HYPERCONVEX ULTRAMETRIC SPACES AND FIXED POINT THEORY

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(Received May 2002)

Abstract. This paper is inspired by the work of Sine and Soardi [9], [10]. We introduce the notion of hyperconvexity in generalized ultrametric spaces i.e. where the distance has values in an arbitrary partially ordered set. Then we give a fixed point theorem in these spaces for contracting or nonexpansive mappings which leads to some corollaries.

1. Introduction

The hyperconvexity is a recent notion introduced by Aronsjazn and Panitchpakdi [1] in 1956. It was intensively studied for nonexpansive maps. In 1976, Sine and Soardi [9], [10] proved that fixed point property holds for nonexpansive mappings in bounded hyperconvex spaces. Recently, Jawhari, Misane and Pouzet [3] showed that Sine and Soardi Fixed Point Theorem is equivalent to the classical Tarski’s theorem via the notion of generalized metric spaces.

In this paper, the notion of hyperconvexity is defined on generalized ultrametric spaces and is also successfully used in fixed point theory.

Generalized ultrametric spaces are a common generalization of Partially ordered sets and ordinary ultrametric spaces, Lawrence 1973, Rutten 1995. The topology for generalized metric spaces extend both the Scott topology for algebraic complete partial orders and the Alexandroff topology for ordinary metric spaces reduced to the $\varepsilon$–ball topology. A new fixed point theorem holds for nonexpansive mappings and leads to an analogue of Banach fixed point theorem in ultrametric spaces.

We discuss in this context the consequences which have been made so far of this theorem and its corollaries.

2. Hyperconvex Ultrametric Spaces

Let $X$ be a set and let $(\Gamma, \leq)$ be a complete lattice with a least element 0 and a greatest element 1 (a complete lattice is an ordered set where each non–empty subset has a supremum and an infimum).

Let $d : X \times X \to \Gamma$ be a mapping which satisfies the following properties:

- For all $x, y, z \in X$ and $\gamma \in \Gamma$:
  - (1) $d(x, y) = 0$ if and only if $x = y$
  - (2) $d(x, y) = d(y, x)$
  - (3) if $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$, then $d(x, z) \leq \gamma$.

1991 Mathematics Subject Classification 06F30, 46A16, 46A19.

Key words and phrases: Ultrametric spaces, hyperconvexity, nonexpansive mappings, fixed points.
The triple \((X,d,\Gamma)\) is called an ultrametric space.

Let \(0 < \gamma \in \Gamma\) and \(a \in X\). The set \(B_\gamma(a) = \{x \in X; d(x,a) \leq \gamma\}\) is called a ball.

We can also use the notation \(B(a,\gamma)\) instead of \(B_\gamma(a)\).

We note the following property for an ultrametric space \((X,d,\Gamma)\).

1.1. If \(0 \neq \alpha \leq \beta\) and \(B_\alpha(a) \cap B_\beta(b) \neq \emptyset\), then \(B_\alpha(a) \subseteq B_\beta(b)\).

**Proof.** Let \(x \in B_\alpha(a)\), then \(d(x,a) \leq \alpha\). On the other hand, there exists \(y \in B_\beta(b)\), hence \(d(y,a) \leq \alpha\) and \(d(y,b) \leq \beta\). Thus by (1), (2), \(d(x,y) \leq \alpha \leq \beta\) and therefore \(d(x,b) \leq \beta\). So \(x \in B_\beta(b)\).

Let \(E\) a set. A family \(F = (E_i)_{i \in I}\) of subsets of \(E\) has the 2-Helly’s property if for all sub-family \(F' = (E'_i)_{i \in I'}\), the intersection of this family \(\bigcap\{E_i: i \in I'\}\) is non-empty if and only if for all \(i, j \in I'\) the intersections \(E_i \cap E_j\) are not empty.

In the same way, a set \(F\) of subsets of \(E\) has the 2-Helly’s property if for all subset \(F'\), the intersection of subsets in \(F'\) is non-empty if and only if the intersection of two arbitrary subsets in \(F'\) is non-empty. In fact, \(F\) has the 2-Helly’s property if and only if each subset of elements of \(F\) has the 2-Helly’s property.

**Definition 2.1.** Let \((X,d,\Gamma)\) be an ultrametric space. \(X\) is called hyperconvex if it satisfies the two following properties:

(i) the set of blls has the 2-Helly’s property: ie For each family \((B(x_i,\gamma_i))_{i \in I}\) of closed balls of \(X\):

If \(B(x_i,\gamma_i) \cap B(x_j,\gamma_j) \neq \emptyset\) for all \(i, j\), then \(\bigcap_{i \in I} B(x_i,\gamma_i) \neq \emptyset\).

(ii) Convexity: For all \(x, y \in X\) and \(\gamma_1, \gamma_2 \in \Gamma\):

If \(d(x,y) \leq \sup(\gamma_1,\gamma_2)\), then there exists \(z \in X\) such that \(d(x,z) \leq \gamma_1\) and \(d(z,y) \leq \gamma_2\).

**Proposition 2.2.** Let \((X,d,\Gamma)\) be an ultrametric space. \(X\) is hyperconvex if and only if each family of closed balls \((B(x_i,\gamma_i))_{i \in I}\) of \(X\) satisfies the property \(\mathcal{P}\):

\[\bigcap_{i \in I} B(x_i,\gamma_i) \neq \emptyset\quad \text{if and only if}\quad d(x_i,x_j) \leq \sup(\gamma_i,\gamma_j)\quad \text{for all }\quad i, j \in I.\]

**Proof.** We first show that:

\[\bigcap_{i \in I} B(x_i,\gamma_i) \neq \emptyset\quad \text{implies that}\quad d(x_i,x_j) \leq \sup(\gamma_i,\gamma_j)\quad \text{for all }\quad i, j \in I.\]

Let \(z \in \bigcap_{i \in I} B(x_i,\gamma_i)\) then for all \(i, j \in I\) \(d(x_i,z) \leq \gamma_i\) and \(d(x_j,z) \leq \gamma_j\) and so \(d(x_i,x_j) \leq \sup(\gamma_i,\gamma_j)\).

Now, let \(X\) be a hyperconvex space and \((B(x_i,\gamma_i))_{i \in I}\) a family of balls of \(X\) with \(d(x_i,x_j) \leq \sup(\gamma_i,\gamma_j)\) for all \(i, j \in I\), then for all \(i, j \in I\) there exists \(z \in X\) such that \(d(x_i,z) \leq \gamma_i\) and \(d(x_j,z) \leq \gamma_j\) (from the convexity). Thus \(\bigcap_{i \in I} B(x_i,\gamma_i) \neq \emptyset\) from 2-Helly.

Conversely, suppose that all the families of closed balls of \(X\) satisfy the property \(\mathcal{P}\) and let \((B(x_i,\gamma_i))_{i \in I}\) a family of balls of \(X\) with \(B(x_i,\gamma_i) \cap B(x_j,\gamma_j) \neq \emptyset\) for all \(i, j \in I\). Let \(z \in B(x_i,\gamma_i) \cap B(x_j,\gamma_j)\) then \(d(x_i,z) \leq \gamma_i\) and \(d(x_j,z) \leq \gamma_j\). So \(d(x_i,x_j) \leq \sup(\gamma_i,\gamma_j)\) for all \(i, j \in I\) and thus \(\bigcap_{i \in I} B(x_i,\gamma_i) \neq \emptyset\) from the property \(\mathcal{P}\).

Let \(x, y \in X\) and \(\gamma_1, \gamma_2 \in \Gamma\) such that \(d(x,y) \leq \sup(\gamma_1,\gamma_2)\) then \(B(x,\gamma_1) \cap B(y,\gamma_2) \neq \emptyset\) from \(\mathcal{P}\). Let \(z \in B(x,\gamma_1) \cap B(y,\gamma_2)\) then \(d(x,z) \leq \gamma_1\) and \(d(z,y) \leq \gamma_2\).
Proposition 2.3. The non-empty intersection of closed balls of an hyperconvex space is hyperconvex.

Proof. Let \((X,d,\Gamma)\) be an hyperconvex ultrametric space and let \(B = B(x,r)\) be a ball of \(X\).

Let \((B_r(x_i,r_i))_{i\in I}\) a family of balls of \(B \ (B_r(x_i,r_i) = B(x_i,r_i))\) which satisfies: \(d(x_i,x_j) \leq \sup(r_i,r_j)\). Let \(i \in I\), we have \(B_r(x_i,r_i) \neq \emptyset\). Let \(y_i \in B_r(x_i,r_i)\), then \(B(y_i,r_i) = B(x_i,r_i)\) from (1.1). Then we can assume without loss of generality that all \(x_i\) are in \(B\). The balls \(B(x_i,r_i)\) satisfy the property \(\mathcal{P}\) in \(X\) hyperconvex, hence \(\bigcap_{i\in I} B(x_i,r_i) \neq \emptyset\). Let \(y\) in this intersection, then \(d(y,x_i) \leq r_i \leq \sup(r,r_i)\) for all \(i \in I\). Hence \(y \in B\). That shows that \(B\) is hyperconvex.

We will proceede in the similar way for the intersection of two balls and so one for an arbitrary family of balls. \(\square\)

Let \((X,d,\Gamma)\) be an ultrametric space and \(T : X \rightarrow X\) a mapping. \(T\) is called nonexpansive or contracting if \(d(T(x),T(y)) \leq d(x,y)\) for all \(x,y \in X\). If for all \(x,y \in X\), \(x \neq y\), \(d(T(x),T(y)) < d(x,y)\) then \(T\) is strictly contracting. Let \(x \in X\), the orbit of \(x\) by \(T\) is the set \(\{x,T(x),T^2(x),\ldots\}\). \(T\) is called strictly contracting on orbits when for every \(x \in X\) such that \(T(x) \neq x\), we have \(d(T^2(x),T(x)) < d(T(x),x)\).

We define \(B\) as the set of the nonempty intersections of closed balls of \(X\) and we denote \(\mathcal{A} = \bigcap \{B \in B ; A \subseteq B\}\) for all \(A \subseteq X\) and \(\delta(A) = \sup\{d(x,y); x,y \in A\}\) the diameter of \(A\).

Theorem 2.4. Let \((X,d,\Gamma)\) be a hyperconvex ultrametric space and \(T : X \rightarrow X\) be a nonexpansive mapping, then \(T\) has a fixed point or there exists a nonempty hyperconvex subset \(S\) of \(X\) such that \(T(S) \subseteq S\), \(S = \overline{T(S)}\) and \(\delta(S) = d(x,T(x))\) for all \(x \in S\).

Proof. Suppose that \(T\) doesn’t have a fixed point.

Let \(B_T = \{B \in B ; T(B) \subseteq B\}\). \(B_T \neq \emptyset\) as \(X = B(x,1)\) for all \(x \in X\), then \(X \in B_T\). Let us show that \(B_T\) has a minimal element. From Zorn’s lemma, it suffices to show that \(B_T\) is inductively ordered or equivalently that each chain of \(B_T\) has a lower bound (a chain is a family of elements of an ordered set such that each two elements of the chain are comparative).

Let \((B_k)_{k \in K}\) a chain of \(B_T\) \((B_k = \bigcap_{i \in I_k} B(x_i^k,\gamma_i^k))\).

Let \(k \in K\). \(B_k \neq \emptyset\). Let \(z \in B_k\) then \(d(x_i^k,z) \leq \gamma_i^k\) and \(d(x_j^k,z) \leq \gamma_j^k\) for all \(i,j \in I_k\). Thus \(d(x_i^k,x_j^k) \leq \sup(\gamma_i^k,\gamma_j^k)\) for all \(i,j \in I_k\).

Let \(k_1,k_2 \in K\) \((k_1 \neq k_2)\): Suppose that \(B_{k_1} \subseteq B_{k_2}\) (as \((B_k)\) is a chain). If \(z \in B_{k_1}\) then \(d(x_i^{k_1},z) \leq \gamma_i^{k_1}\) and \(d(z,x_j^{k_2}) \leq \gamma_j^{k_2}\) for all \(i \in I_{k_1}\) and \(j \in I_{k_2}\). So \(d(x_i^{k_1},x_j^{k_2}) \leq \sup(\gamma_i^{k_1},\gamma_j^{k_2})\).

We conclude that the family \((B(x_i^k,\gamma_i^k))_{k \in K, i \in I_k}\) satisfies the property \(\mathcal{P}\) given in Proposition 2.2 in \(X\) hyperconvex. Thus \(\bigcap_{k \in K} B_k = \bigcap_{k \in K} \bigcap_{i \in I_k} B(x_i^k,\gamma_i^k) \neq \emptyset\) and \(T(\bigcap_{k \in K} B_k) = T(\bigcap_{k \in K} \bigcap_{i \in I_k} B(x_i^k,\gamma_i^k)) \subseteq \bigcap_{k \in K} T(\bigcap_{i \in I_k} B(x_i^k,\gamma_i^k)) = \bigcap_{k \in K} T(B_k) \subseteq \bigcap_{k \in K} B_k\) as all the \(B_k\) are in \(B_T\). Thus from Zorn’s lemma, \(B_T\) has a minimal element \(S\). \(\square\)
Let \( T : X \to X \) be a nonexpansive mapping. Assume furthermore that for every \( k \in K \), \( T(x) \in S \), then \( d(T(x), x) \leq \gamma_k \) as \( d \) is ultrametric. We take \( B_x = B(x, d(x, T(x))) \). \( T(x) \neq x \), then \( B_x \subseteq B(x_k, \gamma_k) \) for all \( k \in K \) from (1.1) and so \( B_x \subseteq S \). Let \( y \in B_x \) then \( d(T(y), T(x)) \leq d(y, x) \leq d(x, T(x)) \) as \( T \) is nonexpansive and so \( d(T(y), x) \leq d(x, T(x)) \) as \( d \) is ultrametric. Then we have \( T(B_x) \subseteq B_x \) and so \( B_x \subseteq B_T \). Thus \( S = B_x \) as \( S \) is minimal in \( B_T \). Let \( y, z \in S \) then \( d(y, x) \leq d(x, T(x)) \) and \( d(z, x) \leq d(x, T(x)) \). So \( d(y, z) \leq d(x, T(x)) \). It follows that \( \delta(S) = \sup \{d(y, z); y, z \in S \} \leq d(x, T(x)) \leq \delta(S) \).

From the Theorem 2.4, we obtain the following corollary:

**Corollary 2.5.** Let \( (X, d, \Gamma) \) be a hyperconvex ultrametric space and \( T : X \to X \) be a nonexpansive mapping. Assume furthermore that for every \( x \in X \) with \( T(x) \neq x \), there exists an integer \( n \geq 0 \) such that \( d(T^{n+1}(x), T^n(x)) < d(T^n(x), T^{n+1}(x)) \) (we take \( T^0(x) = x \)). Then \( T \) has a fixed point and the set of fixed points of \( T \) is hyperconvex.

**Proof.** We can deduce easily from the theorem 2.4 that \( T \) has a fixed point as we can’t find such an hyperconvex subset \( S \) of \( X \). Let \( \text{Fix}(T) \) be the set of fixed points of \( T \) and let \( \{B_{\text{Fix}(T)}(x_i, \gamma_i)\}_{i \in I} \) a family of balls of \( \text{Fix}(T) \) with \( d(x_i, x_j) \leq \sup(\gamma_i, \gamma_j) \) for all \( i, j \in I \) \( B_{\text{Fix}(T)}(x_i, \gamma_i) = B(x_i, \gamma_i) \cap \text{Fix}(T) \). We have \( B(x_i, \gamma_i) \cap \text{Fix}(T) \neq \emptyset \). Let \( y \in B(x_i, \gamma_i) \cap \text{Fix}(T) \), so \( B(y_i, \gamma_i) = B(x_i, \gamma_i) \) from (1.1). Then, we can suppose without loss of generality that \( x_i \in \text{Fix}(T) \) for all \( i \in I \). The balls \( B(x_i, \gamma_i) \) satisfy the property \( P \) given in Proposition 2.2 on \( X \) hyperconvex, then \( F = \bigcap \{B(x_i, \gamma_i) / i \in I \} \neq \emptyset \) and as an intersection of balls of a hyperconvex space, \( F \) is hyperconvex, moreover \( T(F) \subseteq F \) since \( x_i \in \text{Fix}(T) \), so the theorem 2.4 applied to \( F \) gives an \( x \in \text{Fix}(T) \cap F \). Thus the last intersection is not empty, and \( \text{Fix}(T) \) is hyperconvex.

This result generalizes the following theorem.

**Theorem 2.6.** Let \( (X, d, \Gamma) \) be a hyperconvex ultrametric space and let \( T : X \to X \) be contracting and strictly contracting on orbits. Then \( T \) has a fixed point. If moreover \( T \) is strictly contracting on \( X \), then the fixed point is unique.

**References**