NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 33 (2004), 121–127

UNIFICATION OF MODULAR TRANSFORMATIONS FOR CUBIC THETA FUNCTIONS

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(Received April 2003)

Abstract. We obtain a modular transformation for the function

$$a(q,\zeta,z) = \Sigma q^{n^2+nm+m^2}\zeta^{n+m}z^{n-m}, \qquad \text{namely,}$$

$$a(e^{-2\pi t}, e^{i\varphi}, e^{i\theta}) = \frac{1}{t\sqrt{3}}\exp[-(\frac{\varphi^2+3\theta^2}{6\pi t})]\, a(e^{-\frac{2\pi}{3t}}, e^{\frac{\theta}{t}}, e^{\frac{\varphi}{3t}}),$$

and demonstrate how this unifies cubic modular transformations recently found by S. Cooper and their precursors established by J.M. Borwein and P.B. Borwein. Here and throughout the paper, unless otherwise stated, it is understood that the summation index or indices range over all integral values.
We employ some properties of $a(q,\zeta,z)$ and its relation with the classical theta function (in Ramanujan's notation) $f(a,b) = \Sigma a^{n(n+1)/2}b^{n(n-1)/2}$ simply established by the first author in an earlier publication. Also used are some simple properties of $f(a,b)$ and a well known modular transformation for $f(a,b)$ as recorded by Ramanujan.

1. Introduction

The function

$$a(q,\zeta,z) \;=\; \sum q^{n^2+nm+m^2}\zeta^{m+n}z^{n-m}, \tag{1.1}$$

$|q| < 1$, $\zeta \neq 0 \neq z$ was introduced by the first author in [**2**] and was shown to have properties which unified and generalized several known properties of the Hirschhorn–Garvan–Borwein cubic analogues[1] $a(q,z)$, $a'(q,z)$, $b(q,z)$ and $c(q,z)$ of classical theta function [**4**]:

$$a(q,z) \;:=\; \sum q^{n^2+nm+m^2}z^{n-m}, \tag{1.2}$$

$$a'(q,z) \;:=\; \sum q^{n^2+nm+m^2}z^{n}, \tag{1.3}$$

$$b(q,z) \;:=\; \sum q^{n^2+nm+m^2}\omega^{m-n}z^{n}, \tag{1.4}$$

and

$$c(q,z) \;:=\; \sum q^{n^2+nm+m^2+n+m+\frac{1}{3}}\; z^{n-m}, \tag{1.5}$$

2000 *AMS Mathematics Subject Classification:* 54B20, 54E15, 54E55.
Supported by the UGC Grant No. F6–129/2001.
[1]The functions $c(q,z)$ in [**4**] and $c(q,\zeta,z)$ in [**2**] each differ from the ones defined here by means of (1.5) and (1.8), by a factor of $q^{1/3}$.

where $\omega = e^{\frac{2\pi i}{3}}$. That these functions are indeed obtainable from (1.1) can easily by seen by putting $\zeta = 1$ in (1.1) and in the following variants of (1.1).

$$a'(q,\zeta,z) \;:=\; a(q,\zeta^{1/2}z^{1/2},\zeta^{-1/2}z^{1/2}) = \sum q^{n^2+nm+m^2}\zeta^m z^n, \tag{1.6}$$

$$b(q,\zeta,z) \;:=\; a'(q,\zeta\omega,z\omega^2) = \sum q^{n^2+nm+m^2}\omega^{m-n}\zeta^m z^n, \tag{1.7}$$

and

$$c(q,\zeta,z) \;:=\; q^{1/3}a(q,q\zeta,z) = \sum q^{n^2+nm+m^2+n+m+\frac{1}{3}}\,\zeta^{n+m}z^{n-m}. \tag{1.8}$$

Indeed,

$$a(q,z) = a(q,1,z), \qquad a'(q,z) = a'(q,1,z),$$
$$b(q,z) = b(q,1,z) \quad \text{and} \quad c(q,z) = c(q,1,z). \tag{1.9}$$

In his recent paper [**5**], S. Cooper established the following elegant modular transformations for $a'(q,z)$, $a(q,z)$, $b(q,z)$ and $c(q,z)$ which generalize those of J.M. Borwein and P.B. Borwein [**3**] for $a(q) := a(q,1)$, $a'(q) := a'(q,1)$, $b(q) := b(q,1)$ and $c(q) := c(q,1)$:

$$a'(e^{-2\pi t},e^{i\theta}) \;=\; \frac{1}{t\sqrt{3}}\,\exp\left(-\frac{\theta^2}{6\pi t}\right)\,a(e^{-\frac{2\pi}{3t}},e^{\frac{\theta}{3t}}), \tag{1.10}$$

$$a(e^{-2\pi t},e^{i\theta}) \;=\; \frac{1}{t\sqrt{3}}\,\exp\left(-\frac{\theta^2}{2\pi t}\right)\,a'(e^{-\frac{2\pi}{3t}},e^{\frac{\theta}{t}}), \tag{1.11}$$

$$b(e^{-2\pi t},e^{i\theta}) \;=\; \frac{1}{t\sqrt{3}}\,\exp\left(-\frac{\theta^2}{6\pi t}\right)\,c(e^{-\frac{2\pi}{3t}},e^{\frac{\theta}{3t}}), \tag{1.12}$$

and

$$c(e^{-2\pi t},e^{i\theta}) \;=\; \frac{1}{t\sqrt{3}}\,\exp\left(-\frac{\theta^2}{2\pi t}\right)\,b(e^{-\frac{2\pi}{3t}},e^{\frac{\theta}{t}}). \tag{1.13}$$

The main purpose of the present paper is to establish (Section 2) the transformation

$$a(e^{-2\pi t},e^{i\varphi},e^{i\theta}) \;=\; \frac{1}{t\sqrt{3}}\,\exp\left[-\left(\frac{\varphi^2+3\theta^2}{6\pi t}\right)\right]\,a(e^{-\frac{2\pi}{3t}},e^{\frac{\theta}{t}},e^{\frac{\varphi}{3t}}). \tag{1.14}$$

Moreover, we show (Section 3) that this is in fact equivalent to the following generalizations of (1.10)–(1.13),

$$a(e^{-2\pi t},e^{i\varphi},e^{i\theta}) \;=\; \frac{1}{t\sqrt{3}}\,\exp\left[-\left(\frac{\varphi^2+3\theta^2}{6\pi t}\right)\right]\,a'(e^{-\frac{2\pi}{3t}},e^{\frac{2\varphi}{3t}},e^{\frac{3\theta+\phi}{3t}}), \tag{1.15}$$

$$a'(e^{-2\pi t},e^{i\varphi},e^{i\theta}) \;=\; \frac{1}{t\sqrt{3}}\,\exp\left[-\left(\frac{\varphi^2-\varphi\theta+\theta^2}{6\pi t}\right)\right]\,a(e^{-\frac{2\pi}{3t}},e^{\frac{\varphi}{2t}},e^{\frac{2\theta-\varphi}{6t}}), \tag{1.16}$$

$$b(e^{-2\pi t},e^{i\varphi},e^{i\theta}) \;=\; \frac{1}{t\sqrt{3}}\,\exp\left[-\left(\frac{\varphi^2-\varphi\theta+\theta^2}{6\pi t}) + \frac{\varphi}{3t}\right)\right]\,c(e^{-\frac{2\pi}{3t}},e^{\frac{\varphi}{2t}},e^{\frac{2\theta-\varphi}{6t}}), \tag{1.17}$$

and

$$\exp(\frac{-2i\varphi}{3})\ c(e^{-2\pi t}, e^{-i\varphi}, e^{i\theta}) = \frac{1}{t\sqrt{3}}\ \exp\left[-\left(\frac{\varphi^2+3\theta^2}{6\pi t}\right)\right] \times b(e^{-\frac{2\pi}{3t}}, e^{\frac{2\varphi}{3t}}, e^{\frac{3\theta+\varphi}{3t}}). \quad (1.18)$$

That Cooper's identities (1.10)–(1.13) are special cases of (1.15)–(1.18) is immediate by putting $\varphi = 0$ in the latter and using (1.9).

We end the paper by applying (Section 4) the transforms (1.15)–(1.18) to a pair of identities in [**2**], thereby obtaining new identities involving $a'(q, \zeta, z)$.

2. The Main Transformation

We first note some simple relations between the function (1.1) and the classical theta function (in Ramanujan's notation [Eqn. (18.1), Ch.−16, 1]),

$$f(a,b) \ := \ \sum a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1 \quad (2.1)$$

that we will need in the course of this paper. It was indeed shown by the first author in [Eqns. (1.29) and (4.2), 2] that simple series manipulations yield, among other results

$$a(q,\zeta,z) \ = \ q^{3\lambda^2+3\lambda\mu+\mu^2}\ \zeta^{2\lambda+\mu}\ z^{\mu}\ a(q, q^{3(2\lambda+\mu)/2}\zeta, q^{\mu/2} z), \quad (2.2)$$

(λ and μ: integers),

$$a(q,\zeta,z) \ = \ f(q\zeta z^{-1}, q\zeta^{-1}z) f(q^3\zeta z^3, q^3\zeta^{-1}z^{-3}) + \ q\,\zeta\, z\, f(q^2\zeta z^{-1}, \zeta^{-1}z)\ f(q^6\zeta z^3, \zeta^{-1}z^{-3}). \quad (2.3)$$

A special case of (2.2), with $-\lambda = 1 = \mu$ and with ζ changed to $q\zeta$, that will be of our use is

$$a(q, q\zeta, z) \ = \ \zeta^{-1}\, z\, a(q, q^{-1/2}\zeta, q^{1/2}z).$$

Or, what is the same, on using (1.8),

$$c(q,\zeta,z) \ = \ q^{1/3}\zeta^{-1}\, z\, a(q, q^{-1/2}\zeta, q^{1/2}z). \quad (2.2')$$

We also need the classical theta function transformation [Entry 20, 1]

$$\sqrt{\alpha} f(e^{-\alpha^2+n\alpha}, e^{-\alpha^2-n\alpha}) \ = \ e^{n^2/4}\sqrt{\beta} f(e^{-\beta^2+in\beta}, e^{-\beta^2-in\beta}), \quad (2.4)$$

provided $\alpha\beta = \pi$ and $\mathcal{Re}(\alpha^2) > 0$. In fact the following two special cases of (2.4) will be used:

$$f(e^{-\pi t+i\theta}, e^{-\pi t-i\theta}) \ = \ \frac{1}{\sqrt{t}}\ \exp\left(-\frac{\theta^2}{4\pi t}\right)\ f(e^{-\frac{\pi+\theta}{t}}, e^{-\frac{\pi-\theta}{t}}), \quad (2.5)$$

and

$$f(e^{-\pi t+i\theta}, e^{-i\theta}) \ = \ \sqrt{\frac{2}{t}}\ \exp\left(\frac{\pi t}{8} - \frac{i\theta}{2} - \frac{\theta^2}{2\pi t}\right)\ f(-e^{-\frac{2\pi+2\theta}{t}}, -e^{-\frac{2\pi-2\theta}{t}}). \quad (2.6)$$

Indeed, to obtain (2.5) from (2.4), we put $\alpha = \sqrt{\frac{\pi}{t}}$, $\beta = \sqrt{\pi t}$, and $n = \frac{\theta}{\sqrt{\pi t}}$. To obtain (2.6) from (2.4), we substitute $\alpha = \sqrt{\frac{\pi t}{2}}$, $\beta = \sqrt{\frac{2\pi}{t}}$, and $n = -\sqrt{\frac{\pi t}{2}} + i\sqrt{\frac{2}{\pi t}}\,\theta$.

We also need the following simple consequences of the definition (2.1) , noted by Ramanujan (Entries 30(ii)and 30(iii), 31 of [**1**]):

$$f(a,b)+f(-a,-b) \;=\; 2f(a^3b,ab^3), \tag{2.7}$$

and

$$f(a,b)-f(-a,-b) \;=\; 2af\left(\frac{b}{a},\frac{a}{b}\;a^4b^4\right). \tag{2.8}$$

Lastly, we need the following identity which easily follows from the definition (1.1),

$$a\left(q,x^3y,\frac{x}{y}\right) \;=\; a(q,y^2,x^2). \tag{2.9}$$

Indeed, we have

$$\begin{aligned} a\left(q,x^3y,\frac{x}{y}\right) &= \sum q^{n^2+nm+m^2}(x^3y)^{n+m}\left(\frac{x}{y}\right)^{n-m} \\ &= \sum q^{n^2+nm+m^2}x^{4n+2m}y^{2m}. \end{aligned} \tag{2.10}$$

This at once transforms to (2.9) on putting $m=i+j$ and $n=-j$ and using (1.1) again.

We are now in a position to state and prove our master formula.

Theorem 2.1. *With* $a(q,\zeta,z)$ *defined by* (1.1)*, the transformation formula* (1.14) *holds.*

Proof. We have from (2.3) and repeated use of (2.5)–(2.6)

$$\begin{aligned} a(e^{-2\pi t},e^{i\varphi},e^{i\theta}) \;&=\; f(e^{-2\pi t+i(\varphi-\theta)},e^{-2\pi t-i(\varphi-\theta)})f(e^{-6\pi t+i(\varphi+3\theta)},e^{-6\pi t-i(\varphi+3\theta)}) \\ &\quad + e^{-2\pi t+i(\varphi+\theta)}f\big(e^{-4\pi t+i(\varphi-\theta)},e^{-i(\varphi-\theta)}\big)f\big(e^{-12\pi t+i(\varphi+3\theta)},e^{-i(\varphi+3\theta)}\big) \\ &=\; \frac{1}{2\sqrt{3}t}\;\exp\left[-\left(\frac{\varphi^2+3\theta^2}{6\pi t}\right)\right](\alpha\beta+\alpha'\beta') \end{aligned} \tag{2.11}$$

where

$$\alpha=f(e^{-\frac{\pi+\varphi-\theta}{2t}},e^{-\frac{\pi-\varphi+\theta}{2t}}), \qquad \beta=f(e^{-\frac{\pi+\varphi+3\theta}{6t}},e^{-\frac{\pi-\varphi-3\theta}{6t}})$$

and

$$\alpha'=f(-e^{-\frac{\pi+\varphi-\theta}{2t}},-e^{-\frac{\pi-\varphi+\theta}{2t}}), \qquad \beta'=f(-e^{-\frac{\pi+\varphi+3\theta}{6t}},-e^{-\frac{\pi-\varphi-3\theta}{6t}}).$$

On using (2.7) and (2.8) repeatedly and the trivial identity $2(\alpha\beta + \alpha'\beta') = (\alpha+\alpha')(\beta+\beta') + (\alpha-\alpha')(\beta-\beta')$, equation (2.11) becomes

$$\begin{aligned} a(e^{-2\pi t}, e^{i\varphi}, e^{i\theta}) &= \frac{1}{\sqrt{3}t}\exp\left[-\left(\frac{\varphi^2+3\theta^2}{6\pi t}\right)\right] \\ &\quad\times [f(e^{-\frac{2\pi-\theta+\varphi}{t}}, e^{-\frac{2\pi+\theta-\varphi}{t}})f(e^{-\frac{2\pi+3\theta+\varphi}{3t}}, e^{-\frac{2\pi-3\theta-\varphi}{3t}}) \\ &\qquad + e^{-\frac{2\pi+2\theta}{3t}} f(e^{\frac{\varphi-\theta}{t}}, e^{-\frac{4\pi-\theta+\varphi}{t}})f(e^{\frac{\varphi+3\theta}{3t}}, e^{-\frac{4\pi+\varphi+3\theta}{3t}})] \\ &= \frac{1}{t\sqrt{3}}\exp\left[-\left(\frac{\varphi^2+3\theta^2}{6\pi t}\right)\right] \\ &\quad\times [f(q^3\zeta z^3, q^3\zeta^{-1}z^{-3})f(q\zeta z^{-1}, q\zeta^{-1}z) \\ &\qquad + q\zeta z f(q^6\zeta z^3, \zeta^{-1}z^{-3})f(q^2\zeta z^{-1}, \zeta^{-1}z)] \end{aligned} \tag{2.12}$$

with $q = e^{-\frac{2\pi}{3t}}, \zeta = e^{-\frac{\theta+\varphi}{2t}}, z = e^{\frac{3\theta-\varphi}{6t}}$.

This is same as the identity

$$a(e^{-2\pi t}, e^{i\varphi}, e^{i\theta}) = \frac{1}{t\sqrt{3}}\exp\left[-\left(\frac{\varphi^2+3\theta^2}{6\pi t}\right)\right] a(e^{-\frac{2\pi}{3t}}, e^{\frac{\varphi+\theta}{2t}}, e^{\frac{\varphi-3\theta}{6t}}), \tag{2.13}$$

on employing (2.3) and the property $a(q,\zeta,z) = a(q,\zeta^{-1},z^{-1})$ which is an easy consequence of the definition (1.1). Now, (1.14) follows immediately from (2.13) on using (2.9) with $x = e^{\frac{\varphi}{6t}}$, $y = e^{\frac{\theta}{2t}}$. □

3. Equivalent Transformations

Theorem 3.1. *The transformation formulas* (1.15)–(1.18) *are all equivalent to the main transformation* (1.14).

Proof. That (1.14) and (1.15) are equivalent follows immediately on using the equivalence of (1.14) with (2.13), the symmetry $a'(q,\zeta,z) = a'(q,z,\zeta)$ (which trivially follows from the series form in (1.6)) in the right side of (1.15), and finally the first part of (1.6).

Now, (1.16) follows from (1.15) on putting

$$t = \frac{1}{3t'}, \quad \varphi = \frac{-i\varphi'}{2t'} \quad \text{and} \theta = \frac{-i(2\theta'-\varphi')}{6t'}$$

and then using the trivial identity $a'(q,\zeta,z) = a'(q,\zeta^{-1},z^{-1})$ (which again easily follows from the series form in (1.6)).

To prove (1.17), we have from (1.7)

$$b(e^{-2\pi t}, e^{i\varphi}, e^{i\theta}) = a'\big(e^{-2\pi t}, e^{i(\varphi+\frac{2\pi}{3})}, e^{i(\theta-\frac{2\pi}{3})}\big).$$

Applying (1.16) to this gives, on some simplification

$$\begin{aligned} &b(e^{-2\pi t}, e^{i\varphi}, e^{i\theta}) \\ &= \frac{1}{t\sqrt{3}}\exp\left[-\frac{1}{6\pi t}\left\{\left(\theta-\frac{2\pi}{3}\right)^2 - \left(\theta-\frac{2\pi}{3}\right)\left(\varphi+\frac{2\pi}{3}\right) + \left(\varphi+\frac{2\pi}{3}\right)^2\right\}\right] \\ &\quad\times\ a(q, q^{-1/2}\zeta, q^{1/2}z) \end{aligned}$$

with $q = e^{-\frac{2\pi}{3t}}$, $\zeta = e^{\frac{\varphi}{2t}}$ and $z = e^{\frac{2\theta-\varphi}{6t}}$. However, this at once becomes (1.17) on employing (2.2)$'$.

Lastly, (1.18) follows from (1.17) in exactly the same fashion as (1.16) from (1.15) but with the substitutions

$$t = \frac{1}{3t'}, \quad \varphi = \frac{-2i\varphi'}{3t'} \quad \text{and} \quad \theta = \frac{-i(\varphi' + 3\theta')}{3t'}.$$

We will also need the trivial identity $c(q, \zeta, z) = c(q, \zeta, z^{-1})$, which of course follows from the series form in (1.8). □

4. New Identities Involving $a'(q, \zeta, z)$

We now apply the results of the previous section to identities (1.31) and (1.32) in [**2**], namely

$$\begin{aligned} &2\,a(q, \zeta, z)a(q^2, \zeta^2, z^2) \\ &\quad = \; b(q^2)[b(q, \zeta^2, \zeta z^3) + b(q, \zeta^{-2}, \zeta^{-1} z^3)] \\ &\qquad + \; \zeta^2 c(q, \zeta, z)c(q^2, \zeta^2, z^2) + \zeta^{-2} c(q, \zeta^{-1}, z)c(q^2, \zeta^{-2}, z^2) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &3z\,a(q, \zeta, z)f(q^3\zeta^{-1}, \zeta)f(qz^{-1}, z) \\ &\quad = \; f(q\zeta^{-1}, \zeta)f^3(qz^{-1}, z) \\ &\qquad + \; b(q)f(q^3 z^{-3}, z^3)[3\,q\zeta^{-1} f(q^9\zeta^{-3}, \zeta^3) - f(q\zeta^{-1}, \zeta)]. \end{aligned} \tag{4.2}$$

We may clarify that we have indeed taken in (4.2) a version of (1.32) of [**2**] convenient for our purpose here. The equivalence is easily established on repeated use of the triple product identity for the theta function (Entry 19, [**1**]),

$$f(a, b) \; = \; (-a, ab)_\infty (-b; ab)_\infty (ab; ab)_\infty \tag{4.3}$$

and the product form of $b(q)$ $\big($Equation (1.6) of [**3**]$\big)$

$$b(q) \; = \; \frac{(q; q)_\infty^3}{(q^3; q^3)_\infty}, \tag{4.4}$$

where, as usual,

$$(\lambda; q)_\infty \; := \; \prod_{n=0}^{\infty} (1 - \lambda q^n).$$

Theorem 4.1.

$$\begin{aligned} &2a'(q, \zeta^2, \zeta z)a'(q^2, \zeta^2, \zeta z) \\ &\quad = \; [b(q, \zeta^2, \zeta z)b(q^2, \zeta^2, \zeta z) \\ &\qquad + \; b(q, \zeta^{-2}, \zeta^{-1} z)b(q^2, \zeta^{-2}, \zeta^{-1} z)] \\ &\qquad + \; c(q) \times [\zeta^2 c(q^2, \zeta^3, z) + \zeta^{-2} c(q^2, \zeta^{-3}, z)] \end{aligned} \tag{4.5}$$

$$\begin{aligned} &3a'(q^6, \zeta^2, \zeta z)f(q^3\zeta^{-1}, q^3\zeta)f(q^9 z^{-1}, q^9 z) \\ &\quad = \; 3f(q^9\zeta^{-3}, q^9\zeta^3)f^3(q^9 z^{-1}, q^9 z) \\ &\qquad + \; c(q^6)f(q^3 z^{-1}, q^3 z)[f(q\zeta^{-1}, q\zeta) - f(q^9\zeta^{-3}, q^9\zeta^3)]. \end{aligned} \tag{4.6}$$

Proof. Put $q = e^{-2\pi t}$, $\zeta = e^{i\varphi}$, $z = e^{i\theta}$ in (4.1) and apply (1.15)–(1.18), including the special case of (1.12) with $\theta = 0$. Some simplification and change of variables ($e^{-\frac{\pi}{3t}}, e^{\frac{\varphi}{3t}}, e^{\frac{\theta}{t}}$ into q, ζ, z respectively) gives (4.2). Similarly, applying (1.15) and (2.6) repeatedly to (4.2) yields (4.6). We omit the details. □

Remark 4.2. By putting $\zeta = 1$ in (4.5) we obtain an identity of Cooper $\big($(6.3) of [**5**]$\big)$,

$$a'(q,z)a'(q^2,z) \;=\; b(q,z)b(q^2,z) + c(q)c(q^2,z).$$

Remark 4.3. We can recast (4.6) into a form analogous to that of (1.32) of [**2**],

$$\begin{aligned} a'(q^2,\zeta^2,\zeta z) \\ = \frac{(-q^3\zeta^{-3};q^6)_\infty(-q^3\zeta^3;q^6)_\infty(-q^3z^{-1};q^6)_\infty^2(-q^3z;q^6)_\infty^2(q^6;q^6)_\infty^3}{(-q\zeta^{-1};q^2)_\infty(-q\zeta;q^2)_\infty(q^2;q^2)_\infty} \\ + \frac{q^{2/3}(-qz^{-1};q^2)_\infty(-qz;q^2)_\infty(q^6;q^6)_\infty^2\left(\sum \zeta^n q^{n^2/3} - \sum \zeta^{3n}q^{3n^2}\right)}{(-q\zeta^{-1};q^2)_\infty(-q\zeta;q^2)_\infty(-q^3z^{-1};q^6)_\infty(-q^3z;q^6)_\infty(q^2;q^2)_\infty}. \end{aligned} \quad (4.7)$$

One need only make repeated use of (4.3) and the dual of (4.5) (Equation (1.7) of [**3**])

$$c(q) \;=\; 3q^{1/3}\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\,.$$

We omit the details.

Acknowledgement 1. The authors are grateful to the referee for his valuable suggestions which considerably improved the quality of the paper.

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