A NEW LOWER BOUND FOR MATHIEU’S SERIES

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Abstract. In this paper, we establish a new lower bound for Mathieu’s series and present a simple and short proof of the Mathieu’s inequality. We also give a simple derivation of the integral form of the series.

1. Introduction

The Mathieu series was first defined in [10] as

\[ S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2}, \quad r > 0. \]  

(1.1)

In 1890, Mathieu conjectured [10] that

\[ \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} < \frac{1}{r^2} \]  

(1.2)

and used this inequality in his work on elasticity of solid bodies, though without giving a proof of the inequality, which Schröder pointed out in [14]. This inequality was proved only in 1952 by Berg [2]. The Berg’s proof was difficult enough; in his proof Berg relied on the Schröder’s inequality [14]

\[ S(r) < \frac{1}{(1 + r^2)^2} + \frac{2}{(4 + r^2)^2} + \frac{1}{(4 + r^2)}, \]

on the Euler-Maclaurin formula and on certain auxiliary estimates based on the properties of derivatives of the function \( f(x) = x/(x^2 + r^2)^2 \), and also on the Bernoulli polynomial of fourth order.

It was Makai [9] who gave in 1957 an elegant and elementary proof for (1.2) and obtained the following double inequality

\[ \frac{1}{r^2 + 1/2} < \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} < \frac{1}{r^2}. \]  

(1.3)

The study of Mathieu series and its inequality has been growing steadily in the last decades, having a much rich literature; various papers appeared providing new interesting refinements and extensions of Mathieu’s inequality, as well as new approximations, integral representations, and open problems [5, 7, 11, 12, 13, 15].
Alzer, Brenner and Ruehr [1] showed that for the Mathieu series defined by (1.1) the best constants \( k_1 \) and \( k_2 \) in the two-sided estimation
\[
\frac{1}{r^2 + k_1} < S(r) < \frac{1}{r^2 + k_2}, \quad r \neq 0 \tag{1.4}
\]
are given by \( k_1 = 1/(2\zeta(3)) \) and \( k_2 = 1/6 \), where \( \zeta(s) \) is the Riemann zeta function.

Recently, F. Qi et al. [13], have estimated Mathieu’s series as
\[
A(r) \leq S(r) \leq B(r), \tag{1.5}
\]
for \( r > 0 \), where
\[
A(r) = \frac{4(1 + r^2)(e^{-\pi/r} + e^{-\pi/(2r)}) - 4r^2 - 1}{(e^{-\pi/r} - 1)(1 + r^2)(1 + 4r^2)},
\]
and
\[
B(r) = \frac{(1 + 4r^2)(e^{-\pi/r} - e^{-\pi/(2r)}) - 2(r^2 + 1)}{(e^{-\pi/r} - 1)(1 + r^2)(1 + 4r^2)}.
\]

The lower bound in (1.4) is better that the estimate (1.5), however it should be noted that the upper bound in (1.5) is better than the bound of (1.4) for all \( r > 0 \), except for \( r \approx 0 \), where the upper bound of Alzer et al. is still better.

In [8] Lampret, using Euler-Maclaurin summation formula of fourth order, have found new approximation to Mathieu series as
\[
a(1, x) < S(x) < b(1, x), \tag{1.6}
\]
where
\[
a(1, x) = \frac{(x^4 + 2x^2 + 2)(6x^4 + 17x^2 + 15)}{3(1 + x^2)^3(2x^4 + 4x^2 + 5)},
\]
\[
b(1, x) = \frac{(x^4 + 2x^2 + 3)(6x^4 + 17x^2 + 15)}{3(1 + x^2)^3(2x^4 + 4x^2 + 5)}.
\]

The estimate (1.6) is sharp if considered asymptotically, but it fails for \( r \approx 0 \), where the lower bound in (1.4) is better; on the contrary, for larger values of \( r \) the lower estimate (1.4) is worse than (1.6) (see Fig. 1). Moreover, the Lampret’s and Alzer’s approximations are better than the lower estimate of F. Qi on the entire \( \mathbb{R} \); and the lower bound \( A(r) \) becomes negative as \( r \) increases, consequently useless for larger values of \( r \). More recently, Mortici [11] found more accurate approximations.

In this note, by utilizing the elementary tools of analysis, we present the proof more simple and short for the upper bound in the double inequality (1.3), and establish the lower bound which asymptotically coincides with the lower bound of Lampret, since the difference between both estimates tends to zero.

2. Main Result

The main result of this paper is the following proposition.

Proposition 2.1. For all \( r > 0 \), we have
\[
\left(1 - \frac{1}{3r^2}\right) \frac{1}{r^2} < \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} < \frac{1}{r^2}. \tag{2.1}
\]
Figure 1. The graph of function $a(1, x)$ (red line) and the graphs of functions $A(x)$ (black line) and $1/(x^2 + 1/(2\zeta(3)))$ (blue line).

**Proof.** We first prove the upper bound of (2.1), that is the inequality (1.2), and then obtain the lower bound.

We now recall that as it is known from integral calculus \[6\]
\[
\int_0^\infty e^{-\lambda x} \cos (\mu x + q) \, dx = \frac{1}{\lambda^2 + \mu^2} (\lambda \cos(q) - \mu \sin(q)), \quad \lambda > 0. \tag{2.2}
\]

Setting $q = 0$ and putting in (2.2) $\lambda = n$ and $\mu = r = \sqrt{h}$, $h > 0$, and then differentiating on $h$, we get

\[
2\sqrt{h} \frac{n}{(n^2 + h)^2} = \int_0^\infty x e^{-nx} \sin (\sqrt{hx}) \, dx. \tag{2.3}
\]

Hence, from (2.3) we obtain

\[
\sqrt{h} S(\sqrt{h}) = \int_0^\infty \frac{x}{e^x - 1} \sin (\sqrt{hx}) \, dx \tag{2.4}
\]

or

\[
r S(r) = \int_0^\infty \frac{x}{e^x - 1} \sin (rx) \, dx, \tag{2.5}
\]

which is the integral representation of Mathieu series that was first given in \[3, 4\].

Now we use in (2.5) the formula of integration by parts. Applying the formula thrice and denoting

\[
f(x) = -\frac{d}{dx} \left( \frac{x}{e^x - 1} \right),
\]

we find

\[
r^2 S(r) = 1 - \frac{1}{r^2} \int_0^\infty f''(x) [1 - \cos(rx)] \, dx, \tag{2.6}
\]

where in the process of integration we have

\[
f''(x) = \frac{e^x \left[(x-3)e^{2x} + 4xe^x + x + 3\right]}{(e^x - 1)^2}.
\]

Now employing the expansion of exponential function into Maclaurin series, we obtain

\[
(x - 3)e^{2x} + 4xe^x + x + 3 = \sum_{n=5}^\infty \frac{1}{(n-1)!} \left[ 4 + \left( 1 - \frac{6}{n} \right) 2^{n-1} \right] x^n > 0
\]
for \( x > 0 \). Whence we immediately get
\[
S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} < \frac{1}{r^2},
\]
which proves the inequality (1.2).

Further, the first derivative of \( f(x) \) is
\[
f'(x) = -e^x \left[ (x - 2)e^x + x + 2 \right] \frac{1}{(e^x - 1)^3},
\]
thus we have \( \lim_{x \to 0} f'(x) = -1/6 \), that is, \( f'(0) = -1/6 \). Hence from (2.6) it follows that
\[
S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} > \left( 1 - \frac{1}{3r^2} \right) \frac{1}{r^2}.
\]
Combining the last two inequalities, we finally obtain
\[
\left( 1 - \frac{1}{3r^2} \right) \frac{1}{r^2} < S(r) < \frac{1}{r^2},
\]
which establishes a new lower bound of \( S(r) \) and proves that the double inequality (2.7) holds for all \( r \) from the interval \( (0; \infty) \).

\[\square\]

3. Discussion

It should be noted that the lower estimate in (2.1) is worse than the estimates of (1.3) and (1.4) when \( r < 1 \) and \( r < 1/(3 - 2\zeta(3))^{1/2} \approx (5/3)^{1/2} = 1.292... \), respectively (see Fig. 2).

In addition, the left side term of (2.1) becomes negative when \( r < 0.577... \) (see Fig. 3), consequently the estimate is useless for smaller values of \( r \).

However, for all \( r > 1 \) the lower bound in (2.1) is better than that of Makai (1.3) and for all \( r > 1/(3 - 2\zeta(3))^{1/2} \approx 1.292... \) the lower bound of (2.1) is better than the lower bound given by Alzer (1.4), that is, it is better than the mentioned estimates.
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for all $r$ in the interval $(1.292; \infty)$ (see Fig. 2). Moreover, for $x > \approx 0.6643$ the lower bound (2.1) is better than the estimate of F. Qi (see Fig. 4).

Although the lower estimate of Lampret (1.6) is still better than our lower bound (2.1) for all $x > 0$ (see Fig. 5), they, nonetheless, practically coincide, since as $x$ becomes larger, the difference between both estimates tends to zero. For example, using Mathematica, we have that for $x = 4$ the difference is equal to $0.000538372$ and for $x = 10$ the difference is $0.0000161726$, for $x = 100$ it is already equal to $1.66617 \times 10^{-9}$, and for larger $x$ and $x \to \infty$ Mathematica yields 0.

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Figure 5. The graph of function \( (1 - 1/(3x^2))/x^2 \) (black) and the graph of function \( a(1, x) \) (red).

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