Another strengthening of the d-Wijsman topology occurs if we replace the family of distance functionals $\{d(x, \cdot) : x \in X\}$ by a family of gap functionals $\{D_d(S, \cdot) : S \in \mathscr{S}\}$ or a family of excess functionals $\{e_d(S, \cdot) : S \in \mathscr{S}\}$ where $\mathscr{S}$ is a subfamily of $\mathscr{P}_0(X)$ containing the singleton subsets $s(X)$, owing to the fact that for each $x \in X$,

$$D_d(\{x\}, \cdot) = d(x, \cdot) = e_d(\{x\}, \cdot)$$

regarded as functions on $\mathscr{P}_0(X)$.

Over the last twenty five years, there have been a number of papers considering gap and excess topologies and their position in the general theory of hyperspace topologies (see, e.g., [**4, 7, 8, 12, 19**]). Often functionals of different types are used in combination to generate standard hyperspace topologies as weak topologies, following [**8**]. The authors recently completed a study of gap topologies [**3**], the major result being identification of necessary and sufficient conditions on families of subsets $\mathscr{T}$ and $\mathscr{S}$ each containing the singletons and compatible metrics ρ and d such that the gap topology determined by $\mathscr{T}$ and ρ is coarser than the gap topology determined by $\mathscr{S}$ and d. These conditions are suggested by necessary and sufficient conditions for the inclusion $\tau_{W_\rho} \leq \tau_{W_d}$ discovered more than twenty years ago by Costantini, Levi and Zieminska [**10**] involving the ability to place between two concentric ρ-balls a finite union of d-balls of prescribed radii and d-balls with the same centers but with slightly larger radii as well.

In this article we principally study topologies on $\mathscr{P}_0(X)$ induced by excess functionals with fixed left arguments running over a family of nonempty subsets $\mathscr{S}$ of X often required to contain the singletons - a standard assumption (see, e.g., [**2, 12**]). For one thing, the assumption guarantees that the topology be Hausdorff when restricted to nonempty closed subsets as it would contain the Wijsman topology. Of course, we are interested in determining when one such topology is coarser than another, varying metrics as well as the family $\mathscr{S}$. Our method, as with previous investigations of Wijsman and gap topologies, involves splitting such excess topologies into upper and lower halves. But unlike our previous studies, separate inclusion of both upper halves and lower halves while of course sufficient for inclusion of the two-sided left excess topologies is not in general necessary!

We also study excess topologies where the excess functionals have fixed right argument, giving in one stroke necessary and sufficient conditions for inclusion of the two-sided excess topologies. We note that the family $\{e_d(\cdot, S) : S \in \mathscr{C}_0(X)\}$ generates as a weak topology the well-studied *d-proximal topology* on $\mathscr{C}_0(X)$ [**2, 4, 8**] which is also generated by $\{D_d(\cdot, S) : S \in \mathscr{C}_0(X)\}$. On the other hand, in contrast to benign gap topologies and topologies determined by excess functionals with fixed left argument, the pathology here is remarkable.

2. Preliminaries

Let $\langle X, d\rangle$ be a metric space. If $\mathscr{S} \subseteq \mathscr{P}_0(X)$, we write $\sum(\mathscr{S})$ for the family of finite unions of members of $\mathscr{S}$ and $\downarrow \mathscr{S}$ for $\{E \in \mathscr{P}_0(X) : \exists S \in \mathscr{S} \text{ with } E \subseteq S\}$.

If A and B are nonempty subsets of $\langle X, d\rangle$, the *Hausdorff distance* between them [**2, 13**] is given by

$$H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\} = \sup_{x\in X}|d(x, A) - d(x, B)|.$$

Hausdorff distance so defined is an extended real-valued pseudometric on $\mathscr{P}_0(X)$ which becomes an extended real-valued metric when restricted to $\mathscr{C}_0(X)$. Note that for $\{a, b\} \subseteq X, d(a, b) = H_d(\{a\}, \{b\})$. The proof of the following folk theorem is left to the reader.

Proposition 2.1. *Let $\langle X, d\rangle$ be a metric space and let S be a nonempty subset. Then $A \mapsto e_d(S, A)$, $A \mapsto e_d(A, S)$ and $A \mapsto D_d(S, A)$ are H_d-continuous on $\mathscr{P}_0(X)$.*

Realizing that the topologies on a prescribed set form a lattice with respect to inclusion, we adopt the standard notation $\tau_1 \leq \tau_2$ if $\tau_1 \subseteq \tau_2$ and $\tau_1 \vee \tau_2$ for the join of the two topologies in the lattice, i.e., the topology generation by $\tau_1 \cup \tau_2$.

We call a hyperspace topology τ - that is, a topology on $\mathscr{P}_0(X)$ - an *upper topology* (resp. *lower topology*) provided whenever $\mathscr{V} \in \tau$, $A \in \mathscr{V} \Rightarrow$ each nonempty subset (resp. superset) of A again lies in $\mathscr{V}$.

We can break the H_d-topology τ_{H_d} on $\mathscr{P}_0(X)$ into upper and lower halves $\tau^+_{H_d}$ and $\tau^-_{H_d}$ [**13**]; a local base at $A \in \mathscr{P}_0(X)$ for the former consists of $\{\mathscr{H}^+_d(A, \varepsilon) : \varepsilon > 0\}$ where $\mathscr{H}^+_d(A, \varepsilon) := \{C : C \subseteq B_d(A, \varepsilon)\}$. Dually, a local base for $\tau^-_{H_d}$ at A consists of $\{\mathscr{H}^-_d(A, \varepsilon) : \varepsilon > 0\}$ where $\mathscr{H}^-_d(A, \varepsilon) := \{C : A \subseteq B_d(C, \varepsilon)\}$. Of course, $\tau_{H_d} = \tau^-_{H_d} \vee \tau^+_{H_d}$.

Borrowing the notation of [**5**], we denote the closure of $\mathscr{S} \subseteq \mathscr{P}_0(X)$ with respect to $\tau^-_{H_d}$ by $\mathscr{S}^*_d$; thus $A \in \mathscr{S}^*_d$ if and only if for some $\varepsilon > 0$ and $S \in \mathscr{S}$, we have $A \subseteq B_d(S, \varepsilon)$.

Similarly, it is customary to split the d-Wijsman topology into upper and lower halves $\tau^+_{W_d}$ and $\tau^-_{W_d}$, the former being generated by all sets of the form $\{E : d(x, E) > \alpha\}$ where $x \in X$ and $\alpha > 0$ and the later being generated by all sets of the form $\{E : d(x, E) < \alpha\}$ where $x \in X$ and $\alpha > 0$ [**11, 15**]. As is well known, $\tau^-_{W_d}$ coincides with the *lower Vietoris topology*, having as a subbase all sets of the form $\{E : E \cap V \neq \emptyset\}$ where V runs over the nonempty open subsets of X. Thus, the lower d-Wijsman topology is unchanged under replacing d by an equivalent metric. Convergence of a net $\langle A_j\rangle_{j\in J}$ in the upper Wijsman topology means that for each $x \in X$, $\liminf_{j\in J} d(x, A_j) \geq d(x, A)$, while $\tau^-_{W_d}$-convergence means for each $x \in X$, $\limsup_{j\in J} d(x, A_j) \leq d(x, A)$.

3. Excess Functions with Fixed Left Argument

Given a family of nonempty subsets $\mathscr{S}$ of a metric space $\langle X, d\rangle$, the *left excess topology* on $\mathscr{P}_0(X)$ determined by $\mathscr{S}$ is the weakest topology such that for each $S \in \mathscr{S}$,

$$E \mapsto e_d(S, E)$$

is continuous as an extended real-valued function. Denoting this hyperspace topology by $\mathcal{LE}_{\mathscr{S},d}$, we can decompose it as

$$\mathcal{LE}^-_{\mathscr{S},d} \vee \mathcal{LE}^+_{\mathscr{S},d}$$

where $\mathcal{LE}^-_{\mathscr{S},d}$ is generated by of all sets of the form $\{E : e_d(S, E) < \alpha\}$ $(S \in \mathscr{S}, \alpha > 0)$ and $\mathcal{LE}^+_{\mathscr{S},d}$ is generated by of all sets of the form $\{E : e_d(S, E) > \alpha\}$ $(S \in \mathscr{S}, \alpha > 0)$.

Our first result says in part that we may assume without loss of generality that our family $\mathscr{S}$ consists of nonempty closed sets and that the family is stable under finite unions.

Lemma 3.1. *Suppose $\mathscr{S}$ and $\mathscr{T}$ are families of nonempty subsets of a metric space $\langle X, d\rangle$. Then*
(1) *if $\mathscr{T} \subseteq \mathscr{S}$, then $\mathcal{LE}^-_{\mathscr{T},d} \leq \mathcal{LE}^-_{\mathscr{S},d}$ and $\mathcal{LE}^+_{\mathscr{T},d} \leq \mathcal{LE}^+_{\mathscr{S},d}$;*

(2) *if $\mathscr{T} = \{\overline{S} : S \in \mathscr{S}\}$, then $\mathcal{LE}^-_{\mathscr{T},d} = \mathcal{LE}^-_{\mathscr{S},d}$ and $\mathcal{LE}^+_{\mathscr{T},d} = \mathcal{LE}^+_{\mathscr{S},d}$;*

(3) *$\mathcal{LE}^-_{\mathscr{S},d} = \mathcal{LE}^-_{\sum(\mathscr{S}),d}$ and $\mathcal{LE}^+_{\mathscr{S},d} = \mathcal{LE}^+_{\sum(\mathscr{S}),d}$.*

Proof. Claim (1) is obvious and claim (2) follows immediately from $e_d(S, E) = e_d(\overline{S}, E)$ whatever S and E may be. For (3) we just look at the lower half of the hyperspace topology. If $\{S_1, S_2, \ldots, S_n\} \subseteq \mathscr{S}$ and $\alpha > 0$, then

$$\{E : e_d(\cup_{i=1}^n S_i, E) < \alpha\} = \{E : \max_{i\leq n} e_d(S_i, E) < \alpha\}$$
$$= \cap_{i=1}^n \{E : e_d(S_i, E) < \alpha\},$$

and this shows that $\mathcal{LE}_{\mathscr{S},d^-} \geq \mathcal{LE}^-_{\sum(\mathscr{S}),d}$. □

The lower excess topology determined by a family $\mathscr{S}$ can be made properly finer replacing $\mathscr{S}$ by $\downarrow \mathscr{S}$.

Example 3.2. Let $X = \mathbb{R}$ equipped with the usual metric, and let $\mathscr{S} = \{\mathbb{R}\} \cup s(X)$. Clearly, $\downarrow \mathscr{S} = \mathscr{P}_0(X)$. Let $A_n = [0, n]$ and let $A = [0, \infty)$. If $x < 0$ then $\forall n \in \mathbb{N}$, $e_d(\{x\}, A_n) = e_d(\{x\}, A) = |x|$, while if $x > 0$,

$$\lim_{n\to\infty} e_d(\{x\}, A_n) = 0 = e_d(\{x\}, A).$$

Also, $\forall n$, $e_d(\mathbb{R}, A_n) = e_d(\mathbb{R}, A) = \infty$. Thus, we actually have $\mathcal{LE}_{\mathscr{S},d}$-convergence of $\langle A_n\rangle$ to A. Since $\mathbb{R} \in \mathscr{S}$, we have $[0, \infty) \in \downarrow S$, and $\lim_{n\to\infty} e_d([0, \infty), A_n) = \infty$ while $e_d([0, \infty), A) = 0$. This shows that $\mathcal{LE}^-_{\downarrow\mathscr{S},d}$-convergence fails.

The next lemma says that one-sided left excess topologies on $\mathscr{P}_0(X)$ are not enlarged by replacing our family $\mathscr{S}$ by its closure in the Hausdorff pseudometric topology and is anticipated by [**2**, Lemma 4.1.2].

Lemma 3.3. *Suppose $\mathscr{S}$ and $\mathscr{T}$ are families of nonempty subsets of $\langle X, d\rangle$ such that $\forall\varepsilon > 0\ \forall T \in \mathscr{T}$, $\exists S \in \mathscr{S}$ with $H_d(S,T) \leq \varepsilon$. Then $\mathcal{LE}^-_{\mathscr{T},d} \leq \mathcal{LE}^-_{\mathscr{S},d}$ and $\mathcal{LE}^+_{\mathscr{T},d} \leq \mathcal{LE}^+_{\mathscr{S},d}$.*

Proof. We just prove inclusion for the lower halves, leaving the upper halves to the interested reader. Let $T \in \mathscr{T}$ and $\alpha > 0$ be arbitrary and suppose $A \in \{E : e_d(T,E) < \alpha\}$. Put $e_d(T,A) := \beta < \alpha$. Choosing $S \in \mathscr{S}$ with $H_d(S,T) < \frac{\alpha-\beta}{2}$, let $\mathscr{A} = \{E : e_d(S,E) < \frac{\alpha+\beta}{2}\}$. Evidently $A \in \mathscr{A}$ because

$$e_d(S,A) \leq e_d(S,T) + e_d(T,A) < \frac{\alpha-\beta}{2} + \beta = \frac{\alpha+\beta}{2},$$

while $\mathscr{A} \subseteq \{E : e_d(T,E) < \alpha\}$ because if $E \in \mathscr{A}$, then

$$e_d(T,E) \leq e_d(T,S) + e_d(S,E) < \frac{\alpha-\beta}{2} + \frac{\alpha+\beta}{2} = \alpha.$$

This shows that $\{E : e_d(T,E) < \alpha\}$ contains a $\mathcal{LE}^-_{\mathscr{S},d}$-neighborhood of each of its points. □

Corollary 3.4. *Suppose $\mathscr{S}$ and $\mathscr{T}$ are families of nonempty subsets of $\langle X, d\rangle$ that have the same H_d-closures in $\mathscr{P}_0(X)$. Then $\mathcal{LE}^-_{\mathscr{T},d} = \mathcal{LE}^-_{\mathscr{S},d}$ and $\mathcal{LE}^+_{\mathscr{T},d} = \mathcal{LE}^+_{\mathscr{S},d}$.*

As a particular application, let $\mathscr{S} = s(x)$ and let $\mathscr{T} =$ the family $\mathscr{K}_0(X)$ of nonempty compact subsets of X. The family of nonempty finite subsets $\mathscr{F}_0(X) = \sum(s(x))$ of X generates the same left-excess topologies as does $s(X)$ and $\mathscr{F}_0(X)$ and $\mathscr{K}_0(X)$ have the same H_d-closures in $\mathscr{P}_0(X)$, namely the family of d-totally bounded nonempty subsets. We may conclude that $s(X)$ and $\mathscr{K}_0(X)$ determine the same left-excess topologies.

The next lemma explains more concretely what convergence in the lower topology entails

Lemma 3.5. *Suppose $\langle X, d\rangle$ is a metric space and $\mathscr{S}$ is a family of nonempty subsets of X. For a net $\langle A_j\rangle_{j\in J}$ of nonempty subsets of X and $A \in \mathscr{P}_0(X)$, the following conditions and equivalent:*

(1) *$\langle A_j\rangle_{j\in J}$ is $\mathcal{LE}^-_{\mathscr{S},d}$-convergent to A;*

(2) *$\forall S \in \mathscr{S}$, lim $\text{sup}_{j\in J}\ e_d(S,A_j) \leq e_d(S,A)$;*

(3) *whenever $S \in \mathscr{S}$ and $0 < \varepsilon < \alpha$, $S \subseteq B_d(A,\varepsilon) \Rightarrow S \subseteq B_d(A_j,\alpha)$ eventually.*

Proof. (1) $\Rightarrow$ (2). Assume (1) holds, and fix $S \in \mathscr{S}$. If $e_d(S,A) = \infty$, there is nothing to show. Otherwise, let $\alpha > e_d(S,A)$ be arbitrary. Since $\{E : e_d(S,E) < \alpha\}$ is a $\mathcal{LE}^=_{\mathscr{S},d}$-neighborhood of A, we must have $A_j \in \{E : e_d(S,E) < \alpha\}$ eventually and condition (2) follows.

(2) $\Rightarrow$ (3). Suppose (2) holds and $S \in \mathscr{S}$ satisfies $S \subseteq B_d(A,\varepsilon)$. Clearly, $e_d(S,A) \leq \varepsilon$ so that by (2) eventually $e_d(S,A_j) < \alpha$ which implies $S \in B_d(A_j,\alpha)$.

(3) $\Rightarrow$ (1). Suppose $A \in \{E : e_d(S, A) < \alpha\}$. Then for some $\varepsilon \in (0, \alpha)$, we have $e_d(S, A) < \varepsilon \Rightarrow S \subseteq B_d(A, \varepsilon)$. By condition (3), eventually $S \subseteq B_d(A_j, \frac{1}{2}(\varepsilon + \alpha))$ so that eventually, $e_d(S, A_j) \leq \frac{1}{2}(\varepsilon + \alpha) < \alpha$. □

This corollary sharpens [**2**, Lemma 4.1.1].

Corollary 3.6. *Suppose $\langle X, d\rangle$ is a metric space and let $\mathscr{S}$ be subfamily of $\mathscr{P}_0(X)$ containing the singletons. The following conditions are equivalent for a sequence $\langle x_n\rangle$ in X and a point $p \in X$:*
(1) $lim_{n\to\infty} d(x_n, p) = 0$*;*

(2) $\langle\{x_n\}\rangle$ *converges to* $\{p\}$ *in* $\mathcal{LE}_{\mathscr{S},d}$*;*

(3) $\langle\{x_n\}\rangle$ *converges to* $\{p\}$ *in* $\mathcal{LE}^-_{\mathscr{S},d}$*.*

Proof. If $\lim_{n\to\infty} d(x_n, p) = 0$, then by Proposition 2.1, for each $S \in \mathscr{S}$, we have $\lim_{n\to\infty} e_d(S, \{x_n\}) = e_d(S, \{p\})$. This proves (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is trivial because $\mathcal{LE}^-_{\mathscr{S},d}$ is a coarser topology. For (3) $\Rightarrow$ (1), if we have convergence of $\langle\{x_n\}\rangle$ to $\{p\}$ in $\mathcal{LE}^-_{\mathscr{S},d}$, then since $\{p\} \in \mathscr{S}$, by Lemma 3.5 we have

$$\limsup_{n\to\infty} d(p, x_n) = \limsup_{n\to\infty} e_d(\{p\}, \{x_n\}) \leq e_d(\{p\}, \{p\}) = 0,$$

which shows that $\lim_{n\to\infty} x_n = p$ back in the metric space. □

The easy proof of the following companion to Lemma 3.5 is left to the reader.

Lemma 3.7. *Suppose $\langle X, d\rangle$ is a metric space and $\mathscr{S}$ is a family of nonempty subsets of X. For a net $\langle A_j\rangle_{j\in J}$ of nonempty subsets of X and $A \in \mathscr{P}_0(X)$, the following conditions are equivalent:*

(1) $\langle A_j\rangle_{j\in J}$ *is* $\mathcal{LE}^+_{\mathscr{S},d}$*convergent to* A*;*

(2) $\forall S \in \mathscr{S}$*,* $lim\ inf_{j\in J}\ e_d(S, A_j) \geq e_d(S, A)$*;*

(3) *whenever* $S \in \mathscr{S}$ *and* $0 < \varepsilon < \alpha$*,* $S \not\subseteq B_d(A, \alpha) \Rightarrow S \not\subseteq B_d(A_j, \varepsilon)$ *eventually.*

As we said at the outset, we are most interested in families that contain the singletons, and in this case, the structure of the upper excess topologies becomes more transparent. The next result is a special case of [**7**, Lemma 2] which was stated but not proved there. As a courtesy to the reader, we include a proof here.

Proposition 3.8. *Let $\langle X, d\rangle$ be a metric space and let $\mathscr{S}$ contain the singletons. Then $\mathcal{LE}^+_{\mathscr{S},d} = \tau^+_{W_d}$ on $\mathscr{P}_0(X)$.*

Proof. Since $\tau^+_{W_d} = \mathcal{LE}^+_{s(X),d}$ and $s(X) \subseteq \mathscr{S}$, one inclusion is immediate. For the other inclusion $\mathcal{LE}^+_{\mathscr{S},d} \leq \tau^+_{W_d}$, let $S_0 \in \mathscr{S}$ and $\alpha > 0$ be arbitrary, and let $E_0 \in \{E : e_d(S_0, E) > \alpha\}$. By the definition of excess, there exists $s_0 \in S_0$ with $d(s_0, E) > \alpha$. Clearly, $E_0 \in \{E : d(s_0, E) > \alpha\} \subseteq \{E : e_d(S_0, E) > \alpha\}$ as required. □

Our next result gives necessary and sufficient conditions for $\mathcal{LE}^-_{\mathscr{T},\rho} \leq \mathcal{LE}^-_{\mathscr{S},d}$ on $\mathscr{P}_0(X)$. There is no need in our arguments to assume that either family of subsets contains the singletons or that d and ρ determine the same topology on X.

Theorem 3.9. *Let d and ρ be metrics on a set X and let $\mathscr{S}$ and $\mathscr{T}$ be families of nonempty subsets. The following conditions are equivalent:*
(1) $\mathcal{LE}^-_{\mathscr{T},\rho} \leq \mathcal{LE}^-_{\mathscr{S},d}$ *on* $\mathscr{P}_0(X)$;

(2) $\forall T \in \mathscr{T}$, *whenever* $0 < \varepsilon < \alpha$ *and* $A \in \mathscr{H}^-_\rho(T,\varepsilon)$ *where* $\mathscr{H}^-_\rho(A,\alpha) \neq \mathscr{P}_0(X)$, $\exists\ \{S_1, S_2, \ldots, S_n\} \subseteq \mathscr{S}$ *and* $0 < \lambda_i < \sigma_i$ *for* $i = 1, 2, \ldots, n$ *with*

$$A \in \cap_{i=1}^n \mathscr{H}^-_d(S_i,\lambda_i) \subseteq \cap_{i=1}^n \mathscr{H}^-_d(S_i,\sigma_i) \subseteq \mathscr{H}^-_\rho(T,\alpha).$$

Proof. At the outset, we explain what $\mathscr{H}^-_\rho(A,\alpha) \neq \mathscr{P}_0(X)$ really means: there exists $p \in X$ with $A \nsubseteq B_\rho(p,\alpha)$; alternatively, $\exists p \in X\ \exists a \in A$ with $\rho(a,p) \geq \alpha$.

(2) $\Rightarrow$ (1). Assume $\langle A_j \rangle_{j\in J}$ is a net of nonempty subsets $\mathcal{LE}^-_{\mathscr{S},d}$-convergent to A. To verify $\mathcal{LE}^-_{\mathscr{T},\rho}$-convergence, we use the third criterion of Lemma 3.5. To this end, suppose $T \in \mathscr{T}$ satisfies $T \subseteq B_\rho(A,\varepsilon)$ and $\alpha > \varepsilon$ is arbitrary. If $\mathscr{H}^-_\rho(A,\alpha) = \mathscr{P}_0(X)$. then for each $j \in J, T \subseteq B_\rho(A_j,\alpha)$ and we are done. Otherwise, choose $\{S_1, S_2, \ldots, S_n\} \subseteq \mathscr{S}$ and $0 < \lambda_i < \sigma_i$ that satisfy condition (2). Since $S_i \subseteq B_d(A,\lambda_i)$ and $\lambda_i < \sigma_i$ for $i \leq n$, for some $j_0 \in J, j \geq j_0 \Rightarrow \forall i \leq n,\ S_i \subseteq B_d(A_j,\sigma_i)$, i.e.,

$$A_j \in \cap_{i=1}^n \mathscr{H}^-_d(S_i,\sigma_i) \qquad\qquad (j \geq j_0).$$

By the last inclusion in condition (2) we get $T \subseteq B_\rho(A_j,\alpha)$ whenever $j \geq j_0$ as required.

(1) $\Rightarrow$ (2). We prove the contrapositive. If condition (2) fails, $\exists T \in \mathscr{T},\ \exists 0 < \varepsilon < \alpha, \exists A$ with $T \subseteq B_\rho(A,\varepsilon)$ and $\exists p \in X$ with $A \nsubseteq B_\rho(p,\alpha)$ such that for every choice of $S_1, S_2, \ldots, S_n$ in $\mathscr{S}$ and $0 < \lambda_i < \sigma_i$ for $i = 1, 2, \ldots, n$, either

$$A \notin \cap_{i=1}^n \mathscr{H}^-_d(S_i,\lambda_i) \text{ or } \cap_{i=1}^n \mathscr{H}^-_d(S_i,\sigma_i) \nsubseteq \mathscr{H}^-_\rho(T,\alpha).$$

Put $\mathscr{B} = \{S \in \mathscr{S} : \exists\mu > 0 \text{ such that } S \subseteq B_d(A,\mu)\}$. If $\mathscr{B} = \emptyset$, then each net in $\mathscr{P}_0(X)$ is $\mathcal{LE}^-_{\mathscr{S},d}$-convergent to A because $\forall S \in \mathscr{S},\ e_d(S,A) = \infty$. On the other hand, the constant sequence $\{p\}, \{p\}, \{p\}, \ldots$ cannot $\mathcal{LE}^-_{\mathscr{T},\rho}$-converge to A because $e_\rho(T,A) < \varepsilon$ while $e_\rho(T,\{p\}) \geq \alpha$.

Thus we are left with the case $\mathscr{B}$ nonempty. For each $S \in \mathscr{B}$ put $r_S = \inf\{\mu > 0 : S \subseteq B_d(A,\mu)\}$. Let $\mathfrak{F}$ be the family of nonempty finite subsets of $\mathscr{B}$ partially ordered by inclusion, and let $J := \mathfrak{F} \times \mathbb{N}$ directed by the product partial order. For each $\{S_1, S_2, \ldots, S_n\} \in \mathfrak{F}$ and $k \in \mathbb{N}$, $A \in \cap_{i=1}^n \mathscr{H}^-_d(S_i, r_{S_i} + 1/k)$. Putting $j = (\{S_1, S_2, \ldots, S_n\}, k)$ by assumption there must exists

$$B_j \in \cap_{i=1}^n \mathscr{H}^-_d(S_i, r_{S_i} + 2/k) \backslash \mathscr{H}^-_\rho(T,\alpha).$$

The net $\langle B_j \rangle_{j\in J}$ evidently does not converge to A with respect to $\mathcal{LE}^-_{\mathscr{T},\rho}$. To see that we get $\mathcal{LE}^-_{\mathscr{S},d}$ convergence to A, fix $S_0 \in \mathscr{S}$ and let $0 < \beta < \mu$ where $S_0 \subseteq B_d(A,\beta)$. Clearly $S_0 \in \mathscr{B}$ and $r_{S_0} \leq \beta$. Choose $k_0 \in \mathbb{N}$ with $\frac{2}{k_0} < \mu - \beta$. Then for each $k \geq k_0$ and $\{S_1, S_2, \ldots, S_n\} \in \mathfrak{F}$, with $j = (\{S_0, S_1, S_2, \ldots, S_n\}, k)$, we have

$$B_j \in \cap_{i=0}^n \mathscr{H}_d^-(S_i, r_{S_i} + \frac{2}{k}) \subseteq \mathscr{H}_d^-(S_0, r_{S_0} + \frac{2}{k}),$$

and since $r_{S_0} + \frac{2}{k} < \mu$, we get $S_0 \subseteq B_d(B_j, \mu)$ whenever $j \geq \{\{S_0\}, k_0\}$. □

While we did not assume that either family contains the singletons, if $\mathscr{T}$ happens to contain the singletons, then $\mathcal{LE}^-_{\mathscr{T},\rho} \leq \mathcal{LE}^-_{\mathscr{S},d}$ on $\mathscr{P}_0(X)$ easily implies that the d-topology is finer than the ρ-topology on X. To see this, let $\langle x_n \rangle$ be a sequence in X with $\lim_{n\to\infty} d(p, x_n) = 0$. Then by Proposition 2.1, $\forall S \in \mathscr{S}$, $\lim_{n\to\infty} e_d(S, \{x_n\}) = e_d(S, \{p\})$. Since $\{p\} \in \mathscr{T}$, we conclude that

$$\limsup_{n\to\infty} e_\rho(\{p\}, \{x_n\}) \leq e_\rho(\{p\}, \{p\}) = 0,$$

that is, $\lim_{n\to\infty} \rho(p, x_n) = 0$.

Assuming that our families $\mathscr{S}$ and $\mathscr{T}$ both contain the singletons, then $\mathcal{LE}^+_{\mathscr{T},\rho} \leq \mathcal{LE}^+_{\mathscr{S},d}$ occurs if and only if $\tau^+_{W_\rho} \leq \tau^+_{W_d}$. Necessary and sufficient conditions for the latter inclusion were identified by Costantini, Levi and Zieminska [**10**] in a paper that can be viewed as the provenance of the current paper as well as [**3**].

Definition 3.10. Let $\langle X, d \rangle$ be a metric space and let A and B be nonempty subsets. We say that B *strictly d-includes* A provided there exists $\{b_1, b_2, b_3, \ldots, b_n\} \subseteq B$ and numbers $0 < \lambda_i < \sigma_i$ for $i = 1, 2, \ldots, n$ with

$$A \subseteq \cup_{i=1}^n B_d(b_i, \lambda_i) \subseteq \cup_{i=1}^n B_d(b_i, \sigma_i) \subseteq B.$$

Their characterization of inclusion of the upper Wijsman topologies goes as follows: $\tau^+_{W_\rho} \leq \tau^+_{W_d}$ if and only if whenever $p \in X$ and $0 < \varepsilon < \alpha$ and $B_\rho(p, \alpha) \neq X$, then $B_\rho(p, \varepsilon)$ is strictly d-included in $B_\rho(p, \alpha)$. In plain English: given two concentric ρ-balls with different radii, if the one with larger radius is a proper subset of X, then the one with smaller radius is strictly d-included in one with larger radius. Their condition easily implies that the d-topology is finer than the ρ-topology, and the condition is necessary and sufficient for $\tau_{W_\rho} \leq \tau_{W_d}$ as well (see also [**2**, p. 40]).

Given that each two-sided left excess topology is the supremum of its upper and lower halves, we immediately get sufficient conditions for $\mathcal{LE}_{\mathscr{T},\rho} \leq \mathcal{LE}_{\mathscr{S},d}$ provided both families contain the singletons.

Theorem 3.11. *Let X be a set equipped with metrics d and ρ and let $\mathscr{S} \subseteq \mathscr{P}_0(X)$ and $\mathscr{T} \subseteq \mathscr{P}_0(X)$ where $s(X) \subseteq \mathscr{S} \cap \mathscr{T}$. Suppose the following two conditions hold:*

(1) *$\forall T \in \mathscr{T}$, whenever $0 < \varepsilon < \alpha$ and $A \in \mathscr{H}^-_\rho(T, \varepsilon)$ where $\mathscr{H}^-_\rho(A, \alpha) \neq \mathscr{P}_0(X)$, there exists $\{S_1, S_2, \ldots, S_n\} \subseteq \mathscr{S}$ and $0 < \lambda_i < \sigma_i$ for $i = 1, 2, \ldots, n$ with*

$$A \in \cap_{i=1}^n \mathscr{H}_d^-(S_i, \lambda_i) \subseteq \cap_{i=1}^n \mathscr{H}_d^-(S_i, \sigma_i) \subseteq \mathscr{H}_\rho^-(T, \alpha);$$

(2) *whenever $p \in X$ and $0 < \varepsilon < \alpha$ and $B_\rho(p, \alpha) \neq X$, then $B_\rho(p, \varepsilon)$ is strictly d-included in $B_\rho(p, \alpha)$.*

Then $\mathcal{LE}_{\mathscr{T},\rho} \leq \mathcal{LE}_{\mathscr{S},d}$ and the d-topology is finer than the ρ-topology on X.

As we have seen, if $\mathscr{S}$ is a family of nonempty subsets containing the singletons, then $\mathcal{LE}^+_{\mathscr{S},d}$ agrees with the upper d-Wijsman topology, so trivially, if $\mathscr{T}$ is another such family, then the upper left excess topologies agree (note that the metrics are both d here). Remarkably, in this setting, the inclusion $\mathcal{LE}_{\mathscr{T},d} \leq \mathcal{LE}_{\mathscr{S},d}$ can hold while $\mathcal{LE}^-_{\mathscr{T},d} \leq \mathcal{LE}^-_{\mathscr{S},d}$ can fail, showing that there is some interaction between the upper and lower halves of the hyperspace topologies.

Example 3.12. Let $X = \{a, b, c\} \cup \mathbb{N}$ where the $\{a, b, c\} \cap \mathbb{N} = \emptyset$, equipped with the metric d defined for $x \neq y$ by

$$d(x, y) = \begin{cases} 1 & \text{if either } \{x, y\} = \{a, b\} \text{ or } c \in \{x, y\}, \\ 2 & \text{otherwise.} \end{cases}$$

Notice that c has distance 1 from every other point. Put $A = \{a\} \cup \mathbb{N}$, $B = \{b\} \cup \mathbb{N}$, and $C = \{c\} \cup \mathbb{N}$, and then let $\mathscr{T} = \{A\} \cup s(X)$ and $\mathscr{S} = \{B, C\} \cup s(X)$. Let us first show that $\mathcal{LE}^-_{\mathscr{T},d} \nleq \mathcal{LE}^-_{\mathscr{S},d}$.

To this end, consider $\{E : e_d(A, E) < \frac{1}{2}\} = \{A, A \cup \{b\}, A \cup \{c\}, X\}$, which is evidently a subbasic open set in $\mathcal{LE}^-_{\mathscr{T},d}$. We show that this contains no $\mathcal{LE}^-_{\mathscr{S},d}$-neighborhood of A. Without loss of generality, we can assume such a neighborhood looks like $\cap_{i=1}^n \{E : e_d(S_i, E) < \varepsilon_i\}$ where $\{S_1, S_2, \ldots, S_n\} \subseteq \mathscr{S}$ and each $\varepsilon_i > 0$. It suffices to show that for each $i \leq n$, $\{E : e_d(S_i, E) < \varepsilon_i\}$ contains a singleton subset $\{x_i\}$, for then $\{x_1, x_2, \ldots, x_n\}$ would lie in the intersection while $\{A, A \cup \{b\}, A \cup \{c\}, X\}$ contains no finite set as a member.

If $S_i = \{x_i\}$, then this does the job. Otherwise either $S_i = B$ or $S_i = C$, and the condition $A \in \{E : e_d(S_i, E) < \varepsilon_i\}$ in either case implies $\varepsilon_i > 1$, so that $\{c\} \in \{E : e_d(S_i, E) < \varepsilon_i\}$.

To complete the argument, we show that nevertheless $\mathcal{LE}_{\mathscr{T},d} \leq \mathcal{LE}_{\mathscr{S},d}$. In view of the canonical subbases for these two-sided excess topologies, it suffices to show that for each $\varepsilon > 0$, $\{E : e_d(A, E) < \varepsilon\} \in \mathcal{LE}_{\mathscr{S},d}$. There are three cases to consider for $\varepsilon > 0$:

(1) if $0 < \varepsilon \leq 1$, then $\{E : e_d(A, E) < \varepsilon\} = \{A, A \cup \{b\}, A \cup \{c\}, X\}$;

(2) if $1 < \varepsilon \leq 2$, then $\{E : e_d(A, E) < \varepsilon\} = \{A, B, A \cup \{b\}\} \cup \{E : c \in E\}$;

(3) if $\varepsilon > 2$, then $\{E : e_d(A, E) < \varepsilon\} = \mathscr{P}_0(X)$.

As for the first case, we compute

$$\{A, A \cup \{b\}, A \cup \{c\}, X\} = \{A \cup \{c\}, X\} \cup \{A, A \cup \{b\}\},$$

while

$$\{A \cup \{c\}, X\} = \{E : e_d(C, E) < \frac{1}{2}\} \cap \{E : e_d(\{a\}, E) < \frac{1}{2}\}$$

and

$$\{A, A \cup \{b\}\} = \{E : e_d(B, E) < \frac{3}{2}\} \cap \{E : e_d(\{a\}, E) < \frac{1}{2}\} \cap \{E : e_d(\{c\}, E) > \frac{1}{2}\}.$$

The second case is easier:

$$\{A, B, A \cup \{b\}\} \cup \{E : c \in E\} = \{E : e_d(B, E) < \frac{3}{2}\}.$$

The last case is of course trivial.

Proposition 3.13. *Suppose* $\mathcal{LE}^-_{\mathscr{T},\rho} \leq \mathcal{LE}^-_{\mathscr{S},d}$. *Then* $\mathscr{T} \subseteq (\sum(\mathscr{S}))^*_\rho$.

Proof. If not, there exists $T \in \mathscr{T}$ and $\varepsilon > 0$ such that for each $S \in \sum(\mathscr{S})$, $e_\rho(T, S) > \varepsilon$. Now $X \in \{E : e_\rho(T, E) < \varepsilon\}$. We show that this contains no $\mathcal{LE}^-_{\mathscr{S},d}$ neighborhood of X. Such a neighborhood can be assumed to be of the form $\cap_{i=1}^n \{E : e_d(S_i, E) < \varepsilon_i\}$ where $S_1, S_2, \ldots, S_n \in \mathscr{S}$ and $\varepsilon_i > 0$. Since $\cup_{i=1}^n S_i$ belongs to the neighborhood and by assumption, $e_\rho(T, \cup_{i=1}^n S_i) > \varepsilon$, we have $\cup_{i=1}^n S_i \notin \{E : e_\rho(T, X) < \varepsilon\}$, a contradiction. □

We next provide some counterexamples around our last proposition when $d = \rho$.

Example 3.14. The condition $\mathscr{T} \subseteq (\sum(\mathscr{S}))^*_d$ is not sufficient for $\mathcal{LE}^-_{\mathscr{T},d} \leq \mathcal{LE}^-_{\mathscr{S},d}$. This occurs for example whenever $X \in \sum(\mathscr{S})$ as is the case for the space $\{a, b, c\} \cup \mathbb{N}$ of Example 3.12.

Example 3.15. The stronger requirement that $\mathscr{T}$ be contained in the closure of $\sum(\mathscr{S})$ with respect to Hausdorff distance H_d while sufficient for both $\mathcal{LE}^-_{\mathscr{T},d} \leq \mathcal{LE}^-_{\mathscr{S},d}$ and for $\mathcal{LE}_{\mathscr{T},d} \leq \mathcal{LE}_{\mathscr{S},d}$ is necessary for neither. Consider $X = \{a, b\} \cup \mathbb{N}$ where the $\{a, b\} \cap \mathbb{N} = \emptyset$, equipped with the metric d defined for $x \neq y$ by

$$d(x, y) = \begin{cases} 1 & \text{if } \{x, y\} = \{a, b\} \\ 2 & \text{otherwise.} \end{cases}$$

Put $A = \{a\} \cup \mathbb{N}$ and $B = \{b\} \cup \mathbb{N}$, and then let $\mathscr{T} = \{A\} \cup s(X)$ and $\mathscr{S} = \{B\} \cup s(X)$.

(1) if $0 < \varepsilon \leq 1$, then $\{E : e_d(A, E) < \varepsilon\} = \{A, X\}$;

(2) if $1 < \varepsilon \leq 2$, then $\{E : e_d(A, E) < \varepsilon\} = \{A, B, X\}$;

(3) if $\varepsilon > 2$, then $\{E : e_d(A, E) < \varepsilon\} = \mathscr{P}_0(X)$.

In case (1), $\{A, X\} = \{E : e_d(B, E) < \frac{3}{2}\} \cap \{E : E_d(\{a\}, E) < \frac{1}{2}\}$, and in case (2) $\{A, B, X\} = \{E : e_d(B, E) < \frac{3}{2}\}$. From this we get inclusion of (and by symmetry, equally of) the lower excess topologies and thus the two-sided excess topologies. But there is no $S \in \sum(\mathscr{S})$ for which $H_d(A, S) < 1$.

The situation becomes much simpler with respect to inclusion of left excess topologies when $\mathscr{S}$ is assumed to be a *bornology*, that is, a family that is hereditary, contains the singletons, and stable under finite unions [**5, 6**]. Nothing need be assumed about the second family $\mathscr{T}$.

Theorem 3.16. *Let* $\mathscr{S}$ *be a bornology on a metric space* $\langle X, d\rangle$ *and let* $\mathscr{T} \subseteq \mathscr{P}_0(X)$. *Then the following conditions are equivalent:*

(1) $\mathcal{LE}^-_{\mathscr{T},d} \leq \mathcal{LE}^-_{\mathscr{S},d}$ *on* $\mathscr{P}_0(X)$;

(2) $\mathcal{LE}_{\mathscr{T},d} \leq \mathcal{LE}_{\mathscr{S},d}$ *on* $\mathscr{P}_0(X)$;

(3) $\mathscr{T} \subseteq (\mathscr{S})^*_d$;

(4) $\forall \varepsilon > 0\ \forall T \in \mathscr{T},\ \exists S \in \mathscr{S}$ *with* $H_d(S,T) \leq \varepsilon$.

Proof. (1) ⇒ (2). This follows immediately from the fact that

$$\mathcal{LE}^+_{\mathscr{T},d} \leq \mathcal{LE}^+_{\mathscr{T}\cup s(X),d} = \tau^+_{W_d} = \mathcal{LE}^+_{\mathscr{S},d},$$

the last equality holding because $\mathscr{S}$ contains the singletons.

(2) ⇒ (3). The proof of Proposition 3.13 goes through, noting that $\sum(\mathscr{S}) = \mathscr{S}$ and that a local base for $\mathcal{LE}_{\mathscr{S},d}$ at X consists of all sets of the form $\cap_{i=1}^n \{E : e_d(S_i, E) < \varepsilon_i\}$ where $S_1, S_2, \ldots, S_n \in \mathscr{S}$ and $\varepsilon_i > 0$ because $e_d(S,X) > 0$ is impossible whatever $S \in \mathscr{S}$ may be.

(3) ⇒ (4). Let $\varepsilon > 0$ and $T \in \mathscr{T}$ be arbitrary. By condition (3), choose $S \in \mathscr{S}$ with $T \subseteq B_d(S,\varepsilon)$. Next put $S_0 := B_d(T,\varepsilon) \cap S \in \mathscr{S}$; clearly both $S_0 \subseteq B_d(T,\varepsilon)$ and $T \subseteq B_d(S_0,\varepsilon)$ so that $H_d(S_0,T) \leq \varepsilon$.

(4) ⇒ (1). This follows from Lemma 3.3 which holds more generally. □

4. Excess Functionals with Fixed Right Argument

Given a family of nonempty subsets $\mathscr{S}$ of a metric space $\langle X, d\rangle$, we now look at topologies on $\mathscr{P}_0(X)$ which are weak topologies generated by a family of excess functionals with fixed right arguments running over $\mathscr{S}$, i.e., $\{e_d(\cdot, S) : S \in \mathscr{S}\}$. We denote such a weak topology by $\mathcal{RE}_{\mathscr{S},d}$. When $\mathscr{S} = s(X)$, we get the weakest topology on $\mathscr{P}_0(X)$ such that for each $x \in X$, $A \mapsto \sup_{a\in A} d(x,a)$ is continuous. It is is natural to call $\mathcal{RE}_{s(X),d}$ the *dual Wijsman topology* determined by d. Dual Wijsman topologies are not in general Hausdorff when restricted to $\mathscr{C}_0(X)$; for example, in the real line equipped with the usual metric, for each $x \in \mathbb{R}$, we have $e_d(\{0,1\},\{x\}) = e_d([0,1],\{x\})$.

Since $e_d(E,S)$ increases with E for fixed S, the upper and lower halves of the topology are formally reversed from what we saw in the last section. The lower right excess topology $\mathcal{RE}^-_{\mathscr{S},d}$ is generated by all sets of the form $\{E : e_d(E,S) > \alpha\}$ where S runs over $\mathscr{S}$ and $\alpha > 0$, while $\mathcal{RE}^+_{\mathscr{S},d}$ is generated by all sets of the form $\{E : e_d(E,S) < \alpha\}$ where S runs over $\mathscr{S}$ and $\alpha > 0$.

As we promised, the analysis presented in this section will be more informal than that in the previous section. Our first two lemmas describe what upper and lower convergence look like locally. Their proofs are left as easy exercises.

Lemma 4.1. *Suppose $\langle X, d\rangle$ is a metric space and $\mathscr{S}$ is a family of nonempty subsets of X. For a net $\langle A_j\rangle_{j\in J}$ of nonempty subsets of X and $A \in \mathscr{P}_0(X)$, the following conditions are equivalent:*

(1) $\langle A_j\rangle_{j\in J}$ *is $\mathcal{RE}^+_{\mathscr{S},d}$-convergent to A;*

(2) $\forall S \in \mathscr{S}$, *lim sup*$_{j\in J}$ $e_d(A_j,S) \leq e_d(A,S)$;

(3) *whenever* $S \in \mathscr{S}$ *and* $0 < \varepsilon < \alpha$, $A \subseteq B_d(S,\varepsilon) \Rightarrow A_j \subseteq B_d(S,\alpha)$ *eventually.*

Lemma 4.2. *Suppose $\langle X, d\rangle$ is a metric space and $\mathscr{S}$ is a family of nonempty subsets of X. For a net $\langle A_j\rangle_{j\in J}$ of nonempty subsets of X and $A \in \mathscr{P}_0(X)$, the following conditions are equivalent:*

(1) *$\langle A_j\rangle_{j\in J}$ is $\mathcal{RE}^-_{\mathscr{S},d}$-convergent to A;*

(2) *$\forall S \in \mathscr{S}$, $\liminf_{j\in J} e_d(A_j, S) \geq e_d(A, S)$;*

(3) *whenever $S \in \mathscr{S}$ and $0 < \varepsilon < \alpha$, $A \nsubseteq B_d(S, \alpha) \Rightarrow A_j \nsubseteq B_d(S, \varepsilon)$ eventually.*

The next result, whose proof is also left to the reader, parallels Corollary 3.6

Corollary 4.3. *Suppose $\langle X, d\rangle$ is a metric space and let $\mathscr{S}$ be subfamily of $\mathscr{P}_0(X)$ containing the singletons. The following conditions are equivalent for a sequence $\langle x_n\rangle$ in X and a point $p \in X$:*
(1) *$\lim_{n\to\infty} d(x_n, p) = 0$;*

(2) *$\langle\{x_n\}\rangle$ converges to $\{p\}$ in $\mathcal{RE}_{\mathscr{S},d}$;*

(3) *$\langle\{x_n\}\rangle$ converges to $\{p\}$ in $\mathcal{RE}^+_{\mathscr{S},d}$.*

One-sided or two sided $\mathcal{RE}_{\mathscr{S},d}$-convergence is made stronger by replacing $\mathscr{S}$ by a larger family of subsets and is left unchanged replacing $\mathscr{S}$ by any other family with the same closure with respect to the Hausdorff pseudometric topology. The next example shows at once that $\mathcal{RE}_{\mathscr{S},d}$-convergence can be properly weaker than either $\mathcal{RE}_{\downarrow\mathscr{S},d}$-convergence or $\mathcal{RE}_{\sum(\mathscr{S}),d}$-convergence.

Example 4.4. We work with a three-point set $X = \{a, b, c\}$ equipped with the zero-one discrete metric. For $\mathscr{S}$, we take $\{\{a\}, \{b\}, \{c\}, X\}$. Put $A_n = \{a, b\}$ and $A = X$. For each $x \in X$ and each $n \in \mathbb{N}$, we have $e_d(A_n, \{x\}) = 1 = e_d(A, \{x\})$ while $e_d(A_n, X) = 0 = e_d(A, X)$. On the other hand, $\sum(\mathscr{S}) = \downarrow\mathscr{S} = \mathscr{P}_0(X)$, and $\forall n \in \mathbb{N}, e_d(A_n, \{a, b\}) = 0$ while $e_d(A, \{a, b\}) = 1$

There is also no hope of getting an analog of Theorem 3.16 for right excess topologies. While in general the ability to approximate each element of $\mathscr{T}$ by elements of $\mathscr{S}$ in d-Hausdorff distance is sufficient for $\mathcal{RE}_{\mathscr{T},d} \leq \mathcal{RE}_{\mathscr{S},d}$, this condition is not necessary even when $\mathscr{S}$ is a bornology.

Example 4.5. Consider $\mathbb{R}$ equipped with the usual metric d; let $\mathscr{S}$ be the bornology of finite nonempty subsets $\mathscr{F}_0(\mathbb{R})$ and let $\mathscr{T} = \mathscr{F}_0(\mathbb{R}) \cup \{\mathbb{R}\}$. Since for each $E \in \mathscr{P}_0(X)$, $e_d(E, \mathbb{R}) = 0$, we get for arbitrary $A \in \mathscr{P}_0(\mathbb{R})$ and an arbitrary net of nonempty subsets $\langle A_j\rangle_{j\in J}$

$$\forall S \in \mathscr{S}, \lim_{j\in J} e_d(A_j, S) = e_d(A, S) \Rightarrow \forall T \in \mathscr{T},\ \lim_{j\in J} e_d(A_j, T) = e_d(A, T).$$

This shows that $\mathcal{RE}_{\mathscr{T},d} = \mathcal{RE}_{\mathscr{S},d}$, while $\forall S \in \mathscr{S}, H_d(\mathbb{R}, S) = \infty$.

We come to the main result of this section. It stands in stark contrast to the results of Section 3.

Theorem 4.6. *Let $\mathscr{S}$ and $\mathscr{T}$ be two families of nonempty subsets of a set X equipped with metrics d and ρ. The following conditions are equivalent:*

(1) *both* $\mathcal{RE}^+_{\mathscr{T},\rho} \le \mathcal{RE}^+_{\mathscr{S},d}$ *and* $\mathcal{RE}^-_{\mathscr{T},\rho} \le \mathcal{RE}^-_{\mathscr{S},d}$ *on* $\mathscr{P}_0(X)$*;*

(2) $\mathcal{RE}_{\mathscr{T},\rho} \le \mathcal{RE}_{\mathscr{S},d}$ *on* $\mathscr{P}_0(X)$*;*

(3) $\forall T \in \mathscr{T}$, *whenever* $0 < \varepsilon < \alpha$ *with* $B_\rho(T,\alpha) \ne X$, $\exists S_1, S_2, \ldots, S_n \in \mathscr{S}$ $\exists 0 < \lambda_i < \sigma_i$ $(i = 1, 2, 3, ..., n)$ *with*

$$B_\rho(T,\varepsilon) \subseteq \cap_{i=1}^n B_d(S_i,\lambda_i) \subseteq \cap_{i=1}^n B_d(S_i,\sigma_i) \subseteq B_\rho(T,\alpha).$$

Proof. (3) $\Rightarrow$ (1). Suppose condition (3) holds and we have $C = \mathcal{RE}^+_{\mathscr{S},d}-\lim_{j\in J} C_j$ in $\mathscr{P}_0(X)$. Let $T \in \mathscr{T}$ and $0 < \varepsilon < \alpha$ be given with $C \subseteq B_\rho(T,\varepsilon)$. If $B_\rho(T,\alpha) = X$, then for each index $j \in J$, we have $C_j \subseteq B_\rho(T,\alpha)$. Otherwise, choose $S_1, S_2, \ldots, S_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n, \sigma_1, \sigma_2, \ldots, \sigma_n$ consistent with condition (3) above. Then $\forall i \le n$, $C \subseteq B_d(S_i,\lambda_i)$ so that by Lemma 4.1 eventually $\forall i \le n$, $C_j \subseteq B_d(S_i,\sigma_i)$. The inclusion $\cap_{i=1}^n B_d(S_i,\sigma_i) \subseteq B_\rho(T,\alpha)$ now guarantees eventually $C_j \subseteq B_\rho(T,\alpha)$, establishing $\mathcal{RE}^+_{\mathscr{T},\rho}$ convergence.

Next, assume we have $C = \mathcal{RE}^-_{\mathscr{S},d}-\lim_{j\in J} C_j$ in $\mathscr{P}_0(X)$. Fix $T \in \mathscr{T}$ and $0 < \varepsilon < \alpha$ where $C \cap B_\rho(T,\alpha)^c \ne \emptyset$. Choose $S_1, S_2, \ldots, S_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n, \sigma_1, \sigma_2, \ldots, \sigma_n$ again consistent with (3). From DeMorgan's laws, we conclude

$$C \cap (\cup_{i=1}^n B_d(S_i,\sigma_i)^c) \ne \emptyset,$$

and so we may choose $i_0 \le n$ with $C \cap B_d(S_{i_0},\sigma_{i_0})^c \ne \emptyset$. By Lemma 4.2, $\exists j_0 \in J$ such that for all $j \ge j_0$, $C \cap B_d(S_{i_0},\lambda_{i_0})^c \ne \emptyset \Rightarrow \forall j \ge j_0$, $C_j \cap B_\rho(T,\varepsilon)^c \ne \emptyset$.

(1) $\Rightarrow$ (2). This is trivial.

(2) $\Rightarrow$ (3). Suppose (3) fails for some $T \in \mathscr{T}$ and $0 < \varepsilon < \alpha$ where $B_\rho(T,\alpha) \ne X$. Consider this subfamily of $\mathscr{E}$ of $\mathscr{S}$:

$$\mathscr{E} := \{S \in \mathscr{S} : \exists \mu > 0 \text{ with } B_\rho(T,\varepsilon) \subseteq B_d(S,\mu)\}.$$

We consider two possible cases ; (i) $\mathscr{E} = \emptyset$; and (ii) $\mathscr{E} \ne \emptyset$. We will show that in each case, condition (2) fails.

In case (i), fix $x_0 \in B_\rho(T,\alpha)^c$ and consider the constant sequence of nonempty sets

$$C_k = B_\rho(T,\varepsilon) \qquad (k = 1, 2, 3, \ldots).$$

We claim that $\langle C_k \rangle$ is $\mathcal{RE}_{\mathscr{S},d}$-convergent to $C := \{x_0\} \cup B_\rho(T,\varepsilon)$. By the emptyness of $\mathscr{E}$, for each element $S \in \mathscr{S}$ there are points of $B_\rho(T,\varepsilon)$ arbitrarily far from S in d-distance, and as a result

$$e_d(C,S) = \infty = \lim e_d(C_k,S).$$

establishing the claim. But $\mathcal{RE}_{\mathscr{T},\rho}$-convergence fails because $\forall k \in \mathbb{N}$

$$e_\rho(C_k,T) = e_\rho(B_\rho(T,\varepsilon),T) \le \varepsilon$$

whereas

$$e_\rho(C,T) = e_\rho(B_\rho(T,\varepsilon) \cup \{x_0\},T) \ge \rho(x_0,T) \ge \alpha.$$

In case (ii), for each $S \in \mathscr{E}$, put

$$q_S := \inf\{\mu > 0 : B_\rho(T,\varepsilon) \subseteq B_d(S,\mu)\} \geq 0.$$

Note that if $\{S_1, S_2, \dots S_n\} \subseteq \mathscr{E}$ and $k \in \mathbb{N}$

$$B_\rho(T,\varepsilon) \subseteq \cap_{i=1}^n B_d(S_i, q_{S_i} + \frac{1}{2k})$$

and therefore by the failure of condition (3) with respect to T, α and ε we have

$$\cap_{i=1}^n B_d(S_i, q_{S_i} + \frac{1}{k}) \nsubseteq B_\rho(T,\alpha).$$

Let $\mathfrak{F}$ denote the finite subsets of $\mathscr{E}$ and direct $\mathfrak{F}\times\mathbb{N}$ in the usual manner. Put $C = B_\rho(T,\varepsilon)$ and for each $(\{S_1, S_2, \dots, S_n\}, k) \in \mathfrak{F} \times \mathbb{N}$ pick $x(\{S_1, S_2, \dots, S_m\}, k) \in \cap_{i=1}^n B_d(S_i, q_{S_i} + \frac{1}{k}) \backslash B_\rho(T,\alpha)$. We now consider the net defined on $\mathfrak{F} \times \mathbb{N}$ by

$$(\{S_1, S_2, \dots, S_n\}, k) \mapsto \{x(\{S_1, S_2, \dots, S_n\}, k)\} \cup C,$$

which we intend to show to be $\mathcal{RE}_{\mathscr{S},d}$-convergent to C but not $\mathcal{RE}_{\mathscr{T},\rho}$-convergent to C. Since C is contained in each term of the net, we obviously have $\mathcal{RE}^-_{\mathscr{S},d}$-convergence. To show $\mathcal{RE}^+_{\mathscr{S},d}$-convergence, we rely on Lemma 4.1. Fix $S_0 \in \mathscr{S}$ and $0 < \beta < \mu$ with $C \subseteq B_d(S_0,\beta)$. Pick $k_0 \in \mathbb{N}$ with $\frac{1}{k_0} < \mu - \beta$. If $k \geq k_0$, then we have $\frac{1}{k} \leq \frac{1}{k_0}$, and as a result if $\{S_1, S_2, \dots, S_n\} \subseteq \mathscr{S}$, we get

$$x((\{S_0, S_1, S_2, \dots, S_n\}, k) \in \cap_{i=0}^n B_d(S_j, q_{S_j} + \frac{1}{k}) \subseteq B_d(S_0, q_{S_0} + \frac{1}{k_0}).$$

Now as $C \subseteq B_d(S_0,\beta)$, we have $\beta \geq q_{S_0}$ and so

$$q_{S_0} + \frac{1}{k_0} \leq \beta + \frac{1}{k_0} < \mu.$$

Thus $C \cup \{x(\{S_0, S_1, S_2, \dots, S_n\}, k)\} \subseteq B_d(S_0,\mu)$, and the net is $\mathcal{RE}^+_{\mathscr{S},d}$-convergent to C. But $\mathcal{RE}^+_{\mathscr{T},\rho}$ and hence $\mathcal{RE}_{\mathscr{T},\rho}$ convergence to C fail, because $e_\rho(C,T) \leq \varepsilon$ whereas whenever $\{S_1, S_2, \dots, S_n\} \subseteq \mathscr{E}$ and $k \in \mathbb{N}$, we have

$$e_\rho(\{x(\{S_1, S_2, \dots, S_n\}, k)\} \cup C, T) \geq \rho(x(\{S_1, S_2, \dots, S_n\}, k), T) \geq \alpha. \quad \square$$

When $\mathscr{T} = \mathscr{S} = s(X)$, condition (3) in the above statement is an analog of the strict d-inclusion condition of Costantini, Levi, and Zieminska [**10**], replacing unions with intersections, and characterizes inclusion of the two-sided dual Wijsman topology determined by ρ in the one determined by d.

Condition (3) of our last result can be recast in a way that resembles condition (2) of Theorem 3.9, replacing condition (3) by this equivalent condition:

(3*) $\forall T \in \mathscr{T}$, whenever $0 < \varepsilon < \alpha$ where $\mathscr{H}^+_\rho(T,\alpha) \neq \mathscr{P}_0(X)$, $\exists\,\{S_1, S_2, \dots, S_n\} \subseteq \mathscr{S}$ and $0 < \lambda_i < \sigma_i$ for $i = 1, 2, \dots, n$ such that whenever $A \in \mathscr{H}^+_\rho(T,\varepsilon)$

$$A \in \cap_{i=1}^n \mathscr{H}^+_d(S_i,\lambda_i) \subseteq \cap_{i=1}^n \mathscr{H}^+_d(S_i,\sigma_i) \subseteq \mathscr{H}^+_\rho(T,\alpha).$$

The difference in the two conditions is that in the former, the S_i, λ_i, and σ_i depend on A while in the latter they do not, i.e., they can be determined by reference to T alone.

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