INTEGRATION OPERATORS FROM CAUCHY INTEGRAL TRANSFORMS TO WEIGHTED DIRICHLET SPACES

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Abstract. The boundedness and compactness of integration operators between Cauchy integral transforms and weighted Dirichlet spaces are characterized.

1. Introduction

Let \( \mathbb{D} \) be the open unit disk in the complex plane \( \mathbb{C} \), \( \partial \mathbb{D} \) its boundary, \( H(\mathbb{D}) \) the class of all holomorphic functions on \( \mathbb{D} \), \( dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta \) the normalized area measure on \( \mathbb{D} \), \( H^\infty(\mathbb{D}) \) the space of all bounded analytic function on \( \mathbb{D} \) with the norm \( \| f \|_\infty = \sup_{z \in \mathbb{D}} |f(z)| \) and \( \mathfrak{M} \) the space of all complex Borel measures on \( \partial \mathbb{D} \). For \( 0 < p < \infty \), the Hardy space \( H^p \) is the space of all \( f \in H(\mathbb{D}) \) such that \( |f|^p \) admits a harmonic majorant. If we take as the norm of \( f \in H^p \) the \( p \)-th root of the value at some fixed point \( z \in \mathbb{D} \) of the least harmonic majorant of \( |f|^p \), then \( H^p \) is a Banach space for \( 1 \leq p < \infty \) (\( p \)-Banach space for \( 0 < p < 1 \)). Moreover, regardless of \( z \) all these norms are equivalent. It is, however, more customary to work with another definition of \( H^p \) and with another equivalent norm. Recall that \( f \in H(\mathbb{D}) \) belongs to \( H^p \) if and only if the integrals
\[
M_p(r, f) := \int_{\partial \mathbb{D}} |f(r \zeta)|^p dm(\zeta), \quad 0 < r < 1,
\]
are bounded. In this case
\[
\| f \|_p := \sup_{0 < r < 1} M_p(r, f)^{1/p}
\]
is a norm (\( p \)-norm if \( 0 < p < 1 \)) on \( H^p \) which is equivalent to norms described above. Also it is well known that if \( 0 < p < q < \infty \), then \( H^q \subset H^p \).

Let \( \omega \) be a positive integrable function. If we extend it on \( \mathbb{D} \) by \( \omega(z) = \omega(|z|), z \in \mathbb{D} \), we call it a weight or a weight function. By \( \mathcal{D}_\omega \) we denote the weighted Dirichlet space consisting of all \( f \in H(\mathbb{D}) \) such that
\[
\| f \|^2_{\mathcal{D}_\omega} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < \infty.
\]
The Bergman space \( \mathcal{A}_\omega^2 \) is a Hilbert space of holomorphic functions on \( \mathbb{D} \) with the norm
\[
\| f \|^2_{\mathcal{A}_\omega^2} = \int_{\mathbb{D}} |f(z)|^2 \omega(z) dA(z) < \infty.
\]

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A simple computation shows that a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) belongs to \( D_\omega \) if and only if
\[
\sum_{n=0}^{\infty} |a_n|^2 \omega_n < \infty,
\]
where \( \omega_0 = 1 \) and
\[
\omega_n = \omega(n) = 2n^2 \int_0^1 r^{2n-1} \omega(r) dr, \quad n \in \mathbb{N}.
\]
The sequence \( (\omega_n)_{n \in \mathbb{N}_0} \) is called the weight sequence of the weighted Dirichlet space \( D_\omega \). The properties of the weighted Dirichlet space with the weight sequence \( (\omega_n)_{n \in \mathbb{N}_0} \), clearly depends upon \( \omega_n \).

We may assume that the weight \( \omega \) satisfies the following three properties:

1. \( \omega \) is non-increasing;
2. \( \frac{\omega(r)}{(1-r)^{-\delta}} \) is non-decreasing for some \( \delta > 0 \);
3. \( \lim_{r \to 1} \omega(r) = 0 \).

If \( \omega \) also satisfies one of the following properties:

4. \( \omega \) is convex and \( \lim_{r \to 1} \omega(r) = 0 \); or
5. \( \omega \) is concave,

then such a weight function is called admissible (see [9]). If \( \omega \) satisfies conditions (W1), (W2), (W3) and (W4), then it is said that \( \omega \) is I-admissible. If \( \omega \) satisfies conditions (W1), (W2), (W3) and (W5), then it is said that \( \omega \) is II-admissible. I-admissibility corresponds to the case \( \mathcal{H}^2 \subseteq \mathcal{H}_\omega \subset A^2 \) for some \( \alpha > -1 \), whereas II-admissibility corresponds to the case \( D \subseteq \mathcal{H}_\omega \subset \mathcal{H}^2 \). If we say that a weight is admissible it means that it is I-admissible or II-admissible.

A function \( f \) in \( H(\mathbb{D}) \) is in the space of Cauchy integral transforms \( K \) if
\[
f(z) = \int_{\partial \mathbb{D}} \frac{d\mu(\zeta)}{1 - \zeta z}, \quad z \in \mathbb{D},
\]
for some \( \mu \in \mathfrak{M} \). The space \( K \) becomes a Banach space under the norm
\[
\|f\|_K = \inf \left\{ \|\mu\| : f(z) = \int_{\partial \mathbb{D}} \frac{d\mu(\zeta)}{1 - \zeta z} \right\},
\]
where \( \|\mu\| \) denotes the total variation of measure \( \mu \). It is well known that
\[
H^1 \subset K \subset \cap_{0<r<1} H^p.
\]
Let \( g \in H(\mathbb{D}), n \in \mathbb{N} \cup \{0\} \) and \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \). We define the generalized integration operator as follows
\[
I_{g,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta)) g(\zeta) d\zeta, \quad z \in \mathbb{D}.
\]
Operator (3) is an extension of many operators appearing in the literature. For example, if \( n = 0 \), then is obtained an operator, which is a natural extension of the integral operator by Pommerenke [10]. If \( n = 1 \) then is obtained, so called generalized composition operator, which is a natural extension of the integral operator by Yoneda [29]. Recently, several authors have studied these operators along
with composition and weighted composition operators on different spaces of analytic functions. For example, one can refer to ([1]-[9], [11]-[29] and the related references therein for the study of these operators on different spaces of analytic functions. Here we provide complete characterizations of when $g$ and $\varphi$ induce bounded or compact integration operator $I_{g,\varphi}$ from the space $\mathcal{K}$ of Cauchy integral transforms into weighted Dirichlet spaces.

Throughout this paper constants are denoted by $C$, they are positive and not necessarily the same at each occurrence. We write $A \simeq B$ if there is a positive constant $C$ such that $CA \leq B \leq A/C$.

2. Boundedness and Compactness of $I_{g,\varphi}^{(n)} : \mathcal{K} \to \mathcal{D}_\varphi$

**Theorem 1.** Let $g \in H(\mathbb{D})$, $n \in \mathbb{N} \cup \{0\}$ and $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then $I_{g,\varphi}^{(n)} : \mathcal{K} \to \mathcal{D}_\varphi$ is bounded if and only if the family

$$\{g/(1 - \bar{\zeta}\varphi)^{n+1} : \zeta \in \partial \mathbb{D}\}$$

is a norm bounded subset of $\mathcal{A}_D^2$, that is there exists a constant $M > 0$ such that

$$\sup_{\zeta \in \partial \mathbb{D}} \int_{\partial \mathbb{D}} \frac{|g(z)|^2}{|1 - \zeta\varphi(z)|^{2(n+1)}} dA(z) \leq M < \infty. \quad (4)$$

**Proof.** First suppose that (4) holds. If $f \in \mathcal{K}$, then there is $\mu \in \mathfrak{M}$ with $\|\mu\| = \|f\|_\mathcal{K}$ such that

$$f(z) = \int_{\partial \mathbb{D}} \frac{1}{1 - \zeta z} d\mu(\zeta).$$

Thus we have

$$f^{(n)}(z) = n! \int_{\partial \mathbb{D}} \frac{(\bar{\zeta})^n}{(1 - \bar{\zeta}z)^{n+1}} d\mu(\zeta), \quad n \in \mathbb{N}. \quad (5)$$

Replacing $z$ in (5) by $\varphi(z)$, using Jensen’s inequality and multiplying such obtained inequality by $|g(z)|^2 \omega(z)$, we obtain

$$|g(z)|^2 |f^{(n)}(\varphi(z))|^2 \omega(z) \leq (n!)^2 \|\mu\|^2 \int_{\partial \mathbb{D}} \frac{|g(z)|^2}{|1 - \zeta\varphi(z)|^{2(n+1)}} \omega(z) \frac{d\|\mu\|}{\|\mu\|}, \quad (6)$$

Integrating (6) with respect to $dA(z)$ and applying Fubini’s theorem yield

$$\int_{\partial \mathbb{D}} |g(z)|^2 |f^{(n)}(\varphi(z))|^2 \omega(z) dA(z) \leq (n!)^2 \|\mu\| \int_{\partial \mathbb{D}} \left[ \int_{\partial \mathbb{D}} \frac{|g(z)|^2}{|1 - \zeta\varphi(z)|^{2(n+1)}} \omega(z) dA(z) \right] d\|\mu\|/\|\mu\|, \quad (7)$$

Since $I_{g,\varphi}^{(n)} f(0) = 0$ and $(I_{g,\varphi}^{(n)} f)'(z) = g(z) f^{(n)}(\varphi(z))$, so by (4), the inequality in (7) reduces to

$$\|I_{g,\varphi}^{(n)} f\|_{\mathcal{D}_\varphi}^2 \leq (n!)^2 M \|\mu\| \int_{\partial \mathbb{D}} d\|\mu\|/\|\mu\| = (n!)^2 M \|\mu\|^2 = (n!)^2 M \|f\|_{\mathcal{K}}^2.$$  

Thus $I_{g,\varphi}^{(n)} : \mathcal{K} \to \mathcal{D}_\varphi$ is bounded.

Conversely, suppose that $I_{g,\varphi}^{(n)} : \mathcal{K} \to \mathcal{D}_\varphi$ is bounded. Let $f_\zeta(z) = 1/1 - \bar{\zeta}z$. Then the fact that $\|f_\zeta\|_\mathcal{K} = 1$ for each $\zeta \in \partial \mathbb{D}$ and the boundedness of $I_{g,\varphi}^{(n)} : \mathcal{K} \to \mathcal{D}$. 

implies that $I_{g,\varphi}^{(n)} f_\zeta \in \mathcal{D}_\omega$ for every $\zeta \in \partial \mathbb{D}$. In particular, the fact that $I_{g,\varphi}^{(n)} f_\zeta(0) = 0$ asserts that

$$\frac{g}{(1 - \bar{\zeta} \varphi)^{n+1}} \in \mathcal{A}_\omega$$

for every $\zeta \in \partial \mathbb{D}$. Moreover,

$$\sup_{\zeta \in \partial \mathbb{D}} \int_{\mathbb{D}} \frac{|g(z)|^2}{1 - \bar{\zeta} \varphi(z)}^2 \omega(z) dA(z)$$

and so (4) holds, as desired.

To prove the next theorem, we need the following lemma.

**Lemma 1.** Let $g \in H(\mathbb{D})$, $n \in \mathbb{N} \cup \{0\}$ and $\varphi$ be a holomorphic self-map of $\mathbb{D}$. Then $I_{g,\varphi}^{(n)} : K \to \mathcal{D}_\omega$ is compact if and only if for any sequence $\{f_m\}$ in $K$ with $\|f_m\|_{K} \leq 1$ and which converges to zero locally uniformly, we have $\lim_{m \to \infty} \|I_{g,\varphi}^{(n)} f_m\|_{\mathcal{D}_\omega} = 0$.

Since the unit ball of $K$ is a compact subset of $H(\mathbb{D})$, the family of measures $I_{g,\varphi}^{(n)} : K \to \mathcal{D}_\omega$ is bounded. Then the following statements are equivalent:

1. $I_{g,\varphi}^{(n)} : K \to \mathcal{D}_\omega$ is compact;
2. The integral

$$\int_{\mathbb{D}} \frac{|g(z)|^2}{1 - \bar{\zeta} \varphi(z)}^2 \omega(z) dA(z)$$

is a continuous function of $\zeta \in \partial \mathbb{D}$.
3. The family of measures $\{\nu_\zeta : \zeta \in \partial \mathbb{D}\}$ defined by

$$\nu_\zeta(E) = \int_E \frac{|g(z)|^2}{1 - \bar{\zeta} \varphi(z)}^2 \omega(z) dA(z)$$

is equi-absolutely continuous. That is, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\nu_\zeta(E) < \varepsilon$ for all $\zeta \in \partial \mathbb{D}$, whenever $\mathcal{A}(E) < \delta$.
4. $g \in \mathcal{A}_\omega^2$

and

$$\lim_{r \to 1, \zeta \in \partial \mathbb{D}} \sup_{|\varphi(z)| > r} \frac{|g(z)|^2}{1 - \bar{\zeta} \varphi(z)}^2 \omega(z) dA(z) = 0.$$  \hspace{1cm} (8)

**Proof.** (1) $\Rightarrow$ (2). Let $\zeta_m \in \partial \mathbb{D}$ with $\zeta_m \to \zeta$ as $m \to \infty$, and let

$$f_{\zeta_m}(z) = \frac{1}{1 - \bar{\zeta}_m z}.$$ 

Then $\|f_{\zeta_m}\|_{K} = 1$ and $f_{\zeta_m} \to f_\zeta$ uniformly on compact subsets of $\mathbb{D}$. Since $I_{g,\varphi}^{(n)} : K \to \mathcal{D}_\omega$ is compact, by Lemma 1, we have

$$\|I_{g,\varphi}^{(n)} f_{\zeta_m} - I_{g,\varphi}^{(n)} f_\zeta\|_{\mathcal{D}_\omega} \to 0$$
as \( m \to \infty \). Since \( I_{g;\varphi}^{(n)} : K \to \mathcal{D}_\omega \) is bounded, there is a constant \( L > 0 \) such that
\[
||I_{g;\varphi}^{(n)} f_k||_{\mathcal{D}_\omega} \leq L ||f_k||_K = L \text{ for all } \zeta \in \partial D. \]
Let
\[
d\lambda_{g,\omega}(z) = |g(z)|^2 \omega(z) dA(z).
\]
Thus by Cauchy-Schwarz inequality, we have
\[
\int_D \left( \frac{1}{(1 - \overline{\zeta} \varphi(z))^{n+1}} - \frac{1}{(1 - \overline{\zeta \varphi(z))^{n+1}}} \right)^2 d\lambda_{g,\omega}(z)
\]
\[
\leq C \int_D \left| f_{\zeta}^{(n)}(\varphi(z)) \right|^2 - \left| f_{\zeta}^{(n)}(\varphi(z)) \right|^2 d\lambda_{g,\omega}(z)
\]
\[
\leq C \left( \int_D |f_{\zeta}^{(n)}(\varphi(z)) - f_{\zeta}^{(n)}(\varphi(z))|^2 d\lambda_{g,\omega}(z) \right)^{1/2}
\]
\[
= C ||I_{g;\varphi}^{(n)} f_k - I_{g;\varphi}^{(n)} f_k||_{\mathcal{D}_\omega} \to 0
\]
as \( m \to \infty \). Thus
\[
\int_D \frac{|g(z)|^2}{|1 - \zeta \varphi(z)|^{2(n+1)} \omega(z)} dA(z) \to \int_D \frac{|g(z)|^2}{|1 - \zeta \varphi(z)|^{2(n+1)} \omega(z)} dA(z),
\]
which shows the continuity of the integral in (2).

(2) \( \Rightarrow \) (3). Suppose that (3) does not holds. Then there exists a sequence \( \{\zeta_k\} \) in \( \partial D \) with \( \zeta_k \to \zeta \) and a sequence \( \{E_k\} \) in \( D \) such that \( A(E_k) \to 0 \) as \( k \to \infty \), but \( \nu_{\zeta}(E_k) \geq C > 0 \) for all \( k \in \mathbb{N} \). Note that
\[
|\nu_{\zeta_k}(E_k) - \nu_{\zeta}(E_k)| \leq \int_{E_k} \left| \frac{1}{(1 - \overline{\zeta_k \varphi(z))^{n+1}}} - \frac{1}{(1 - \overline{\zeta \varphi(z)}^{n+1}} \right|^2 d\lambda_{g,\omega}(z).
\]
Thus
\[
\nu_{\zeta_k}(E_k) \leq \int_{E_k} \left| \frac{1}{(1 - \overline{\zeta_k \varphi(z))^{n+1}}} - \frac{1}{(1 - \overline{\zeta \varphi(z))^{n+1}} \right|^2 d\lambda_{g,\omega}(z).
\]
Since \( I_{g;\varphi}^{(n)} : K \to \mathcal{D}_\omega \) is bounded, so equation (4) holds. Therefore, \( \nu_{\zeta}(E_k) \to 0 \) as \( k \to \infty \). Moreover, as in first part first term in (9) is dominated by
\[
\int_{E_k} \left| \frac{1}{(1 - \overline{\zeta_k \varphi(z)}^{n+1}} - \frac{1}{(1 - \overline{\zeta \varphi(z))^{n+1}} \right|^2 d\lambda_{g,\omega}(z).
\]
Therefore, \( \nu_{\zeta_k}(E_k) \to 0 \) as \( k \to \infty \). This contradiction shows that (2) \( \Rightarrow \) (3).

(3) \( \Rightarrow \) (1). Let \( \{f_m\} \) be a sequence in \( K \) such that \( \sup_m ||f_m||_K \leq 1 \) and \( f_m \to 0 \) uniformly on compact subsets of \( D \). We have to show that \( ||I_{g;\varphi}^{(n)} f_m||_{\mathcal{D}_\omega} \to 0 \) as \( m \to \infty \). For each \( m \), we can find \( \mu_m \in \mathfrak{M} \) with \( ||\mu_m|| = ||f_m||_K \) such that
\[
f_m(z) = \int_{\partial D} \frac{1}{1 - \zeta z} d\mu_m(\zeta).
\]
Composing with \( \varphi \) and applying Jensen’s inequality, we have
\[
|f_m^{(n)}(\varphi(w))|^2 \leq ||\mu_m||^2 \int_{\partial D} \frac{1}{|1 - \overline{\zeta \varphi(w)}|^{2(n+1)}} \frac{d|\mu_m||_{\partial D}}{||\mu_m||}.
\]
Integrating with respect to \( d\lambda_{g,\omega}(w) \) and then applying Fubini’s theorem, we have
\[
\int_D |f_m^{(n)}(\varphi(w))|^2 d\lambda_{g,\omega}(w) \leq ||\mu_m|| \int_{\partial D} \int_D \frac{1}{|1 - \overline{\zeta \varphi(w)}|^{2(n+1)}} d\lambda_{g,\omega}(w) d|\mu_m||_{\partial D}.
\]
Let $\epsilon > 0$ be given. Now choose a compact set $F \subset \mathbb{D}$ such that $A(\mathbb{D} \setminus F) < \delta$. Thus
\[
\int_{\mathbb{D}\setminus F} |f^{(n)}_m(\varphi(w))|^2 d\lambda_{g,\omega}(w) \\
\leq ||\mu_m|| \int_{\partial\mathbb{D} \setminus F} \frac{1}{|1 - \bar{z}\varphi(w)|^{2(n+1)}} d\lambda_{g,\omega}(w)d|\mu_m|(|z|) \\
\leq \epsilon ||\mu_m|| \int_{\partial\mathbb{D}} d|\mu_m|(|z|) = \epsilon ||f_m||^2_K < \epsilon.
\]
(10)

On $F$, $|f^{(n)}_m(\varphi(w))|^2 < \epsilon$ as $m \geq m_0$. Moreover, by taking $f(z) = z^n/n! \in K$, the boundedness of $I^{(n)}_{g,\omega} : K \to \mathcal{D}$ gives
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} d\lambda_{g,\omega}(w) \leq C.
\]
Thus
\[
\int_{\mathbb{D}\setminus F} |f^{(n)}_m(\varphi(w))|^2 d\lambda_{g,\omega}(w) < \epsilon C
\]
(11)
as $m \geq m_0$. Therefore, by (10) and (11), $||I^{(n)}_{g,\omega}f_m||_{\mathcal{D}_0} \to 0$ as $m \to \infty$. (1) $\Rightarrow$ (4).

Since $I^{(n)}_{g,\omega} : K \to \mathcal{D}_0$ is bounded, for $f(z) = z^n/n! \in K$, we have that $g \in \mathcal{A}_2$. Let $f_n(z) = z^m$, $m \in \mathbb{N}$. It is a norm bounded sequence in $K$ converging to zero uniformly on compacts of $\mathbb{D}$. Hence by Lemma 1, it follows that $||I^{(n)}_{g,\omega}f_m||_{\mathcal{D}_0} \to 0$ as $m \to \infty$. Thus for every $\epsilon > 0$, there is an $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, we have
\[
\left(\prod_{j=0}^{n-1} (m-j)\right)^2 \int_{\mathbb{D}} |\varphi(z)|^{2(n-m)}|g(z)|^2\omega(z)dA(z) < \epsilon.
\]
(12)

From (12), we have that for each $r \in (0,1)$
\[
r^{2(m-n)} \left(\prod_{j=0}^{n-1} (m-j)\right)^2 \int_{|\varphi(z)|>r} |g(z)|^2\omega(z)dA(z) < \epsilon.
\]
(13)

Hence for $r \in \left[\prod_{j=0}^{n-1} (m-j)\right]^{-\frac{1}{n-2-m}}$, we have
\[
\int_{|\varphi(z)|>r} |g(z)|^2\omega(z)dA(z) < \epsilon.
\]
(14)

Let $f \in B_K$ and $f_t(z) = f(tz)$, $0 < t < 1$. Then $\sup_{0<t<1} \|f_t\|_\mathcal{K} \leq ||f||_\mathcal{K}$. Let $t \in (0,1)$ and $f_t \to f$ uniformly on compacts of $\mathbb{D}$ as $t \to 1$. The compactness of $I^{(n)}_{g,\omega} : K \to \mathcal{D}_0$ implies that
\[
\lim_{t \to 1} ||I^{(n)}_{g,\omega}f_t - I^{(n)}_{g,\omega}f||_{\mathcal{D}_0} = 0.
\]
Hence for every $\epsilon > 0$, there is a $t \in (0,1)$ such that
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_t^{(n)}(\varphi(z)) - f^{(n)}(\varphi(z))|^2|g(z)|^2\omega(z)dA(z) < \epsilon.
\]
(15)
Inequalities (14) and (15), give
\[
\sup_{\alpha \in \mathcal{B}} \int_{|\varphi(z)| > r} |f^{(n)}(\varphi(z))|^2 |g(z)|^2 \omega(z) dA(z)
\leq 2 \int_{\mathbb{D}} |f_z^{(n)}(\varphi(z)) - f_z^{(n)}(\varphi(z))|^2 |g(z)|^2 \omega(z) dA(z) \\
+ 2 \int_{|\varphi(z)| > r} |f_z^{(n)}(\varphi(z))|^2 |g(z)|^2 \omega(z) dA(z)
\leq 2 \varepsilon (1 + \|f_z^{(n)}\|_\infty^2).
\]
Hence for every \( f \in \mathcal{B}_\mathcal{K} \), there is a \( \delta_0 \in (0, 1) \), \( \delta_0 = \delta_0(f, \varepsilon) \), such that for \( r \in (\delta_0, 1) \)
\[
\int_{|\varphi(z)| > r} |f^{(n)}(\varphi(z))|^2 |g(z)|^2 \omega(z) dA(z) < \varepsilon.
\]
From the compactness of \( f_z^{(n)}: \mathcal{K} \to \mathcal{D} \), we have that for every \( \varepsilon > 0 \) there is a finite collection of functions \( f_1, f_2, \ldots, f_k \in \mathcal{B}_\mathcal{K} \) such that for each \( f \in \mathcal{B}_\mathcal{K} \), there is a \( j \in \{1, 2, \ldots, k\} \) such that
\[
\sup_{\alpha \in \mathcal{B}} \int_{\mathbb{D}} |f(j)(\varphi(z)) - f^{(n)}(\varphi(z))|^2 |g(z)|^2 \omega(z) dA(z) < \varepsilon.
\]
On the other hand, from (16) it follows that if \( \delta := \max_{1 \leq j \leq k} \delta_j(f_j, \varepsilon) \), then for \( r \in (\delta, 1) \) and all \( j \in \{1, 2, \ldots, k\} \) we have
\[
\int_{|\varphi(z)| > r} |f(j)(\varphi(z))|^2 |g(z)|^2 \omega(z) dA(z) < \varepsilon.
\]
From (17) and (18) we have that for \( r \in (\delta, 1) \) and every \( f \in \mathcal{B}_\mathcal{K} \)
\[
\int_{|\varphi(z)| > r} |f^{(n)}(\varphi(z))|^2 |g(z)|^2 \omega(z) dA(z) < 4 \varepsilon.
\]
Applying (19) to the functions \( f_\zeta(z) = 1/(1 - \zeta z) \), \( \zeta \in \partial \mathbb{D} \), we obtain
\[
\sup_{\zeta \in \partial \mathbb{D}} \int_{|\varphi(z)| > r} \frac{|g(z)|^2}{|1 - \zeta \varphi(z)|^{2(n+1)}} \omega(z) dA(z) < 4 \varepsilon/(n!)^2,
\]
from which (8) follows.
(4) \( \Rightarrow \) (1). Assume that \( (f_m)_{m \in \mathbb{N}} \) is a bounded sequence in \( \mathcal{K} \), say by \( L \), converging to 0 uniformly on compacts of \( \mathbb{D} \) as \( m \to \infty \). Then by the Weierstrass theorem, \( f_m^{(k)} \) also converges to 0 uniformly on compacts of \( \mathbb{D} \), for each \( k \in \mathbb{N} \). We show that
\[
\|L^{(n)}_{\varphi, \delta} f_m\|_{\mathcal{D}_\omega} \to 0 \quad \text{as} \quad m \to \infty,
\]
and then apply Lemma 1.
For each \( m \in \mathbb{N} \), we can find a \( \mu_m \in \mathcal{M} \) with \( \|\mu_m\| = \|f_m\|_\mathcal{K} \) such that
\[
f_m(z) = \int_{\partial \mathbb{D}} \frac{d\mu_m(\zeta)}{1 - \zeta z}.
\]
Differentiating (20) \( n \) times, composing such obtained equation by \( \varphi \), applying Jensen's inequality, as well as the boundedness of sequence \( (f_m)_{m \in \mathbb{N}} \), we obtain
\[
|f_m^{(n)}(\varphi(w))|^2 \leq L(n!)^2 \int_{\partial \mathbb{D}} \frac{d|\mu_m(\zeta)|}{|1 - \zeta \varphi(w)|^{2(n+1)}}.
\]
By the second condition in (4), we have that for every \( \varepsilon > 0 \), there is an \( r_1 \in (0, 1) \) such that for \( r \in (r_1, 1) \), we have

\[
\sup_{\zeta \in \partial D} \int_{|\varphi(z)| > r} \frac{|g(z)|^2}{|1 - \zeta \varphi(z)|^{2n+2}} \omega(z)dA(z) < \varepsilon.
\] (22)

Now

\[
\|I^{(n)}_{g, \varphi} f_m\|_{D_{\omega}}^2 = \left( \int_{|\varphi(z)| \leq r} |f^{(n)}_m(\varphi(z))|^2 |g(z)|^2 \omega(z)dA(z) + \int_{|\varphi(z)| > r} |f^{(n)}_m(\varphi(z))|^2 |g(z)|^2 \omega(z)dA(z) \right) \int_{|\varphi(z)| \leq r} |g(z)|^2 \omega(z)dA(z)
\]

Using (21), (22), Fubini’s theorem and the fact that \( \sup_{|w| \leq r} |f^{(n)}_m(w)|^2 < \varepsilon \), for sufficiently large \( m \), say \( m \geq m_0 \), we have that for \( m \geq m_0 \)

\[
\|I^{(n)}_{g, \varphi} f_m\|_{D_{\omega}}^2 \leq C \sup_{|\varphi(z)| \leq r} |f^{(n)}_m(\varphi(z))|^2 \int_{|\varphi(z)| \leq r} |g(z)|^2 \omega(z)dA(z) + C \int_{\partial D} \int_{|\varphi(z)| > r} |g(z)|^2 \omega(z)dA(z)d|\mu_m|(|\zeta|)
\]

\[
\leq C \left( M + \int_{\partial D} d|\mu_m|(|\zeta|) \right) \varepsilon \leq C \varepsilon.
\]

From this and the fact that \( \varepsilon \) is arbitrary, the result follows.

References

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