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N–DIMENSIONAL MINKOWSKI SPACE AND SPACE–TIME ALGEBRA

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Abstract. By using an n–Dimensional Minkowski space, the space–time algebra is introduced. It is used for discussing physical problems of special relativity.

Introduction

Clifford algebra was founded by the British mathematician W.K. Clifford in 1878. Since 1980's, it has been applied widely in maths, physics and engineering technology fields etc [**1**]. Especially in geometric calculation (Euclidean geometric calculation, non–Euclidean geometric calculation), geometric automatic reasoning, differential geometry and theoretical physics field [**2**]–[**8**]. In this paper, by using the hyperbolic virtual unit of Clifford algebra to expose n–dimensional Minkowski space and the concept of space–time algebra is introduced. It can be used for discussing physical problems of special relativity.

1. Definition and Example

Definition 1.1. Suppose S is an n–dimensional linear space over $\mathbb{R}$ (real field). S is an n–dimensional Minkowski space if there exist a bivariant real–valued function $\rho : S \times S \longrightarrow \mathbb{R}$ and a basic vector system $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ of S and satisfying

$$\begin{aligned} 1^\circ. \quad & \rho(\boldsymbol{u},\boldsymbol{v}) = \rho(\boldsymbol{v},\boldsymbol{u}); \\ 2^\circ. \quad & \rho(k\boldsymbol{u},\boldsymbol{v}) = k\rho(\boldsymbol{u},\boldsymbol{v}); \\ 3^\circ. \quad & \rho(\boldsymbol{u}+\boldsymbol{v},\boldsymbol{w}) = \rho(\boldsymbol{u},\boldsymbol{w}) + \rho(\boldsymbol{v},\boldsymbol{w}), \end{aligned} \tag{1.1}$$

and

$$4^\circ. \qquad \rho(\boldsymbol{e}_i,\boldsymbol{e}_j) = \begin{cases} 1, & i = j \le n-1; \\ -1, & i = j = n; \\ 0, & i \ne j, \end{cases} \tag{1.2}$$

for all $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in S$ and $k \in \mathbb{R}$. The bivariant real–valued function ρ is said to be *space–time inner product* or called to be Minkowski inner product (M inner product) if ρ satisfies (1.1) and (1.2). The basic vector system (1.2) is called to be a Minkowski orthogonal basis (M orthogonal basis) over S.

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In what follows we use the notation

$$\rho(\boldsymbol{u}, \boldsymbol{v}) \ = \ \boldsymbol{u} \cdot \boldsymbol{v}. \tag{1.3}$$

Example 1.2. Creation space $\mathbb{R}^{n-1,1}$ of Clifford algebra $\mathrm{Cl}_{n-1,1}$ is an n–dimensional real linear space and its basic set

$$\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n-1}, \boldsymbol{e}_n\}$$

under the inner product of $\mathrm{Cl}_{n-1,1}$ satisfy [1, 2, 3]:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j \ = \ \begin{cases} 1, & i = j \leq n-1; \\ -1, & i = j = n; \\ 0, & i \neq j. \end{cases} \tag{1.4}$$

If the space–time inner product is that of $\mathrm{Cl}_{n-1,1}$, then $\mathbb{R}^{n-1,1}$ is an n–dimensional Minkowski space.

Example 1.3. Suppose i is a imaginary unit of complex numbers.

$$S \ = \ \{(x_1, \cdots, x_{n-1}, ix_n) \mid x_1, \cdots, x_n \in \mathbb{R}\},$$

S is an n–dimensional Minkowski space under the space–time inner product defined as

$$\begin{aligned} (x_1, \cdots, x_{n-1}, ix_n) \cdot (y_1, \cdots, y_{n-1}, iy_n) \ &= \ x_1y_1 + \cdots + x_{n-1}y_{n-1} + ix_niy_n \\ &= \ x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_ny_n, \end{aligned} \tag{1.5}$$

Example 1.4. Suppose j is hyperbolic virtual unit of Clifford algebra $(j^2 = 1, j^* = -j)$ [1, 2, 3]. Let

$$T \ = \ \{(x_1, \cdots, x_{n-1}, jx_n) \mid x_1, \cdots, x_n \in \mathbb{R}\}.$$

Define a space–time inner product

$$\begin{aligned} (x_1, \cdots, x_{n-1}, jx_n) \cdot (y_1, \cdots, y_{n-1}, jy_n) \ &= \ x_1y_1 + \cdots + x_{n-1}y_{n-1} + jx_nj^*y_n \\ &= \ x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_ny_n, \end{aligned} \tag{1.6}$$

then T is an n–dimensional Minkowski space.

Example 1.5. $\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_1, \cdots, x_n \in \mathbb{R}\}$ is also an n–dimensional Minkowski space for this inner product as follows

$$(x_1, \cdots, x_{n-1}, x_n) \cdot (y_1, \cdots, y_{n-1}, y_n) \ = \ x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_ny_n. \tag{1.7}$$

When $n = 4$, $\mathbb{R}^{3,1}$, S, T, $\mathbb{R}^4$ may be applied for discussing physical problems of special relativity [1, 2, 3, 4]. In this paper, we denote $M_n \equiv T$ and discuss some correlated problems from M_n.

2. Space–Time Space

In $M_n = \{(x_1, \cdots, x_{n-1}, jx_n)\}$, $x_1, \cdots, x_{n-1}$ is said to be space component and x_n is time component.

Definition 2.1. Suppose S is an n–dimensional Minkowski space. According to its space–time inner product, the *space–time inner product normal number* (for short *space–time normal number* or *M normal number*) of vector $\boldsymbol{w}(\boldsymbol{w} \in S)$ is defined as

$$\|\boldsymbol{w}\|_M \;=\; \sqrt{|\boldsymbol{w}\cdot\boldsymbol{w}|}, \tag{2.1}$$

n–dimensional Minkowski space S and space–time normal number (2.1) be called together n–dimensional space-time normic space (n–dimensional M normic space). The space is denoted by $(S, \|\cdot\|_M)$.

In order to examine some correlated properties of the space–time normic space and the application in special relativity. Let $x_n = ct$, x_n is time component in M_n. Where c denotes light speed, t is time. So we denote

$$M_n \;=\; \{(x_1, \cdots, x_{n-1}, jct)\}, \tag{2.2}$$

or

$$M_n \;=\; \{\boldsymbol{r} + jct\}, \tag{2.3}$$

where $\boldsymbol{r} = (x_1, \cdots, x_{n-1})$ is $n-1$–dimensional real location vector.

Proposition 2.2.

(1) *Suppose* $(M_n, \|\cdot\|)$ *is a space–time normic space. For any* $\boldsymbol{w} = \boldsymbol{r} + jct \in M_n$, $\|\boldsymbol{w}\|_M = 0$ *iff* $r = |ct|$, $r = \sqrt{\boldsymbol{r}\cdot\boldsymbol{r}}$;

(2) $\|k\boldsymbol{w}\|_M = |k|\|\boldsymbol{w}\|_M$ *for any* $k \in \mathbb{R}$, $\boldsymbol{w} \in M_n$.

$\|\boldsymbol{w}_1 + \boldsymbol{w}_2\|_M \le \|\boldsymbol{w}_1\|_M + \|\boldsymbol{w}_2\|_M$ is not necessary for any $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \in M_n$.

Example 2.3. Choose $\boldsymbol{w}_1 = (1, 0, 0, \cdots, 0, 2j)$, $\boldsymbol{w}_2 = (0, 1, 0, \cdots, 0, 2j) \in M_n$, then

$$\|\boldsymbol{w}_1 + \boldsymbol{w}_2\|_M = \sqrt{14} > \sqrt{3} + \sqrt{3} = \|\boldsymbol{w}_1\|_M + \|\boldsymbol{w}_2\|_M.$$

Let

$$M_n^+ = \{\boldsymbol{w} = \boldsymbol{r} + jct \in M_n \mid ct \ge r, \quad \text{iff when} \quad t = 0, \quad \text{the equality sign is correct}\}. \tag{2.4}$$

M_n^+ is said to be a future timelike region of M_n, $\dot{M}_n^+ = M_n^+ \backslash \{0\}$ is called strictly future timelike region .

Theorem 2.4 (Inverted triangle inequality)**.** *Choose any* $\boldsymbol{w}_1, \boldsymbol{w}_2 \in M_n$. *When* $\boldsymbol{w}_1, \boldsymbol{w}_2 \in M_n^+$, *we have*

$$\|\boldsymbol{w}_1 + \boldsymbol{w}_2\|_M \;\ge\; \|\boldsymbol{w}_1\|_M + \|\boldsymbol{w}_2\|_M, \tag{2.5}$$

and iff $\boldsymbol{w}_1, w_2$ *is linear dependent, the equality holds.*

3. Space–Time Algebra

Definition 3.1. Let $(S, \|\cdot\|_M)$ is an n–dimensional space–time normic space. S is an n–dimensional *space–time algebra* (n–dimensional *space–time normic algebra*. If there exists a multiplication operation $\circ$ defined on S and satisfying $\boldsymbol{a}\circ(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\circ\boldsymbol{b}+\boldsymbol{a}\circ\boldsymbol{c}, (\boldsymbol{b}+\boldsymbol{c})\circ\boldsymbol{a} = \boldsymbol{b}\circ\boldsymbol{a}+\boldsymbol{c}\circ\boldsymbol{a}$; $k(\boldsymbol{a}\circ\boldsymbol{b}) = (k\boldsymbol{a})\circ\boldsymbol{b} = \boldsymbol{a}\circ(k\boldsymbol{b})$ for all $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}\in S$ and $k\in\mathbb{R}$, besides

$$\|\boldsymbol{a}\circ \mathrm{b}\|_M \;=\; \|\boldsymbol{a}\|_M\|\boldsymbol{b}\|_M. \tag{3.1}$$

Example 3.2. Two–dimensional Minkowski space M_2 may be denoted by

$$M_2 \;=\; \{x+jy\}. \tag{3.2}$$

Let the multiplication operation over M_2 be $(x_1+jy_1)(x_2+jy_2) = (x_1x_2+y_1y_2)+j(x_1y_2+x_2y_1)$, then M_2 is two–dimensional real exchange algebra and $(M_2, \|\cdot\|)$ becomes two–dimensional space–time algebra.

Define a multiplication operation $\circ$ over an n–dimensional Minkowski space as follows

$$\circ : (\boldsymbol{r}_1+jct_1)\circ(\boldsymbol{r}_2+jct_2) \;=\; j\big((\boldsymbol{r}_1\cdot\boldsymbol{r}_2+c^2t_1t_2)+j(ct_1\boldsymbol{r}_2+ct_2\boldsymbol{r}_1)\big), \tag{3.3}$$

then M_n is n–dimensional real algebra, but space–time normic space M_n is not space–time algebra under the operation $\circ$.

Example 3.3. Choose $\boldsymbol{w}_1 = \left(\frac{\sqrt{3}}{2}, 0, 0, \cdots, 0, j\right)$, $\boldsymbol{w}_2 = \left(0, \frac{\sqrt{3}}{2}, 0, \cdots, 0, j\right) \in M_n$ then we get

$$\|\boldsymbol{w}_1\circ\boldsymbol{w}_2\|_M = \frac{\sqrt{2}}{2} \neq \frac{1}{4} = \|\boldsymbol{w}_1\|_M\|\boldsymbol{w}_2\|_M.$$

Theorem 3.4. *If $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ is linear dependent, then*

$$\|\boldsymbol{w}_1\circ\boldsymbol{w}_2\|_M \;=\; \|\boldsymbol{w}_1\|_M\|\boldsymbol{w}_2\|_M \tag{3.4}$$

for any $\boldsymbol{w}_1, \boldsymbol{w}_2\in M_n$.

Theorem 3.5. *By using the space–time normal number* (2.1) *and the binary operation* (3.3)*, we introduce a binary operation over M_n as what follows*

$$\odot : (\boldsymbol{r}_1+jct_1)\odot(\boldsymbol{r}_2+jct_2) \;=\; (\boldsymbol{r}_1+jct_1)\circ(\boldsymbol{r}_{2\|}+jct_2)+\|\boldsymbol{w}_1\|_M\boldsymbol{r}_{2\perp}, \tag{3.5}$$

then M_n becomes an n–dimensional real algebra and

$$\|\boldsymbol{w}_1\circ\boldsymbol{w}_2\|_M \;=\; \|\boldsymbol{w}_1\|_M\|\boldsymbol{w}_2\|_M,$$

where $\boldsymbol{r}_{2\|}$ and $\boldsymbol{r}_{2\perp}$ are the component which $\boldsymbol{r}_2$ is parallel and perpendicular with $\boldsymbol{r}_1$ respectively. Thus we can say n–dimensional space–time normic space M_n is an n–dimensional space–time algebra over the binary operation $\odot$.

Theorem 3.6. *For all $\boldsymbol{u}, \boldsymbol{w}\in M_n^+$, we have*

$$\boldsymbol{u}\odot\boldsymbol{w} \;\in\; M_n^+, \tag{3.6}$$

especially, let

$$\boldsymbol{w}' \;=\; -\boldsymbol{u}*\odot\boldsymbol{w}, \tag{3.7}$$

when $\|\boldsymbol{u}\|_M = 1$. We can deduce a Lorentz transformation of n–dimensional Minkowski space–time.

Proof. Denote $\boldsymbol{u} = \boldsymbol{r}_u + jct_u$ for all $\boldsymbol{u}, \boldsymbol{w} \in M_n^+$, $\|\boldsymbol{u}\| = 1$. Then $\boldsymbol{u}$ can be denoted by $\boldsymbol{u} = j(\cosh\varphi + j\boldsymbol{r}_{\boldsymbol{u}}^{\circ}\sinh\varphi)$ [**1**], where $\varphi = \operatorname{arctanh}\left(\frac{r_u}{ct_u}\right)$, $\boldsymbol{r}_u = \frac{\boldsymbol{r}_u}{r_u}$. Let $\boldsymbol{v} = \frac{\boldsymbol{r}_u}{t_u}$, then $\cosh\varphi, \sinh\varphi$ can be written as $\cosh\varphi = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$, $\sinh\varphi = \frac{v}{c}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$. If let $\boldsymbol{w}' = \boldsymbol{r}' + jct' = -\boldsymbol{u}^* \odot \boldsymbol{w}$, we obtain

$$\boldsymbol{r}' + jct' \;=\; j(\cosh\varphi - j\boldsymbol{r}_u^{\circ}\sinh\varphi)\circ(\boldsymbol{r}_{\|} + jct) + \boldsymbol{r}_{\perp}.$$

Let $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$, by the expansion of it, we have

$$\boldsymbol{r}' = \gamma(\boldsymbol{r}_{\|} - \boldsymbol{v}t) + \boldsymbol{r}_{\perp}, t' = \gamma(t - \boldsymbol{v}\cdot\boldsymbol{r}/v^2), \tag{3.8}$$

which satisfies $(ct')^2 - (r')^2 = (ct)^2 - r^2$. □

When $n = 4$, substituting $\boldsymbol{v} = (v_x, v_y, v_z)$, $\boldsymbol{r}_{\|} = (\boldsymbol{r}_u\cdot\boldsymbol{r})\boldsymbol{r}_u/r_u^2$, $\boldsymbol{r}_{\perp} = \boldsymbol{r} - \boldsymbol{r}_{\|}$ into (3.8), we obtain linear equation system

$$\left\{\begin{array}{lcl} x' &=& \left(1 + (\gamma-1)\frac{v_x^2}{v^2}\right)x + \left((\gamma-1)\frac{v_xv_y}{v^2}\right)y + \left((\gamma-1)\frac{v_xv_z}{v^2}\right)z - \gamma v_x t,\\ y' &=& \left((\gamma-1)\frac{v_xv_y}{v^2}\right)x + \left(1 + (\gamma-1)\frac{v_y^2}{v^2}\right)y + \left((\gamma-1)\frac{v_yv_z}{v^2}\right)z - \gamma v_y t,\\ z' &=& \left((\gamma-1)\frac{v_zv_x}{v^2}\right)x + \left((\gamma-1)\frac{v_zv_y}{v^2}\right)y + \left(1 + (\gamma-1)\frac{v_z^2}{v^2}\right)z - \gamma v_z t,\\ t' &=& -\gamma\frac{v_x x}{c^2} - \gamma\frac{v_y y}{c^2} - \gamma\frac{v_z z}{c^2} + \gamma t. \end{array}\right. \tag{3.9}$$

Its corresponding matrix form is

$$\begin{bmatrix} x'\\ y'\\ z'\\ jct' \end{bmatrix} = \begin{bmatrix} 1 + (\gamma-1)\frac{v_x^2}{v^2} & (\gamma-1)\frac{v_xv_y}{v^2} & (\gamma-1)\frac{v_xv_z}{v^2} & \frac{-\gamma j v_x}{c}\\ (\gamma-1)\frac{v_xv_y}{v^2} & 1 + (\gamma-1)\frac{v_y^2}{v^2} & (\gamma-1)\frac{v_yv_z}{v^2} & \frac{-\gamma j v_y}{c}\\ (\gamma-1)\frac{v_xv_z}{v^2} & (\gamma-1)\frac{v_yv_z}{v^2} & 1 + (\gamma-1)\frac{v_z^2}{v^2} & \frac{-\gamma j v_z}{c}\\ \frac{-\gamma j v_x}{c} & \frac{-\gamma j v_y}{c} & \frac{-\gamma j v_z}{c} & \gamma \end{bmatrix} \begin{bmatrix} x\\ y\\ z\\ jct \end{bmatrix}. \tag{3.10}$$

The coefficient matrix [3, 4]

$$A \;=\; \begin{bmatrix} 1 + (\gamma-1)\frac{v_x^2}{v^2} & (\gamma-1)\frac{v_xv_y}{v^2} & (\gamma-1)\frac{v_xv_z}{v^2} & \frac{-\gamma j v_x}{c}\\ (\gamma-1)\frac{v_xv_y}{v^2} & 1 + (\gamma-1)\frac{v_y^2}{v^2} & (\gamma-1)\frac{v_yv_z}{v^2} & \frac{-\gamma j v_y}{c}\\ (\gamma-1)\frac{v_xv_z}{v^2} & (\gamma-1)\frac{v_yv_z}{v^2} & 1 + (\gamma-1)\frac{v_z^2}{v^2} & \frac{-\gamma j v_z}{c}\\ \frac{-\gamma j v_x}{c} & \frac{-\gamma j v_y}{c} & \frac{-\gamma j v_z}{c} & \gamma \end{bmatrix}$$

satisfies

$$AA^H \;=\; E, \tag{3.11}$$

where A^H is the transposed conjugate matrix of A; E is an unit matrix.

Especially, if $v_x = v$, $v_y = v_z = 0$ then equation system (3.10) can be changed into

$$\left\{\begin{array}{lcl} x' &=& \gamma(x - vt), y' = y, z' = z,\\ t' &=& \gamma(t - \frac{vx}{c^2}), \end{array}\right. \tag{3.12}$$

its corresponding matrix form is

$$\begin{bmatrix} x' \\ y' \\ z' \\ jct' \end{bmatrix} = \begin{bmatrix} \cosh\varphi & 0 & 0 & -j\sinh\varphi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -j\sinh\varphi & 0 & 0 & \cosh\varphi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ jct \end{bmatrix}. \qquad (3.13)$$

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