ON MARKOV AND KOLMOGOROV MATRICES AND THEIR
RELATIONSHIP WITH ANALYTIC OPERATORS

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Abstract. In this article, the relationship between sub–Markov and sub–Kolmogorov matrices is investigated. In addition some results about analytic operators and analytic semigroups are exhibited. In particular it is shown that analyticity properties have ramifications for limiting properties of bounded analytic semigroups. These result has some interesting consequences for matrices with Markov and Kolmogorov properties.

1. Introduction

First we introduce some notions and definitions which we are going to use.

Definition 1.1. The matrix \( K = (k_{ij})_{i,j=1}^{n} \) is called a Kolmogorov matrix, if the following two conditions are fulfilled:

\[
k_{ij} \geq 0, \quad i \neq j \quad \text{and} \quad i, j = 1, \ldots, n(1)
\]

and

\[
\sum_{i=1}^{n} k_{ij} = 0, \quad j = 1, \ldots, n. \tag{2}
\]

The matrix \( K \) is called a sub-Kolmogorov matrix if it satisfies (1) and if (2) is replaced with

\[
\sum_{i=1}^{n} k_{ij} \leq 0, \quad j = 1, \ldots, n. \tag{3}
\]

For matrices we introduce the notion of the (positive) maximum principle, as well as the sub-Markov property. In Van Casteren [6] the reader may find some related results for linear operators which satisfy the maximum principle.

Definition 1.2. The matrix \( K = (k_{ij})_{i,j=1}^{n} \) is said to satisfy the positive maximum principle if it possesses the following property: for every row vector \( f = (f_1, \ldots, f_n) \) with \( \max_{1 \leq i \leq n} f_i > 0 \) the inequality

\[
(fK)_{i_0} \leq 0
\]

is satisfied whenever the index \( i_0 \) is chosen in such a way that \( f_{i_0} = \max_{1 \leq i \leq n} f_i \).

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The matrix $K = (k_{ij})_{i,j=1}^n$ is said to satisfy the maximum principle if it possesses the following property: for every row vector $f = (f_1, \ldots, f_n)$ the inequality
\[(fK)_{i_0} \leq 0\] is satisfied whenever the index $i_0$ is chosen in such a way that $f_{i_0} = \max_{1 \leq i \leq n} f_i$.

**Definition 1.3.** A matrix $M = (m_{ij})_{i,j=1}^n$ is called a sub–Markov matrix, if the following two conditions are fulfilled:
\[m_{ij} \geq 0, \quad i,j = 1, \ldots, n\] and
\[\sum_{i=1}^n m_{ij} \leq 1, \quad j = 1, \ldots, n.\]

**Definition 1.4.** Let $K$ be a square matrix.

(a) An eigenvalue $\mu$ of $K$ is called dominant if $\lim_{t \to \infty} \left\| e^{tK} (I - P) \right\| = 0$. Here $P$ is the Dunford projection on the generalized eigenspace corresponding to $\mu$; i.e. $P = \frac{1}{2\pi i} \int_\gamma (\lambda I - K)^{-1} d\lambda$, where $\gamma$ is a (small) positively oriented circle around $\mu$. The disc centered at $\mu$ and with circumference $\gamma$ does not contain other eigenvalues.

(b) An eigenvalue $\mu$ of $K$ is called critical if it is dominant and the zero space of $K - \mu I$ is one–dimensional.

**Remark 1.5.** Here the generalized eigenspace is the set of all vectors $x \in \mathbb{C}^n$ for which $(\mu I - K)^\ell x = 0$ for some $1 \leq \ell \leq n$. Notice that $\sum_{k=0}^n \alpha_k (\mu I - K)^\ell = 0$ for appropriately chosen constants $\alpha_k$, $0 \leq k \leq n$.

In the sequel we consider the following simplex $W$ in space $\mathbb{R}^n$:
\[W = \left\{ x \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_n \geq 0, \sum_{i=1}^n x_i = 1 \right\},\] and the following $(n - 1)$–dimensional subspace $L$ of $\mathbb{R}^n$:
\[L = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0 \right\}.\]

2. The Relationship Between Markov and Kolmogorov Matrices

We consider a time continuous system with a constant matrix $K$:
\[\dot{x}(t) = Kx(t), \quad -\infty < t < \infty.\] Let $K$ be a Kolmogorov matrix. Then 0 is a dominant eigenvalue of $K$; let $P : \mathbb{R}^n \to \mathbb{R}^n$ be the Dunford projection on the eigen–space corresponding to the eigenvalue 0. If the eigenvalue 0 has multiplicity 1, then $P$ projects on the one–dimensional subspace $1_{\mathbb{R}}$. Since 0 is a dominant eigenvalue of $K$, a key spectral estimate of the following form is valid:
\[\left\| fe^{tK} (I - P) \right\| \leq Ce^{-\delta t} \left\| f \right\|, \quad f \in \mathbb{R}^n,\] where $\delta$ is strictly positive, $\left\| \cdot \right\|$ is some (chosen) norm on $\mathbb{R}^n$, and where $C$ is some finite constant which depends on the chosen norm.
In the next proposition we want to exhibit the relationship between matrices which satisfy the maximum principle, which possess the sub–Markov property, and which preserve positivity.

**Proposition 2.1.** Let $K$ be an $n \times n$–matrix.

(a) Suppose that $K$ satisfies the maximum principle. Then the following inequalities hold true:

$$\max_{1 \leq i \leq n} (f_i - s (fK)_i) \geq \max_{1 \leq i \leq n} f_i;$$  \hspace{1cm} (12)

$$\min_{1 \leq i \leq n} (f_i - s (fK)_i) \leq \min_{1 \leq i \leq n} f_i;$$  \hspace{1cm} (13)

where the row vector $(f_1, \ldots, f_n)$ is arbitrary. Consequently, if $s > 0$ the null space of the operator $f \mapsto f (I - sK)$ is trivial, and since we are in the finite-dimensional situation $I - sK$ is invertible. Moreover, if $s > 0$ then

$$1 (I - sK)^{-1} \leq 1;$$  \hspace{1cm} (14)

$$(I - sK)^{-1} \geq 0. \hspace{1cm} (15)$$

(b) If $K$ satisfies the positive maximum principle, then for $s > 0$ the inverses $(I - sK)^{-1}$ exist. Moreover, (14) and (15) are satisfied as well.

(c) The following assertions are equivalent:

(i) For every vector $f$ with $0 \leq f \leq 1$ the inequalities $0 \leq f (I - sK)^{-1} \leq 1$ are valid for all $s > 0$ for which the inverse $(I - sK)^{-1}$ exists.

(ii) For every vector $f$ with $0 \leq f \leq 1$ the inequalities $0 \leq fe^{sK} \leq 1$ are valid for all $s \geq 0$.

(iii) $K$ satisfies the positive maximum principle.

(iv) $K$ satisfies both the inequalities (14) and (15).

**Remark 2.2.** As indicated in (c) the inequalities (14) and (15) can be combined. If $0 \leq s \|K\| < 1$, then $I - sK$ is invertible. The following equalities represent the matrix $e^{sK}$, $s > 0$, as certain limits in terms of $(I - tK)^{-1}$, where $t > 0$ is small:

$$f e^{sK} = \lim_{n \to \infty} f \left( I - \frac{s}{n} K \right)^{-n} = \lim_{n \to \infty} f e^{-ns} e^{ns (I - \frac{1}{n} K)^{-1}}$$

$$= \lim_{n \to \infty} f e^{sK (I - \frac{1}{n} K)^{-1}} = \lim_{n \to \infty} e^{-ns} \sum_{k=0}^{\infty} \frac{(ns)^k}{k!} f \left( I - \frac{1}{n} K \right)^{-k}.$$

The implication (ii) $\Rightarrow$ (i) follows from the useful representation:

$$f (I - sK)^{-1} = \int_0^\infty e^{-t} \left( f e^{tK} \right) dt.$$

**Proof.** (a) Choose $1 \leq i_0 \leq n$ in such a way that $f_{i_0} = \max \{ f_i : 1 \leq i \leq n \}$. Then $(fK)_{i_0} \leq 0$, and thus

$$\max_{1 \leq i \leq n} (f_i - s(fK)_i) \geq f_{i_0} - s(fK)_{i_0} \geq f_{i_0} = \max \{ f_i : 1 \leq i \leq n \}.$$
This proves inequality (12). Upon replacing \( f \) with \(-f\) inequality (13) follows from (12). Upon writing \( g = f(I - sK) \), and thus \( f = g(I - sK)^{-1} \) (12) implies \( \max_{1 \leq i \leq n} (g(I - sK)^{-1}) \leq \max_{1 \leq i \leq n} g_i \). With \( g = 1 \), (14) follows. In the same manner (15) is a consequence of (13).

(b) Let \( k_{ij} \) be the entries of the matrix \( K \). Our first step is to show that the positive maximum principle implies that \( K \) is sub–Kolmogorov; i.e. \( k_{ii0} \geq 0 \) for \( i \neq i_0 \), and \( \sum_{i=1}^{n} k_{i0i} \leq 0 \), for \( 1 \leq i_0 \leq n \). But this is clear since \( 0 < 1 = \max_{1 \leq i \leq n} 1 \) is 1, and hence \( \sum_{i=1}^{n} k_{i0i} = (1K)_{i0} \leq 0 \). Next we fix \( \varepsilon > 0 \) arbitrary, and we suppose that \( i \neq i_0 \), and \( i_0 = 1 \leq i_0 \leq n \). Define the vector \( f_\varepsilon \) as follows: \( f_{\varepsilon, j} = 1 \), for \( j = i \), \( f_{\varepsilon, j} = -\varepsilon \), for \( j = i_0 \), and \( f_{\varepsilon, j} = 0 \), elsewhere. Then \( f_{\varepsilon, j} \) is minimal and strictly negative at \( j = i_0 \). From the positive maximum principle it follows that \( k_{i0i} - \varepsilon k_{i0i} = (f_\varepsilon K)_{i0} \geq 0 \). Hence \( k_{ii0} \geq \varepsilon k_{i0i} \) for every \( \varepsilon > 0 \). Consequently, \( k_{ii0} \geq 0 \), for \( i \neq i_0 \). Suppose \( f(I - sK) \geq 0 \). Assume \( f_{i0} = \min_{1 \leq i \leq n} f_i < 0 \). Then, by the positive maximum principle \((fK)_{i0} \geq 0\), and hence

\[
0 \leq (f - sfK)_{i0} = f_{i0} - s \sum_{i=1}^{n} f_i k_{i0i} = f_{i0} + s \sum_{i=1}^{n} (f_{i0} - f_i) k_{i0i} - sf_{i0} \sum_{i=1}^{n} k_{i0i}
\]

(the matrix \( K \) is sub–Kolmogorov and \( f_{i0} < 0 \))

\[
\leq f_{i0} < 0,
\]

which is a contradiction. Hence, \( f(I - sK) \geq 0 \) implies \( f \geq 0 \). In particular, if \( f(I - sK) = 0 \), then \( f = 0 \). As a result, we see that the null space of \( I - sK \) is zero, whenever \( s > 0 \), and so the operator \( f \mapsto f(I - sK) \) is invertible. Consequently, \((I - sK)^{-1} \geq 0 \). Next let the row vector \( f \) be such that \( f(I - sK) = 1 \). Then, by what we proved above \( f \geq 0 \). Moreover, assume to arrive at a contradiction that, for some \( 1 \leq i_0 \leq n \), \( f_{i0} - 1 = \max_{1 \leq i \leq n} (f_i - 1) > 0 \). Then

\[
0 \geq s((f - 1)K)_{i0} = s(fK)_{i0} - s(1K)_{i0} = f_{i0} - 1 - s(1K)_{i0} \geq f_{i0} - 1 > 0,
\]

which is a contradiction. Hence, the equality \( f(I - sK) = 1 \) implies \( 0 \leq f \leq 1 \). Consequently, \( 0 \leq 1(I - sK)^{-1} \leq 1 \). All this proves assertion (b).

(c) \( (i) \Rightarrow (ii) \). For the implication \( (i) \Rightarrow (ii) \) we write

\[
f e^{sK} = \lim_{n \to \infty} f \left( I - \frac{s}{n} K \right)^{-n}.
\]

The implication \( (ii) \Rightarrow (i) \) follows from the representation:

\[
f(I - sK)^{-1} = \int_{0}^{\infty} e^{-t} (f e^{stK}) dt.
\]

The implication \( (iii) \Rightarrow (iv) \) is proved in (b). The implication \( (iv) \Rightarrow (i) \) is trivial.
(ii) \implies (iii). Let the vector \( f \) be such that \( \max_{1 \leq j \leq n} f_j = f_{j_0} > 0 \). We have to prove \( (fK)_{j_0} \leq 0 \). First we notice that \( f \leq (\max_{1 \leq j \leq n} f_j) \mathbf{1} \), and hence by (ii) \( f e^K \leq (\max_{1 \leq j \leq n} f_j) \mathbf{1} e^K \), \( s > 0 \). Consequently we have

\[
(fK)_{j_0} = \lim_{s \downarrow 0} \frac{(fe^K)_{j_0} - f_{j_0}}{s} \leq \lim_{s \downarrow 0} \max_{1 \leq j \leq n} f_j \left( \frac{1 e^K}{s} \right) - f_{j_0} = \lim_{s \downarrow 0} \frac{f_{j_0} \left( \mathbf{1} e^K \right) - 1}{s} \leq \lim_{s \downarrow 0} \frac{f_{j_0} (1 - 1)}{s} = 0,
\]

where we used \( f_{j_0} \geq 0 \) and the inequality \( \mathbf{1} e^K \leq 1 \), which is a consequence of (ii). This proves the implication (ii) \implies (iii). All this proves assertion (c), and thus Proposition 2.1 follows.

The following theorem shows that matrix operators which satisfy the positive maximum principle are intimately related to sub–Markov chains.

**Theorem 2.3.** Let \( K \) be an \( n \times n \)–matrix. The following assertions are equivalent:

(i) the matrix \( K \) satisfies the positive maximum principle;

(ii) for every \( t > 0 \) the matrix \( e^{tK} \) is sub–Markovian;

(iii) \( K \) is a sub–Kolmogorov matrix.

**Proof.** Let \( t > 0 \) be arbitrary. Representing \( e^{tK} \) as

\[
e^{tK} = \lim_{n \to \infty} \left( I - \frac{t}{n} K \right)^n
\]

(14) implies \( \mathbf{1} e^{tK} \leq 1 \), and (15) implies \( e^{tK} \geq 0 \). Altogether this shows the implication (i) \implies (ii). Conversely, let \( f = (f_1, \ldots, f_n) \) be an arbitrary row vector in \( \mathbb{R}^n \). Choose \( 1 \leq l_0 \leq n \) in such a way that \( m := \max_{1 \leq i \leq n} f_i = f_{l_0} \). Then \( f \leq m \mathbf{1} \), and so

\[
(fK)_{l_0} = \lim_{t \downarrow 0} \frac{1}{t} \left( (fe^{tK})_{l_0} - f_{l_0} \right) \leq \lim_{t \downarrow 0} \frac{1}{t} \left( m (1 e^{tK})_{l_0} - m \right)
\]

(the matrix \( e^{tK} \) is sub-Markov)

\[
\leq \lim_{t \downarrow 0} \frac{m}{t} \left( \mathbf{1} - \mathbf{1} \right)_{l_0} = \lim_{t \downarrow 0} \frac{m}{t} 0 = 0.
\]

This shows the implication (ii) \implies (i). Next we prove (ii) \implies (iii). By (ii) we know that, for all \( t > 0 \), \( \mathbf{1} e^{tK} \leq 1 \). Since

\[
1 K = \lim_{t \downarrow 0} \frac{1}{t} \left( \mathbf{1} e^{tK} - 1 \right) \leq 0,
\]

it follows that \( 1 K \leq 0 \). This proves part of the sub–Kolmogorov property. We still have to show that \( k_{ij} \geq 0 \), for \( j \neq i \). This assertion follows from the equality \( (\delta_{ij}) \) is the Kronecker delta symbol):

\[
k_{ij} = \lim_{t \downarrow 0} \frac{1}{t} \left( (e^{tK})_{ij} - \delta_{ij} \right),
\]

which for \( j \neq i \) shows that \( k_{ij} \geq 0 \), since, by assumption, the matrices \( e^{tK}, t > 0 \), have the sub-Markov property, and thus have non-negative entries.
(iii) $\Rightarrow$ (ii). Again we use the following representation for $e^{tK}$:

$$e^{tK} = \lim_{n \to \infty} \left( I - \frac{t}{n} K \right)^{-n}. \quad (20)$$

From the sub–Kolmogorov property it follows that $(I - sK)^{-1}$, $s > 0$, is positivity preserving. For, let the row vector $f = (f_1, \ldots, f_n)$ be such that $f(I - sK) \geq 0$, and assume that $f_{i_0} = \min_{1 \leq i \leq n} f_i < 0$. Since $-1K \geq 0$, we have for $s > 0$

$$0 \leq f_{i_0} - s(fK)_{i_0} = f_{i_0} - s \sum_{i=1}^{n} f_i k_{i,i_0}$$

$$= f_{i_0} + s \sum_{i=1}^{n} (f_{i_0} - f_i) k_{i,i_0} - sf_{i_0} \sum_{i=1}^{n} k_{i,i_0}$$

$$\leq f_{i_0} - sf_{i_0} (1K)_{i_0} \leq f_{i_0} < 0, \quad (21)$$

which is a contradiction. Consequently, $(I - sK)^{-1} \geq 0$. Moreover, since $-1K \geq 0$, we also obtain:

$$1 - 1 (I - sK)^{-1} = s (-1K) (I - sK)^{-1} \geq 0. \quad (22)$$

Finally, we obtain from (22) together with (20) and the fact that, for $s > 0$ small enough, the operator $(I - sK)^{-1}$ is positivity preserving, the operators $e^{sK}$, $s \geq 0$, have the sub–Markov property. □

The following proposition establishes the relationship between matrices with the Kolmogorov property and matrices with the Markov property.

**Proposition 2.4.** Let $K = (k_{ij})_{i,j=1}^{n}$ be an $n \times n$–matrix. The following assertions are equivalent:

(i) The matrix $K$ has the Kolmogorov property;

(ii) For every $t > 0$ the matrix $e^{tK}$ possesses the Markov property.

**Proof.** For completeness we insert a proof.

(ii) $\Rightarrow$ (i). By hypothesis we know that, for all $t > 0$, $1 e^{tK} = 1$. Since

$$1K = \lim_{t \downarrow 0} \frac{1}{t} (1 e^{tK} - 1) = 0,$$

it follows that $1K = 0$. This proves part of the Kolmogorov property. We still have to show that $k_{ij} \geq 0$, for $j \neq i$. This follows from Theorem 2.3 implication (ii) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (ii). Here we use the following representation for $e^{tK}$:

$$e^{tK} = \lim_{n \to \infty} \left( I - \frac{t}{n} K \right)^{-n}. \quad (23)$$

Since $K$ has the Kolmogorov property, we see $1(I - sK) = 1$, and hence

$$1 = \lim_{n \to \infty} 1 \left( I - \frac{t}{n} K \right)^{-n} = 1 e^{tK}. \quad (24)$$

The positivity of the entries of the matrix $e^{tK}$ again follows from Theorem 2.3 implication (iii) $\Rightarrow$ (ii). □
3. Analytic Operators and Semigroups

The following (important) Theorems 3.1 and 3.6 are inspired by ideas in Nagy [5].

Theorem 3.1. Let $M$ be a bounded linear operator in a Banach space $X$. By definition the sub-space $X_0$ of $X$ is the $\| \cdot \|$ closure of the vector sum of the range and zero-space of $I - M$: $X_0 = \overline{R(I - M) + N(I - M)}$. Suppose that the spectrum of $M$ is contained in the open unit disc union $\{ 1 \}$. The following assertions are equivalent:

(i) $\sup_{|\lambda| < 1} \| (1 - \lambda)(I - \lambda M)^{-1} x \| < \infty$ for every $x \in X_0$;

(ii) $\sup_{n \in \mathbb{N}} \| M^n x \| < \infty$ and $\sup_{n \in \mathbb{N}} (n + 1) \| M^n (I - M) x \| < \infty$ for every $x \in X_0$;

(iii) $\sup_{t > 0} \| e^{t(M-I)} x \| < \infty$ and $\sup_{t > 0} \| t(M-I) e^{t(M-I)} x \| < \infty$ for every $x \in X_0$;

(iv) There exists $\frac{1}{2} \pi < \alpha < \pi$ such that

$$\sup \left\{ \| \lambda (\lambda I - (I - M))^{-1} x \| : -\alpha < \arg(\lambda) < \alpha \right\} < \infty,$$

for all $x \in X_0$;

(v) There exists $\frac{1}{2} \pi < \alpha < \pi$ such that

$$\sup \left\{ \| (I - M)(\lambda I - (I - M))^{-1} x \| : -\alpha < \arg(\lambda) < \alpha \right\} < \infty,$$

for all $x \in X_0$;

(vi) For every $x \in X_0$ the following limits exist

$$P_x := \lim_{n \to \infty} M^n x \quad \text{and} \quad (I - P) x = \lim_{\theta \to 0 \atop 0 < r < 1} (I - M) (I - re^{i\theta} M)^{-1} x.$$ (25)

(vii) For every $x \in X_0$ the following limit exists

$$(I - P) x := \lim_{\theta \to 0 \atop 0 < r < 1} (I - M) (I - re^{i\theta} M)^{-1} x.$$ (26)

Moreover, if $M$ satisfies one of the conditions (i) through (vii), then

$$X_0 = \overline{R(I - M)} + N(I - M).$$

Remark 3.2. The Banach–Steinhaus theorem implies that in (i) through (v) in Theorem 3.1 the vector norms may be replaced with the operator norm restricted to $X_0$; i.e. the operator $M$ must be restricted to $X_0$.

Conditions (a) and (b) of the following corollary are satisfied if the space $X$ is reflexive.

Corollary 3.3. Let $M$ be a bounded linear operator in a Banach space $(X, \| \cdot \|)$. As in Theorem 3.1 let $X_0$ be the closure in $X$ of the sub-space $R(I - M) + N(I - M)$. Suppose that, for $0 < \lambda < 1$, the inverse operators $(I - \lambda M)^{-1}$ exist and are bounded, and that $\sup_{0 < \lambda < 1} (1 - \lambda) \| (I - \lambda M)^{-1} \| < \infty$. If one of the following conditions:
(a) the zero space of the operator \((I - M)^{**}\), which is a sub–space of the bidual space \(X^{**}\) is in fact a subspace of \(X\);

(b) the \(\sigma (X^{*}, X)\)-closure of \(R ((I - M)^{*})\) coincides with its \(\|\cdot\|\)-closure;

(c) the range of \(I - M\) is closed in \(X\);

is satisfied, then the space \(X_0\) coincides with \(X\), and hence all assertions in Theorem 3.1 are equivalent with \(X\) replacing \(X_0\).

**Remark 3.4.** If \(\sup_{n \in \mathbb{N}} \|M^n\| < \infty\), then \(\sup_{0 < \lambda < 1} (1 - \lambda) \left\| (I - \lambda M)^{-1} \right\| < \infty\).

As can be seen from the proof the assertions (i) through (v) in Theorem 3.1 are also equivalent if \(X\) replaces \(X_0\).

**Definition 3.5.** An operator \(M\) which satisfies the equivalent conditions (i) through (v) with \(X\) replacing \(X_0\) of Theorem 3.1 is called an analytic operator in the subspace \(X\). If it satisfies the equivalent conditions (i) through (vii) of Theorem 3.1, then it is called an analytic operator in the subspace \(X_0\).

**Proof of Corollary 3.3.** If the range of \(I - M\) is closed, then by the closed range theorem, the range of \(I - M\) is weak*–closed and hence (c) implies (b). We will prove that (a) as well (b) implies \(X_0 = X\). First we assume (a) to be satisfied. Pick \(x \in X\), and consider

\[
x = (I - M) (I - \lambda M)^{-1} x + (1 - \lambda) M (I - \lambda M)^{-1} x = x - x_\lambda + x_\lambda,
\]

where \(x_\lambda = (1 - \lambda) M (I - \lambda M)^{-1} x\). Then \(\sup_{0 < \lambda < 1} \|x_\lambda\| < \infty\), and consequently the family \(x_\lambda\), \(0 < \lambda < 1\), has a point of adherence \(x^{**}\) in \(X^{**}\); i.e. \(x^{**}\) belongs to the \(\sigma (X^{**}, X^{*})\)-closure of the subset \(\{x_\lambda : 1 - \eta < \lambda < 1\}\), and this for every \(0 < \eta < 1\). Fix \(x^* \in X^{*}\). Then

\[
\left\| \langle (1 - \lambda) M (I - \lambda M)^{-1} x, (I - M)^{*} x^{*} \rangle \right\|
\leq (1 - \lambda) \left\| (I - M) (I - \lambda M)^{-1} x \right\| \left\| M^{*} x^{*} \right\|.
\]

Since \(\sup_{0 < \lambda < 1} (1 - \lambda) \left\| (I - \lambda M)^{-1} \right\| < \infty\), the identity

\[
(I - M) (I - \lambda M)^{-1} = \frac{1}{\lambda} (I - (1 - \lambda) (I - \lambda M)^{-1})
\]

yields that \(\sup_{0 < \lambda < 1} \left\| (I - M) (I - \lambda M)^{-1} \right\| < \infty\). Consequently (28) implies

\[
\langle x^{**}, (I - M)^{*} x^{*} \rangle = \lim_{\lambda \downarrow 1} \langle x_\lambda, (I - M)^{*} x^{*} \rangle
\]

\[
= \lim_{\lambda \downarrow 1} (1 - \lambda) \langle (I - M) (I - \lambda M)^{-1} x, M^{*} x^{*} \rangle = 0.
\]

Hence \(x^{**}\) annihilates \(R ((I - M)^{*})\) and so it belongs to the zero space of the operator \((I - M)^{**}\). By assumption this zero space is a subspace of \(X\). We infer that the vector \(x\) can be written as \(x = x - x_1 + x_1\), where \(x_1\) is a member of \(N(I - M)\), and where \(x - x_1\) belongs to the weak closure of the range of \(I - M\). However this weak closure is the same as the norm–closure of \(R(I - M)\). Altogether this shows \(X = X_0 = \|\cdot\|\)-closure of \(R(I - M) + N(I - M)\).
Next we assume that (b) is satisfied. Let \( x_0^* \) be an element of \( X^* \) which annihilates \( X_0 \); i.e. which has the property that \( \langle x, x_0^* \rangle = 0 \) for all \( x \in X_0 \). Then \( x_0^* \) annihilates \( R(I-M) \), and hence it belongs to zero–space of \( (I-M)^* \). Since \( x_0^* \) also annihilates the zero–space of \( I-M \), it belongs to the weak*–closure of \( \overline{R((I-M)^*)} \). By assumption (b), we see that \( x_0^* \) is a member of its norm–closure; i.e. \( x_0^* \) belongs to the intersection \( N((I-M)^*) \cap \overline{R((I-M)^*)} \). We will show that \( x_0^* = 0 \). By the Hahn–Banach it then follows that \( X_0 = X \). Since \( x_0^* \) belongs to the \( \|\cdot\|\)–closure of \( \overline{(I-M)^*} \), it follows that

\[
x_0^* = \|\cdot\| - \lim_{\lambda \to 1} (I-M)^* \left( (I-\lambda M)^* \right)^{-1} x_0^*.
\]

To see this we first suppose that \( x_0^* = (I-M)^* x_1^* \). Then

\[
(I-M)^* x_0^* = (I-M)^* \left( (I-\lambda M)^* \right)^{-1} (I-M)^* x_1^* = (1-\lambda) M^* \left( (I-\lambda M)^* \right)^{-1} (I-M)^* x_1^*.
\]

Since the family \( M^* \left( (I-\lambda M)^* \right)^{-1} (I-M)^* x_1^* \), \( 0 < \lambda < 1 \), is bounded, we see that (30) is a consequence of (31) provided \( x_0^* \) belongs to the range of \( (I-M)^* \). By the uniform boundedness of the family \( (I-M)^* \left( (I-\lambda M)^* \right)^{-1} \), \( 0 < \lambda < 1 \), the same conclusion is true if \( x_0^* \) belongs to the closure of the range of \( (I-M)^* \). Since, in addition, \( x_0^* \) is a member of \( N((I-M)^*) \), it follows that \( x_0^* = 0 \). This proves Corollary 3.3.

\ Phaser of Theorem 3.1. \( (i) \implies (ii) \). Fix \( 0 < r < 1 \). The following representations from Lyubich [4] are being used:

\[
(n+1)M^n = \frac{1}{2\pi i} \int_{|\lambda|=r} (1-\lambda)^2 (I-\lambda M)^{-2} \frac{d\lambda}{\lambda^{n+1}(1-\lambda)^2},
\]

\[
\frac{1}{2} (n+1)(n+2)M^n(I-M)
\]

\[
= \frac{1}{2\pi i} \int_{|\lambda|=r} (1-\lambda)^2 (I-M)(I-\lambda M)^{-3} \frac{d\lambda}{\lambda^{n+1}(1-\lambda)^2}
\]

\[
= \frac{1}{2\pi i} \int_{|\lambda|=r} (1-\lambda)^2 (I-\lambda M)^{-2} \frac{1}{\lambda^{n+2}(1-\lambda)^2} d\lambda
\]

\[
- \frac{1}{2\pi i} \int_{|\lambda|=r} (1-\lambda)^3 (I-\lambda M)^{-3} \frac{1}{\lambda^{n+2}(1-\lambda)^3} d\lambda.
\]

Put \( C := \sup \left\{ \| (1-\lambda)(I-\lambda M)^{-1} \|_{X_0} : |\lambda| < 1 \right\} \). From (32) we infer

\[
(n+1) \left\| M^n \big|_{X_0} \right\| \leq C^2 \frac{r^n}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|1-re^{i\theta}|^2} d\theta = \frac{C^2}{r^n} \frac{1}{1-\beta^2}.
\]

The choice \( r^2 = \frac{\pi}{\beta+2} \) yields

\[
\left\| M^n \big|_{X_0} \right\| \leq \frac{2}{3} \epsilon C^2.
\]
In the same spirit from (33) we obtain
\[ \frac{1}{2} (n + 1) (n + 2) \left\| M^n (M - I) \right\|_{X_0} \leq \frac{1}{r^{n+1}} \frac{1}{1 - r^2} \cdot \] (36)

The choice \( r^2 = \frac{n+1}{n+3} \) yields the inequality:
\[ (n + 1) \left\| M^n (M - I) \right\|_{X_0} \leq \frac{4e^2}{3} (C^2 + C^3). \] (37)

This proves the implication (i) \( \Rightarrow \) (ii).

(ii) \( \Rightarrow \) (iii). The representations (see Nagy [5])
\[ e^{t(M-I)} = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} M^k \] and \( t (M - I) e^{t(M-I)} = e^{-t} \sum_{k=0}^{\infty} \frac{t^{k+1}}{k!} M^k (M - I) \)
show that (iii) is a consequence of (ii).

(iii) \( \Rightarrow \) (iv). This is a (standard) result in analytic operator semigroup theory: see e.g. Van Casteren [6], Chapter 5, Theorem 5.1.

(iv) \( \Rightarrow \) (v). The equality
\[ (I - M) \left( (\lambda + 1) I - M \right)^{-1} = I - \lambda (I - (M - I))^{-1} \]
shows the equivalence of (iv) and (v).

(v) \( \Rightarrow \) (i). Fix \( x \in X_0 \). The choice \( \lambda = -1 + e^{-i\vartheta} = -2i \sin \left( \frac{1}{2} \vartheta \right) e^{-\frac{i}{2} \vartheta} \), \( |\vartheta| \leq 2\alpha \), yields the boundedness of the function
\[ \vartheta \mapsto (I - M) \left( I - e^{i\vartheta} M \right)^{-1} x \]
on the interval \([-\alpha, \alpha]\). Since, for \( |\lambda| = 1 \), \( \lambda \neq 1 \), the function
\[ \lambda \mapsto (I - M) \left( I - \lambda M \right)^{-1} x \]
is continuous, it follows that the function
\[ \lambda \mapsto (I - M) \left( I - \lambda M \right)^{-1} x \]
is bounded on the unit circle. The maximum modulus theorem shows that this function is bounded on the unit disc, which is assertion (i).

(i) \( \Rightarrow \) (vi). Fix \( x \in X_0 \). For \( 0 < r < 1 \) and \( \vartheta \in \mathbb{R} \) we also have
\[ (I - P \left( re^{i\vartheta} \right)) (I - M) x \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos (\vartheta - t) + r^2} (I - M) \left( I - e^{it} M \right)^{-1} (I - M) x \, dt \]
\[ = (I - M) \left( I - re^{i\vartheta} M \right)^{-1} (I - M) x. \] (38)

In (38) we use the continuity of the boundary function
\[ e^{it} \mapsto (I - M) \left( I - e^{it} M \right)^{-1} (I - M) x \] (39)
to show that
\[ \lim_{re^{i\vartheta} \to 1, 0 \leq r < 1} (I - P \left( re^{i\vartheta} \right)) (I - M) x = (I - P)(I - M)x = (I - M)x \] (40)
exists, and that \( I - P \) is bounded projection. From (i) it follows that the function 
\( \lambda \mapsto (I - M) (I - \lambda M)^{-1} x \) is uniformly bounded on the unit disc, and hence that 
the limit in (40) exists for all \( y \) in the closure of \( R(I - M) \). In addition, for such vectors \( y \) we have 
\( (I - P)y = y \). The limit in (40) trivially exists for \( x \in X \) such that 
\( Mx = x \), we conclude that the limit in (i) exists for all \( x \in X_0 \), because 
\( x = (I - P)x + Px \), where \( (I - P)x \) belongs to the closure of the range of \( I - M \) 
and where 
\[
Px = x - (I - P)x = x - \lim_{\lambda \uparrow 1} (I - M) (I - \lambda M)^{-1} x 
= \lim_{\lambda \uparrow 1} (1 - \lambda) M (I - \lambda M)^{-1} x. \tag{41}
\]
From (41) it follows that \( (I - M)Px = 0 \). In addition, from (ii), which is equivalent 
to (i), we see that \( \lim_{n \to \infty} M^n y = 0 \) for all \( y \) in the range of \( I - M \); here we use the 
boundedness of the sequence \( (n + 1) M^n (I - M) \), \( n \in \mathbb{N} \). The boundedness 
of the sequence \( M^n \), \( n \in \mathbb{N} \), then yields \( \lim_{n \to \infty} M^n y = 0 \) for \( y \in R(I - P) \), 
because the range of \( I - M \) is dense in the range of \( I - P \). An arbitrary \( x \in X_0 \) 
can be written as \( x = (I - P)x + Px \). From the previous arguments it follows 
that \( \lim_{n \to \infty} M^n x = Px \). Fix \( x \in X_0 \). Altogether this shows the implication (v) 
\( \Rightarrow \) (vi), provided we show the continuity of the function in (39) in the sense that 
\( \lim_{t \to 0} (I - M) (I - e^{it} M)^{-1} (I - M) x = (I - M) x \). However, this follows from 
the identity 
\[
(I - M) (I - e^{it} M)^{-1} (I - M) x - (I - M) x 
= (e^{it} - 1) (I - M) (I - e^{it} M)^{-1} Mx,
\]
together with the uniform boundedness (in \( 0 < |t| \leq \pi \)) of the family of operators: 
\( (I - M) (I - e^{it} M)^{-1} \). In the latter we use the (proof of the) implication 
(vi) \( \Rightarrow \) (i).

The implication (vi) \( \Rightarrow \) (vii) being trivial there remains to be shown that (vii) 
implies (i). For this purpose we fix \( x \in X_0 \) and we consider the continuous function 
on the closed unit disc, defined by 
\[
F(\lambda)x := \begin{cases} 
(I - M) (I - \lambda M)^{-1} x & \text{for } |\lambda| \leq 1, \lambda \neq 1, \\
(I - P)x & \text{for } \lambda = 1.
\end{cases} \tag{42}
\]
From (vii) it follows that the function \( F(\lambda)x \) is well-defined and continuous. Hence 
it is bounded. The theorem of Banach–Steinhaus then implies (i). \( \square \)

Theorem 3.1 has the following analogue in the continuous time setting.

**Theorem 3.6.** Let \( K \) be a closed linear operator with a dense domain in a Banach 
space \( X \). Let \( X_0 \) be the closed sub–space of \( X \) defined as the \( \| \cdot \| \)–closure of the 
vector sum of the range and zero–space of \( K \): \( X_0 = R(K) + N(K^-) \). Suppose 
that the spectrum of \( K \) is contained in the open left half plane union \( \{0\} \). The 
following assertions are equivalent:
(i) $\sup_{\Re \lambda > 0} \left\| \lambda (\lambda I - K)^{-1} x \right\| < \infty$ for all $x \in X_0$;

(ii) $\sup_{t > 0} \| e^{tK} x \| < \infty$ and $\sup_{t > 0} \| tK e^{tK} x \| < \infty$ for all $x \in X_0$;

(iii) There exists $\frac{1}{2} \pi < \alpha < \pi$ such that $\sup_{-\alpha \leq \arg(\lambda) \leq \alpha} \left\| \lambda (\lambda I - K)^{-1} x \right\| < \infty$ for all $x \in X_0$;

(iv) For every $x \in X_0$ the following limits exist:

$$P x := \lim_{t \to \infty} e^{tK} x, \quad (I - P) x = \lim_{\lambda \to 0, \Re \lambda > 0} \lambda (\lambda I - K)^{-1} x,$$

and

$$x = \lim_{|\lambda| \to \infty, \Re \lambda > 0} \lambda (\lambda I - K)^{-1} x;$$

(v) For every $x \in X_0$ the following limits exist:

$$(I - P) x = \lim_{\lambda \to 0, \Re \lambda > 0} \lambda (\lambda I - K)^{-1} x,$$

and

$$x = \lim_{|\lambda| \to \infty, \Re \lambda > 0} \lambda (\lambda I - K)^{-1} x.$$

Moreover, if $K$ satisfies one of the conditions (i) through (v), then

$$X_0 = \overline{R(K)} + N(K).$$

**Remark 3.7.** The Banach–Steinhaus theorem implies that in (i) through (iii) in Theorem 3.6 the vector norms may be replaced with the operator norm restricted to $X_0$; i.e. $K$ must be restricted to $X_0 \cap D(K)$.

Conditions (a) and (b) of the following corollary are fulfilled if the space $X$ is reflexive.

**Corollary 3.8.** Let $K$ be a closed densely defined linear operator in a Banach space $(X, \|\cdot\|)$. As in Theorem 3.6 let $X_0$ be the closure in $X$ of the sub–space $R(K) + N(K)$. Suppose that, for $0 < \lambda$, the inverse operators $(\lambda I - K)^{-1}$ exist and are bounded, and that $\sup_{0 < \lambda} \lambda \left\| (\lambda I - K)^{-1} \right\| < \infty$. If one of the following conditions:

(a) the zero space of the operator $(K)^{**}$, which is a sub–space of the bidual space $X^{**}$, is in fact a subspace of $X$;

(b) the $\sigma(X^*, X)$–closure of $R(K^*)$ coincides with its $\|\cdot\|$–closure;

(c) the range of $K$ is closed in $X$;

is satisfied, then the space $X_0$ coincides with $X$, and hence all assertions in Theorem 3.6 are equivalent with $X$ in place of $X_0$.

**Remark 3.9.** If $\sup_{t > 0} \| e^{tK} \| < \infty$, then $\sup_{0 < \lambda} \lambda \left\| (\lambda I - K)^{-1} \right\| < \infty$. 

The proof of Corollary 3.3 can be adapted to yield the result in Corollary 3.8. Instead of (27) we exploit an identity of the form:

\[ x = -K (\lambda I - K)^{-1} x + \lambda (\lambda I - K)^{-1} x. \]

**Definition 3.10.** A densely defined closed linear operator \( K \) satisfying one (and hence all) of the conditions (i) through (iii) in Theorem 3.6 with \( X \) instead of \( X_0 \) is called the generator of a bounded analytic semigroup in \( X_0 \); see Blunck [1].

**Proof of Theorem 3.6.** The proof of Theorem 3.6 follows the same pattern as that of Theorem 3.1. Among other things we use the following equalities:

\[
\begin{align*}
te^{tK} &= \frac{1}{2\pi i} \int_{\omega - i\infty}^{\omega + i\infty} \lambda^2 e^{\lambda t} (\lambda I - K)^{-2} \frac{1}{\lambda^2} d\lambda; \\
\frac{1}{2} t^2 Ke^{tK} &= \frac{1}{2\pi i} \int_{\omega - i\infty}^{\omega + i\infty} \lambda^2 e^{\lambda t} K (\lambda I - K)^{-3} \frac{1}{\lambda^2} d\lambda; \\
(I - P (\omega + i\xi)) K &= \frac{\omega}{\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + (t - \xi)^2} (-K) (it - K)^{-1} K dt \\
&= -K ((\omega + i\xi) I - K)^{-1} K. \quad (43)
\end{align*}
\]

First one shows the equivalence of the assertions in (i) through (iii) in Theorem 3.6: this is a standard result in analytic semigroup theory. In fact, the equivalence of (i), (ii) and (iii) is also true if \( X \) replaces \( X_0 \).

(i) \implies (iv). In order to prove the implication (i) \implies (iv) in (43) we let \( \omega + i\xi \) tend to 0, and we use the continuity of the boundary function \( t \mapsto -K (it - K)^{-1} Kx \) to prove that \( (I - P) Kx \) exists, and that

\[
\|(I - P) |X_0| \leq 1 + \sup_{\lambda > 0} \lambda \left\| (\lambda I - K)^{-1} |X_0| \right\|
\]

Moreover, we have \( (I - P) Kx = Kx \), for \( x \) belonging to the domain of \( K \), and consequently \( (I - P) y = y \), if \( y \) belongs to the closure of \( R(K) \). The latter follows because for \( x \) belonging to the domain of \( K \) we have

\[
-K (it - K)^{-1} Kx = -it K (it - K)^{-1} x. \quad (44)
\]

The equality in (44) in conjunction with the boundedness of the operator-valued function, restricted \( X_0 \),

\[
t \mapsto -K (it - K)^{-1} = I - it (it - K)^{-1}, \quad t \in \mathbb{R} \setminus \{0\},
\]
yields the existence of the second limit in assertion (v).

Next we fix \( x = (I - P) x + Px \in X \), and we consider

\[
e^{tK} x = e^{tK} (I - P) x + e^{tK} Px.
\]

Since by (ii) (and the Banach–Steinhaus theorem) the expression \( \sup_{t > 0} \left\| tKe^{tK} \right\|_{X_0} \) is finite, we get \( \lim_{t \to \infty} e^{tK} y = 0, \ y \in R(K) \). Since, also by (ii), \( \sup_{t > 0} \left\| e^{tK} \right\|_{X_0} < \infty \), we obtain \( \lim_{t \to \infty} e^{tK} (I - P) x = 0 \), because the range of \( K \) is dense in the range of \( I - P \). In addition, we have

\[
e^{tK} Px = K \int_0^1 e^{sK} Px ds =
\]
\( PK \int_0^t e^{tK} x \, ds = 0, \) for all \( t > 0. \) Here we used the fact that \((I - P) K = K\) on the domain of \( K.\) These observations show that \( \lim_{t \to \infty} e^{tK} x = Px \) for all \( x \in X_0. \) Finally (i) also implies that

\[
\lim_{\Re \lambda > 0, |\lambda| \to \infty} \lambda (\lambda I - K)^{-1} x = x
\]

for all \( x \) in the domain of \( K \) restricted to \( X_0.\) Since, by assumption, the domain of \( K \) is dense in \( X, \) the same is true for its restriction to \( X_0.\) Again using (i) implies the equality in (45) for all \( x \in X_0.\)

(iv) \( \implies \) (v). This implication is trivial.

(v) \( \implies \) (i). Fix \( x \in X.\) From (v) together with the continuity of the function \( \lambda \mapsto (\lambda I - K)^{-1} x, \Re \lambda \geq 0, \lambda \neq 0,\) shows that \( \sup_{\Re \lambda > 0} |\lambda| \left\| (\lambda I - K)^{-1} x \right\| < \infty \) for all \( x \in X_0.\) The theorem of Banach–Steinhaus then implies (i). \( \square \)

Finally, we give some remarks. We confine the obtained theorems to the case of Markov and Kolmogorov matrices. Since we did not find proofs of the results in Corollary 3.13 and Corollary 3.14 in the literature, we include them here.

Remark 3.11. Theorem 3.1 may be useful for Markov matrices \( M,\) whereas Theorem 3.6 is more adapted to Kolmogorov matrices \( K.\) If \( M \) is a Markov matrix, then \( \sup_{n \in \mathbb{N}} \| M^n \| = 1, \) provided \( \mathbb{R}^d \) is endowed with the supremum-norm. If \( K \) is a Kolmogorov matrix, and \( \mathbb{R}^d \) is endowed with the supremum-norm, then \( \sup_{t > 0} \| e^{tK} \| = 1.\)

Remark 3.12. A matrix \( M \) is analytic in the sense of Definition 3.5, provided the Jordan block corresponding to the eigenvalue 1 has vanishing side diagonals, and the other eigenvalues of \( M \) belong to the open unit disc. A matrix \( K \) generates a bounded analytic semigroup provided its Jordan block corresponding to the eigenvalue 0 also possesses zero side diagonals. The other eigenvalues are located in the open left half plane.

Corollary 3.13. Let the space \( X \) be finite-dimensional and let \( M \) be a Markov matrix. Then \( M \) is analytic (in the sense of Definition 3.5 and Blunck [2]), and the Kolmogorov matrix \( K := M - I \) generates a bounded analytic semigroup (in the sense of Definition 3.10). In fact more is true. From the hypotheses it follows that, for some \( r > 1, \) the expression \( \sup_{|\lambda| < r} |1 - \lambda| \cdot \left\| (I - \lambda M)^{-1} \right\| \) is finite. It also follows that, for some \( \omega > 0 \) the quantity \( \sup_{|\lambda| > \omega} |\lambda| \cdot \left\| (I - \lambda K)^{-1} \right\| \) is finite.

Proof. Since \( M \) is Markov, it follows that \( \sup_{n \in \mathbb{N}} \| M^n \| < \infty.\) With \( K = M - I \) it also follows that \( \sup_{t > 0} \| e^{tK} \| < \infty.\) Since \( (\lambda I - K)^{-1} = \int_0^\infty e^{-\lambda t} e^{tK} \, dt, \lambda > 0, \) we get \( \sup_{|\lambda| > \omega} |\lambda| \cdot \left\| (I - \lambda K)^{-1} \right\| < \infty.\) Using the Jordan decomposition theorem we see that \( K \) may be written in the form \( K = \sum_{j=1}^m (\alpha_j P_j + N_j).\) The eigenvalues \( \alpha_j, 1 \leq j \leq m, \) belong to the open left half plane \( \{ \lambda \in \mathbb{C} : \Re \lambda < 0 \}, \) and \( \alpha_0 = 0; \) the operators \( \alpha_j P_j + N_j, 0 \leq j \leq m, \) commute; the operators \( P_j, 0 \leq j \leq m, \) are projection operators which commute, and which satisfy \( P_j P_k = 0, j \neq k.\) Finally the operators \( N_j, 0 \leq j \leq m, \) are nilpotent in the sense that \( N_j^d_j = 0 \) for appropriate positive
Remark 3.15. We notice that the operator $(I - P) M^n$ may be written in the form:

$$ (I - P) M^n = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{1}{\lambda^n} (I - \lambda M)^{\lambda-1} \frac{1}{\lambda-1} d\lambda. \quad (48) $$

Corollary 3.14. Let $M$ be a finite-dimensional Markov matrix with the property that $\sup_{n \geq 1} (n + 1) \|M^n (M - I)\| < \infty$ and let $K$ be a finite-dimensional Kolmogorov matrix, with the property that $\sup_{t > 0} \|t K e^{tK}\| < \infty$. Then the following assertions are true:

(i) on the range of $I - M$ the sequence $M^n$ tends to zero exponentially fast;

(ii) on the range of $K$ the family $e^{tK}$ goes to zero exponentially fast.

Although the result in Corollary 3.14 is well-known, a proof is included, because similar arguments show the corresponding result for analytic operators and analytic semigroups in Banach spaces; see Theorem 3.16 below.

Proof. Let $P$ be the projection on the null space of $I - M$ as described in Theorem 3.1. Fix $x \in X$. In order to prove assertion (i) we use the representation

$$ M^n x - P x = M^n (I - P) x = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{1}{\lambda^{n+1}} (I - P)(I - \lambda M)^{-1} x d\lambda, \quad (46) $$

where $r$ is some real number which is strictly larger than 1. Consequently we have:

$$ \|M^n x - P x\| \leq \frac{1}{r^n} \sup_{|\lambda|=r} \| (I - P)(I - \lambda M)^{-1} x \|. $$

Next let $P$ the projection as described in Theorem 3.6. In order to prove assertion (ii), we use the representation:

$$ e^{tK} x - P x = e^{tK} (I - P) x = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{t \lambda} (I - P)(\lambda I - K)^{-1} x d\lambda $$

(integration by parts)

$$ = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{t \lambda} (I - P)(\lambda I - K)^{-2} x d\lambda $$

$$ = \frac{1}{2\pi i} e^{-\omega_0 t} \int_{-\infty}^{\infty} e^{it \xi} (I - P)((-\omega_0 + i \xi)I - K)^{-2} x d\lambda, \quad (47) $$

where $\omega_0$ is some strictly positive real number. Consequently

$$ \|e^{tK} x - P x\| \leq \frac{e^{-\omega_0 t}}{\omega_0 t} \sup_{\Re \lambda = -\omega_0} |\lambda|^2 \| (I - P)(\lambda I - K)^{-2} x \|. $$

The above representations are justified by the results in Corollary 3.13. □

Remark 3.15. We notice that the operator $(I - P) M^n$ may be written in the form:
We also notice the following equalities:

\[(I - P)e^{tK} = \frac{1}{2\pi i} K \int_{-\omega - i\infty}^{-\omega + i\infty} e^{t\lambda} (\lambda I - K)^{-1} \frac{1}{\lambda} d\lambda = \frac{1}{2\pi i} K \int_{-\omega - i\infty}^{-\omega + i\infty} e^{t\lambda} \left\{ \frac{(\lambda I - K)^{-2} \frac{1}{\lambda} + (\lambda I - K)^{-1} \frac{1}{\lambda^2}}{1 - (\lambda I - K)^{-1}} \right\} d\lambda = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{t(\omega + i\xi)} \left\{ ((\omega + i\xi) I - K)^{-2} \frac{1}{(\omega + i\xi)^2} \right\} d\xi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{t(\omega + i\xi)} \left\{ ((\omega + i\xi)^2 I - (\omega + i\xi) I - K)^{-2} - I \right\} \frac{1}{(\omega + i\xi)^2} d\xi. \] (49)

The equalities in (48) and (49) are true whenever $M$ is an analytic operator with 1 an isolated point of the spectrum of $M$, and $K$ generates a bounded analytic semigroup with 0 an isolated point of the spectrum of $K$.

If we replace formula (46) with formula (48), and formula (47) with (49) we obtain a proof of the following result for analytic operators and analytic semigroups in a Banach space.

**Theorem 3.16.** Let $X$ be a Banach space, and let $M$ be an analytic operator such that $\sup_{n \in \mathbb{N}} \|M^n\| < \infty$. In addition let $K$ be the generator of a bounded analytic semigroup. Then the following assertions are true:

(a) If 1 is an isolated point of the spectrum of $M$, then $X = \mathbb{R}(M-I) + N(M-I)$, and there exists a finite constant $C$ and a real number $r > 1$ such that

$$\|M^n (I - P)\| \leq \frac{C}{r^n}, \text{ for all } n \in \mathbb{N}. \quad (50)$$

Here $P$ is the Dunford projection, along the range of $M-I$, on the eigen–space of $M$ corresponding to 1.

(b) If 0 is an isolated point of the spectrum of $K$, then $X = \overline{\mathbb{R}(K)} + N(K)$, and there exists a finite constant $C$ and a constant $\omega > 0$ such that:

$$t \|e^{tK}(I - P)\| \leq Ce^{-\omega t}, \text{ for all } t > 0. \quad (51)$$

Here $P$ is the Dunford projection on the zero–space of $K$ along the closure of the range of $K$.

**Corollary 3.17.** Let $(S,B)$ be a measurable space, where $S$ is a topological Hausdorff space with Borel field. Let $m : B \times S \to [0,1]$ be a sub-probability kernel; i.e. for every $x \in S$, the mapping $B \mapsto m(B,x)$ is supposed to be a sub-probability Borel measure. Consider the linear mapping $M : \mu \mapsto \int m(\cdot, x) d\mu(x)$ from the space of all complex measures $M_b(S,B)$ to itself. Suppose that the operator $M$ is
analytic and has 1 as eigen-value of multiplicity 1. Then the operator $M$ is ergodic in the sense that for every non-negative probability measure $\mu$, the sequence $M^n\mu$ converges exponentially fast to the equilibrium measure, as $n \in \mathbb{N}$ converges to $\infty$.

**Corollary 3.18.** Let $(S, \mathcal{B})$ be a measurable space, where $S$ is a topological Hausdorff space with Borel field. Let $p : [0, \infty) \times \mathcal{B} \times S \to [0, 1]$ be a sub-Markov family of sub-probability kernel; i.e. for every $x \in S$, and for every $t \geq 0$, the mapping $B \mapsto p(t, B, x)$ is supposed to be a sub-probability Borel measure. Suppose that the mapping $p(t, B, x)$ satisfies the Chapman–Kolmogorov identity:

$$
\int p(s, B, x) p(t, dx, y) = p(s + t, B, y).
$$

Put $e^{tK} \mu(B) = \int p(t, B, x) d\mu(x)$, $t \geq 0$, $B \in \mathcal{B}$. Let the semigroup $t \mapsto e^{tK}$ be analytic. Suppose that 0 is an isolated eigen-value of its generator $K$ with multiplicity 1. Then the semigroup $e^{tK}$, $t \geq 0$, is ergodic in the sense that, for every probability measure $\mu$ the family $e^{tK} \mu$ converges exponentially fast to the equilibrium measure, as $t$ tends to $\infty$.

**4. Conclusions**

In this article, we consider the relationship between matrices, which satisfy the maximum principle, the sub-Markov property, and those which preserve positivity (Proposition 2.1). These results allow us to investigate the connection between the matrices with the sub-Markov property and the ones with the sub-Kolmogorov property (Theorem 2.3). In Proposition 2.4 these relationships are employed for pure Kolmogorov and Markov matrices. We also present a Markov matrix $M$ as an analytic operator (Theorem 3.1) and a Kolmogorov matrix $K$ as a generator of a bounded analytic semigroup (Theorem 3.6). We do this by providing conditions on the operator $K$ which guarantee this analyticity and we establish the long time behavior of the corresponding semigroups. We indicate some new properties of these matrices as well. The obtained results yield e.g. Corollary 3.14 about the range of Markov and Kolmogorov matrices. The paper is concluded with similar properties for operators in a Banach space: see Theorem 3.16 and its Corollary 3.17. The proofs rely heavily on Dunford–like projections.

**References**


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