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INVERTIBILITY OF MULTIPLICATION MODULES IV

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Abstract. All rings are commutative with identity and all modules are unital. We give several properties of quasi-invertible submodules of multiplication modules generalizing those of quasi-invertible ideals. We introduce the concepts of generalized Dedekind modules, completely integrally closed modules, $*$-modules and v-multiplication modules and study the relationship among these modules in case they are faithful multiplication modules over integral domains. Finally, we introduce RTP and TPP modules and show that they are equivalent in the class of one dimensional faithful multiplication modules.

1. Introduction.

Throughout this paper all rings are assumed to be commutative with identity and all modules are unital. Let R be a ring and M an R-module. M is called a *multiplication* module if every submodule N and M has the form IM for some ideal I of R. Equivalently, $N = [N : M]\, M, [11]$. A submodule K of M is multiplication if and only if $N \cap K = [N : K]\, K$ for all submodules N of M, [19, Lemma 3.1]. If M is multiplication then

$$[K : M]\, N = [N : M]\, K = [K : M]\, [N : M]\, M,$$

for all submodules K and N of M. If M is a faithful multiplication module then $\text{ann}N = \text{ann}[N : M]$ for all submodules N of M. An R-module M is called a *cancellation* module if $IM = JM$ for some ideals I and J of R then $I = J$. Equivalently, $[IM : M] = I$ for all ideals I of R, see for example [1]. If M is a finitely generated faithful multiplication R-module, then M is cancellation, [20, Corollary to Theorem 9], from which one can easily verify that $[IN : M] = I[N : M]$ for all ideals I of R and all submodules N of M. Anderson, [8], defined $\theta\,(M) = \sum_{m\in M} [Rm : M]$ and showed the usefulness of this ideal in studying multiplication modules. He proved for example that if M is multiplication then $M = \theta\,(M)\, M$ and M is finitely generated multiplication if and only if $\theta\,(M) = R$, [8, Proposition 1 and Theorem 1]. It is shown in [6, Corollary 1.2] that M is multiplication if and only if $Rm = \theta\,(M)\, m$ for each $m \in M$. Equivalently, $R = \theta\,(M) + \text{ann}\,(m)$ for each $m \in M$. Several characterizations of multiplication modules are given in [9, Theorem 2.1]. If R is an integral domain and M a faithful multiplication R-module then M is finitely generated. For, if $m \in M$ then

$$R = \theta\,(M) + \text{ann}\,(m) = \theta\,(M) + \text{ann}\,[Rm : M] \subseteq \theta\,(M) + \text{ann}\,(a)\,,$$

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for each $a \in [Rm : M]$. But R is an integral domain. Thus $\mathrm{ann}\,(a) = 0$, and hence $\theta\,(M) = R$. This shows that M is finitely generated. Multiplication modules recieved in the last thirty years a considerable attention, see for example [5], [6], [8], [9], [12] and [20].

Let R be a ring and S the set of nonzero divisors of R and R_S the total quotient ring of R. If R is an integral domain then $R_S = Q(R)$, the quotient field of R. For a nonzero ideal I of R, let $I^{-1} = \{x \in R_S : xI \subseteq R\} = [R :_{R_S} I]$. I is an invertible ideal of R if $II^{-1} = R$. Let M be an R-module and let

$$T = \{t \in S : \text{ for all } m \in M,\ tm = 0 \text{ implies } m = 0\}.$$

Then T is a multplicatively closed subset of S and if M is torsion-free then $T = S$. In particular, $T = S$ if M is faithful multiplication, [12, Lemma 4.1]. Also, $T = S$ if M is an ideal of R. Let N be a nonzero submodule of M and let $N^{-1} = [M :_{R_T} N]$. Then N^{-1} is an R-submodule of R_T, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. Following [18], N is said to be *invertible in* M if $NN^{-1} = M$. Clearly M is invertible in M. Let I be an ideal of R and $e \in R$ an idempotent such that $\mathrm{ann}\,(e) \subseteq \mathrm{ann}I$ and $e \in II^{-1}$. This is equivalent to $\mathrm{ann}\,(e) = \mathrm{ann}I$ and $II^{-1} = Re$. Naoum called such an ideal quasi-invertible for e. He gave several properties of such ideals in [17]. In this paper, we call a submodule N of M *quasi-invertible for* e if $\mathrm{ann}\,(eM) \subseteq \mathrm{ann}N$ and $eM \subseteq NN^{-1}$ for some idempotent e of R. Suppose M is faithful multiplication. Then N is quasi-invertible for idempotent e if $NN^{-1} = eM$ and $\mathrm{ann}N = \mathrm{ann}\,(e)$. For, $\mathrm{ann}\,(eM) = \mathrm{ann}\,(e) \subseteq \mathrm{ann}N$ and hence $(1-e) \in \mathrm{ann}N$. So $N = eN$. It follows that

$$eM \subseteq NN^{-1} = eNN^{-1} \subseteq eM,$$

so that $NN^{-1} = eM$. Also,

$$\mathrm{ann}\,(e) \subseteq \mathrm{ann}N \subseteq \mathrm{ann}NN^{-1} = \mathrm{ann}\,(eM) = \mathrm{ann}\,(e)\,,$$

so that $\mathrm{ann}N = \mathrm{ann}(e)$. In section 1 of this paper, we investigate quasi-invertible submodules and give several properties for them. We show that quasi-invertible submodules of finitely generated faithful multiplication modules are finitely generated multiplication, and finitely generated projective submodules of finitely generated faithful multiplication modules are quasi-invertible, Theorem 1. Proposition 3 and Theorem 4 give necessary and sufficient conditions for the product, sum and intersecion of quasi-invertible submodules to be quasi-invertible.

Let R be an integral domain and M a faithful multiplication R-module (hence finitely generated). Let I be a nonzero ideal of R and N a nonzero submodule of M. Then $I^{-1} = (IM)^{-1}$ from which it follows that $N^{-1} = [N : M]^{-1}$, [2, Lemma 1]. Let $N_v = \left(N^{-1}\right)^{-1}$. Then $(IM)_v = I_v$ and hence $N_v = [N : M]_v$. Also, for all nonzero submodules K and N of M, $K \subseteq N$ implies $K_v \subseteq N_v$. In [4], the author defined divisorial submodules or v-submodules as a generalization of divisiorial ideals as follows: A nonzero submodule N is *divisorial* if $N = N_vM$. In this case $N_v = [N : M]$. Consequently, $M_v = R$. A submodule N of M is called v-submodule of *finite type* if $N = K_vM$ for some finitely generated submodule K of M. It is clear that N is a v-submodule (of finite type) if and only if $[N : M]$ is a v-ideal (of finite type). If N is invertible then N is v-submodule. For any nonzero submodule N of a multiplication R-module M, $[N : M] \subseteq [N : M]_v$ and hence $N \subseteq N_vM$. Let M be a faithful multiplication module over an integral domain R, I

a nonzero ideal of R and N a nonzero submodule of M. If either I_v is invertible or N_v is invertible then it is easily verified that $(IN)_v = I_vN_v$. A nonzero submodule N is called *v-invertible* if $(NN^{-1})_v = R$ and N_v is invertible if $N_vN_v^{-1} = R$.

Zafrullah [21] defined generalized Dedekind domains and completely integrally closed domains. An integral domain is a generalized Dedekind domain if $(AB)^{-1} = A^{-1}B^{-1}$ for any nonzero fractional ideals A and B of R, and an integral domain is called completely integrally closed if $(AA^{-1})_v = R$, that is, A is v-invertible for all nonzero fractional ideals A of R. Several properties and characterizations of such domains are investigated in [10] and [21]. In this paper, we say that an R-module M is a *generalized Dedekind module* if

$$([K : M]\, N)^{-1} = ([N : M]\, K)^{-1} = K^{-1}N^{-1},$$

for all nonzero submodules K and N of M, and M is a *completely integrally closed module* if N is v-invertible for each nonzero submodule N of M.

In section 2, we prove that if M is a faithful multiplication module over an integral domain then R is a generalized Dedekind (resp. completely integrally closed) domain if and only if M is a generalized Dedekind (resp. completely integrally closed) module, Propositions 9 and 12. In Theorem 14, we show that if M is a faithful multiplication module over an integral domain R, then M is a generalized Dedekind module if and only if M is a completely integrally closed module and $([K : M]\, N)_v = K_vN_v$ for all nonzero submodules K and N of M.

An integral domain R is called a $*$-domain if and only if $(AB)^{-1} = A^{-1}B^{-1} = (A_vB_v)^{-1}$ for all nonzero fractional ideals A and B of R, and it is called v-multiplication if for all proper ideals I and J of R such that $I_v \subseteq J_v$, there exists an ideal A of R such that $I_v = (AJ)_v$, see [10] and [21]. As a generalization of these concepts to the module case, we say that an R-module is a *$*$-module* if

$$([K : M]\, N)^{-1} = K^{-1}N^{-1} = (K_vN_v)^{-1},$$

for all nonzero submodules K and N of M, and M is called a *v-multiplication module* if for all proper submodules K and N of M with $K_v \subseteq N_v$ there exists an ideal I of R such that $K_v = (IN)_v$. We prove in Propositions 16 and 17 that if M is a faithful multiplication module over an integral domain R then M is a generalized GCD module (resp. generalized Dedekind module) if and only if M is a $*$-module and K^{-1} is of finite type (resp. and K_v is of finite type) for all nonzero finitely generated submodules K of M. Theorem 19 shows that if M is a faithful multiplication module over an integral domain then M is v-multiplication if and only if M is completely integrally closed.

An integral domain satisfies RTP (radical trace property) provided $II^{-1} = R$ or $II^{-1} = \sqrt{II^{-1}}$ for each nonzero ideal I of R. A closely related concept to RTP domains is that TPP domains defined as follows: A domain R satisfies the trace property for primary ideals (TPP) if Q is a primary ideal of R, then either Q is invertible or QQ^{-1} is a prime ideal of R, [14]. RTP and TPP are equivalent in the class of Prufer domains. As a generalization of these concepts to the module case we introduce and investigate RTP and TPP modules. We say that M is an *RTP module* if for each nonzero submodule N of M, N is invertible or $NN^{-1} = \mathrm{rad}_M\left(NN^{-1}\right)$, and M is a *TPP module* if for each primary submodule Q of M, Q is invertible or QQ^{-1} is a prime submodule of M. In section 3, we show that if M

is a faithful multiplication module over an integral domain R then R is RTP (resp. TPP) if and only if M is RTP (resp. TPP), Theorem 20. Theorem 23 gives several properties of faithful multiplication TPP modules and Proposition 24 shows that RTP and TPP modules coincide, if M is a one dimensional faithful multiplication module over an integral domain.

All rings in this paper are commutative with identity and all modules are unital. For the basic concepts used, we refer the reader to [13]-[16].

2. Quasi-Invertible Subdmodules of Multiplication Modules.

We start with the following theorem which gives several properties of quasi-invertible submodules of finitely generated faithful multiplication modules. It shows that if M is a finitely generated faithful multiplication R-module, then the class of quasi-invertible submodules of M includes the class of finitely generated projective submodules of M and it is included in the class of finitely generated multiplication submodules of M.

Theorem 1. *Let R be a ring and M a finitely generated faithful multiplication R-module. Let I be an ideal of R and N a submodule of M.*

(1) N is quasi-invertible for idempotent e if and only if $[N : M]$ is quasi-invertible for idempotent e.

(2) I is quasi-invertible for idempotent e if and only if IM is quasi-invertible for idempotent e.

(3) If N is quasi-invertible for idempotent e then N is finitely generated multiplication.

(4) If N is finitely generated projective then N is quasi-invertible for some idempotent e.

Proof. (1) Let N be quasi-invertible for idempotent e. Then $NN^{-1} = eM$ and $\mathrm{ann}N = \mathrm{ann}\,(eM)$. It follows by [2, Lemma 1] and [20, Corollary to Theorem 9] that

$$[N : M]\,[N : M]^{-1} = [N : M]\,N^{-1} = \left[NN^{-1} : M\right] = [eM : M] = Re,$$

and

$$\mathrm{ann}\,[N : M] = \mathrm{ann}N = \mathrm{ann}\,(eM) = \mathrm{ann}\,(e)\,.$$

Hence $[N : M]$ is quasi-invertible for e. Conversely, suppose $[N : M]$ is quasi-invertible for e. Then $[N : M]\,[N : M]^{-1} = Re$ and $\mathrm{ann}\,[N : M] = \mathrm{ann}\,(e)$. So

$$NN^{-1} = [N : M]\,M\,[N : M]^{-1} = eM,$$

and

$$\mathrm{ann}N = \mathrm{ann}\,[N : M] = \mathrm{ann}\,(e) = \mathrm{ann}\,(eM)\,.$$

This shows that N is quasi-invertible for e.

(2) Suppose I is quasi-invertible for e. Then $II^{-1} = Re$ and $\mathrm{ann}I = \mathrm{ann}\,(e)$. This implies that $IMI^{-1} = eM$. But $I^{-1} = (IM)^{-1}$, [2, Lemma 1]. Thus $IM\,(IM)^{-1} = eM$. Also,

$$\mathrm{ann}IM = \mathrm{ann}I = \mathrm{ann}\,(e) = \mathrm{ann}\,(eM)\,,$$

and IM is quasi-invertible for e. Conversely, let IM be quasi-invertible for e. Then $(IM)(IM)^{-1} = eM$. Since $I^{-1} = (IM)^{-1}$, [2, Lemma 1], and M is finitely generated faithful multiplication, hence cancellation, we infer that $II^{-1} = Re$. Moreover,

$$\mathrm{ann}I = \mathrm{ann}IM = \mathrm{ann}(eM) = \mathrm{ann}\,(e)\,,$$

and this shows that I is quasi-invertible for e.

(3) Since N is quasi-invertible for e, it follows by (1) that $[N:M]$ is quasi-invertible for e. So $Re = [N:M]\,[N:M]^{-1}$ and hence $e = \sum_{i=1}^{n} a_i x_i$ for some $a_i \in [N:M]$ and $x_i \in [N:M]^{-1}$. Let $a \in [N:M]$. Then $ae = \sum_{i=1}^{n} a_i\,(ax_i)$. As $a \in [N:M]$, we infer that

$$\mathrm{ann}\,(e) = \mathrm{ann}\,[N:M] \subseteq \mathrm{ann}\,(a)\,,$$

and hence $(1-e) \in \mathrm{ann}\,(a)$ which gives that $a = ae$. So $a = \sum_{i=1}^{n} a_i\,(ax_i)$. But for all $i \in \{1,...,n\}$, $ax_i \in [N:M]\,[N:M]^{-1} \subseteq R$. Thus $\{a_1, a_2, ...a_n\}$ generates $[N:M]$. Now, since M is finitely generated and multiplication, $N = [N:M]\,M$ is finitely generated. Next, let $J \subseteq [N:M]$ be an ideal of R. Then $\mathrm{ann}\,(e) = \mathrm{ann}\,[N:M] \subseteq \mathrm{ann}J$, and hence $(1-e) \in \mathrm{ann}J$. So $(1-e)\,J = 0$, and hence $J = eJ$. It follows that

$$J = eJ = J\,[N:M]^{-1}\,[N:M]\,.$$

Since $J \subseteq [N:M]\,,\, J\,[N:M]^{-1}$ is an ideal of R and hence $[N:M]$ is multiplication. This implies that $N = [N:M]\,M$ is a multiplication submodule of M, see [12, Corollary 1.4].

(4) Suppose N is a finitely generated projective submodule of M. It follows by [13, Proposition 3.30] that $N = (\mathrm{Tr}N)N$, $\mathrm{ann}N = \mathrm{ann}\mathrm{Tr}N$ and $\mathrm{Tr}N$ is a pure ideal of R, where $\mathrm{Tr}N = \sum_{f\in Hom(N,R)} f\,(N)$. (Recall that an ideal I of a ring R is pure if it is locally either zero or R, [15]) Since $N = (\mathrm{Tr}N)\,N$ and N is finitely generated, it follows by [15, Theorem 76] that $R = \mathrm{Tr}N + \mathrm{ann}N = \mathrm{Tr}N + \mathrm{ann}\mathrm{Tr}N$. So $1 = e + f$ for some $e \in \mathrm{Tr}N$ and $f \in \mathrm{ann}\mathrm{Tr}N$. Hence

$$\mathrm{Tr}N = e\mathrm{Tr}N \subseteq Re \subseteq \mathrm{Tr}N,$$

so that $\mathrm{Tr}N = Re$. Since $\mathrm{Tr}N$ is pure, e is idempotent and this gives that

$$\mathrm{ann}(eM) = \mathrm{ann}\,(e) = \mathrm{ann}\mathrm{Tr}N = \mathrm{ann}N.$$

Now, since M is finitely generated faithful multiplication, it follows by [5, Proposition 3.7] and [14, Lemma 4.2.2] that

$$Re = \mathrm{Tr}N = \mathrm{Tr}\,[N:M] = [N:M]\,[N:M]^{-1} = [N:M]\,N^{-1},$$

and hence $eM = [N:M]\,MN^{-1} = NN^{-1}$. This shows that N is quasi-invertible for some idempotent e. □

Before we give the next result we need a lemma which it may be compared with [17, Theorem 1.5].

Lemma 2. *Let R be a ring and M an R-module. If N is quasi-invertible for some idempotent e, then there exists a submodule K of N such that*

$$\mathrm{ann}N = \mathrm{ann}K = \mathrm{ann}(eM).$$

In particular, if M is finitely generated faithful multiplication and N is quasi-invertible in M then there exists $a \in [N : M]$ such that

$$\text{ann}N = \text{ann}(a) = \text{ann}(e).$$

Proof. Since $NN^{-1} = eM$, $em \in NN^{-1}$ for each $m \in M$. Let $em = \sum_{i=1}^{k} n_i x_i$ for some $n_i \in N$ and $x_i \in N^{-1}$. Let $x_i = \dfrac{r_i}{s_i}$ where $r_i \in R$ and $s_i \in T$, and let $s = \prod_{i=1}^{k} s_i$ and $t_i = \prod_{j \neq i} s_j$. Then

$$esm = \sum_{i=1}^{k} n_i x_i s_i t_i = \sum_{i=1}^{k} n_i r_i t_i,$$

where $r_i t_i \in R$. Let $n = \sum_{i=1}^{k} n_i r_i t_i$, Then $n = esm$. Note that n depends on m. So we write $n = n_m$. Hence

$$\sum_{m \in M} R n_m = es \sum_{m \in M}^{k} Rm = esM.$$

Let $K = \sum_{m \in M} R n_m$. Then K is a submodule of M and since $s \in T$ is a nonzero divisor, ann$N$ =ann$K$ =ann(eM). Now, assume that M is finitely generated, faithful and multiplication. It follows by Theorem 1 that $[N : M]$ is quasi-invertible and by [17, Theorem 1.5], there exists $a \in [N : M]$ such that ann$[N : M] =$ ann$(a) =$ ann(e). But ann$N =$ ann$[N : M]$. Thus the result follows. □

The next result gives necessary and sufficient conditions for the product of quasi-invertible submodules and quasi-invertible ideals to be quasi-invertible.

Proposition 3. *Let R be a ring and M a finitely generated faithful multiplication R-module. Let K and N be submodules of M and I an ideal of R.*

(1) If I is quasi-invertible for e and N is quasi-invertible for e then IN is quasi-invertible for e.

(2) If K and N are quasi-invertible for e, then $[K : M]\, N = [N : M]\, K$ is quasi-invertible for e.

(3) If IN is quasi-invertible for e and I is invertible then N is quasi-invertible for e.

(4) If IN is quasi-invertible for e and N is invertible then I is quasi-invertible for e.

Proof. (1) It follows by Lemma 2 and [17, Theorem 1.5] that there exist a submodule K of M and $a \in I$ such that

$$\text{ann}N = \text{ann}K = \text{ann}(eM) = \text{ann}\,(e)\,,$$

and

$$\text{ann}I = \text{ann}\,(a) = \text{ann}\,(e)\,.$$

Obviously, $aK \subseteq IN$ and hence ann$IN \subseteq$ ann(aK). Now, we show that ann$(aK) =$ ann(e). We have, $(1-e) \in$ ann$K$ and $(1-e) \in$ ann(a). So $(1-e)K = 0 = (1-e)a$. Hence $(1-e)aK = (1-e)^2 aK = 0$. It follows that ann$\,(e) = R(1-e) \subseteq$ ann(aK). On the other hand, let $x \in$ ann(aK). Then $xaK = 0$, and hence $xa \in$ ann$K =$ ann$\,(e)$. So $xae = 0$, and hence $xe \in$ ann$\,(a) =$ ann$\,(e)$. It follows that $xe = xe^2 = 0$, and hence $x \in \operatorname{ann}(e)$. This gives that $\operatorname{ann}(aK) \subseteq \operatorname{ann}(e)$, and hence,

$$\operatorname{ann} IN \subseteq \operatorname{ann}(aK) = \operatorname{ann}(e).$$

Also we have that

$$\operatorname{ann} IN \supseteq \operatorname{ann} I \cap \operatorname{ann} N = \operatorname{ann}(e),$$

so that $\operatorname{ann} IN = \operatorname{ann}(e)$. Alternatively, $\operatorname{ann} I = \operatorname{ann}(a) = \operatorname{ann}(e)$ and $\operatorname{ann} N = \operatorname{ann}(b) = \operatorname{ann}(e)$ for some $b \in [N : M]$. By the same argument, we get that $\operatorname{ann} IN = \operatorname{ann}(ab) = \operatorname{ann}(e)$. By Theorem 1(3) we have that I is a finitely generated multiplication ideal of R and N a finitely generated multiplication submodule of M. It follows by [12, Corollary 1.4] that IN is a finitely generated multiplication submodule of M and by [20, Theorem 11], IN is a finitely generated projective submodule of M. The result follows by Theorem 1(4).

(2) Follows by (1) and Theorem 1.

(3) Suppose IN is quasi-invertible for e and I is invertible. It follows that $(IN)(IN)^{-1} = eM$. But $(IN)^{-1} \subseteq I^{-1}N^{-1}$, see the proof of [1, Proposition 2.1 (4)]. Thus $eM = IN(IN)^{-1} \subseteq INI^{-1}N^{-1} = NN^{-1}$. Also, $\operatorname{ann} IN = \operatorname{ann}(e)$, hence $(1-e) \in \operatorname{ann} IN$. So $(1-e)IN = 0$ and hence $(1-e)N = 0$. This implies that $(1-e) \in \operatorname{ann} N$. Hence $\operatorname{ann}(e) \subseteq \operatorname{ann} N$, and N is quasi-invertible for e.

(4) Suppose IN is quasi-invertible for e. Then

$$eM = IN(IN)^{-1} \subseteq INI^{-1}N^{-1} = II^{-1}NN^{-1} \subseteq II^{-1}M.$$

But M is finitely generated faithful multiplication, hence cancellation. Thus $Re \subseteq II^{-1}$. Moreover, $(1-e) \in \operatorname{ann}(e) = \operatorname{ann} IN$. Hence $(1-e)IN = 0$, and hence

$$(1-e)IM = (1-e)INN^{-1} = 0.$$

Since M is faithful, $(1-e)I = 0$, and hence $\operatorname{ann}(e) \subseteq \operatorname{ann} I$ and this shows that I is quasi-invertible for e.

We close this section by a result showing necessary and sufficient conditions for the sum and intersection of quasi-invertible submodules to be quasi-invertible. □

Theorem 4. *Let R be a ring and M a finitely generated faithful multiplication R-module. Let K and N be submodules of M.*

(1) If K and N are quasi-invertible for e_1 and e_2 respectively, then $K+N$ is quasi-invertible for $e_1+e_2-e_1e_2$ if and only if $[K:N]+[N:K]=R$.

(2) If K and N are quasi-invertible for e_1 and e_2 respectively such that $K+N$ is quasi-invertible, then $K\cap N$ is quasi-invertible for e_1e_2.

(3) If $K+N$ and $K\cap N$ are quasi-invertible for e, then each of K and N is quasi-invertible for e.

Proof. (1) Suppose K and N are quasi-invertible for e_1 and e_2 respectively such that $[K:N]+[N:K]=R$. By Theorem 1(3) K and N are finitely generated multiplication submodules of M and by [20, Proposition 4], $K+N$ is a finitely generated multiplication submodule of M. Moreover,

$$\begin{aligned}\operatorname{ann}(K+N) &= \operatorname{ann} K \cap \operatorname{ann} N = \operatorname{ann}(e_1) \cap \operatorname{ann}(e_2)\\ &= R(1-e_1) \cap R(1-e_2) = R(1-e_1)(1-e_2)\\ &= R(1-(e_1+e_2-e_1e_2)) = \operatorname{ann}(e_1+e_2-e_1e_2).\end{aligned}$$

Since $e_1 + e_2 - e_1e_2$ is idempotent, we infer from [20, Theorem 11] that $K + N$ is a finitely generated projective submodule of M, and the result follows by Theorem 1(4). The converse is obvious by the fact that $K+N$ is finitely generated multiplication, [20, Proposition 4].

(2) We infer from Theorem 1(3) that K, N and $K + N$ are finitely generated multiplication submodules of M. It follows by [20, Theorem 8] that $K \cap N$ is a multiplication submodule of M. Also, by (1) we have that $[K : N] + [N : K] = R$. So $1 = x + y$ for some $x \in [K : N]$ and $y \in [N : K]$. It follows that

$$K \cap N = x\,(K \cap N) + y\,(K \cap N) \subseteq xN + yK \subseteq K \cap N,$$

so that $K \cap N = xN + yK$ is finitely generated. Finally, since $[K : N] + [N : K] = R$, we obtain from [7, Corollary 1.2] that

$$\begin{aligned}\mathrm{ann}(K \cap N) &= \mathrm{ann}K + \mathrm{ann}N = \mathrm{ann}\,(e_1) + \mathrm{ann}\,(e_2)\\ &= R\,(1 - e_1) + R\,(1 - e_2) = R\,((1 - e_1) + (1 - e_2) - (1 - e_1)\,(1 - e_2))\\ &= R\,(1 - e_1e_2) = \mathrm{ann}\,(e_1e_2)\,.\end{aligned}$$

As e_1e_2 is idempotent, $K \cap N$ is finitely generated projective, [20, Theorem 11] and by Theorem 1(4), $K \cap N$ is quasi-invertible for e_1e_2.

(3) By Theorem 1(3) each of $K + N$ and $K \cap N$ is a finitely generated multiplication submodule of M. [20, Theorem 8] shows that each of K and N is a multiplication submodule of M. The fact that K and N are finitely generated follows by [16, Ex. 2.3, page 13]. Since

$$\mathrm{ann}(K + N) = \mathrm{ann}K \cap \mathrm{ann}N = \mathrm{ann}\,(e)\,,$$

and by [20, Corollary 3 of Theorem 2 and Proposition 4],

$$\mathrm{ann}\,(K \cap N) = \mathrm{ann}K + \mathrm{ann}N = \mathrm{ann}\,(e)\,,$$

we infer that $\mathrm{ann}K \subseteq \mathrm{ann}\,(K \cap N) = \mathrm{ann}\,(e)$, and $\mathrm{ann}K \supseteq \mathrm{ann}(K + N) = \mathrm{ann}\,(e)$. This shows that $\mathrm{ann}K = \mathrm{ann}\,(e)$. Similarly, $\mathrm{ann}N = \mathrm{ann}\,(e)$. So each of K and N is a finitely generated projective submodule of M, [20, Theorem 11] and the result follows by Theorem 1(4). □

3. Generalized Dedekind Modules.

In this section we explore the relationship among generalized Dedekind modules, completely integrally closed modules, $*$-modules and v-multiplication modules. We start by a technical Lemma.

Lemma 5. *Let R be an integral domain and M a faithful multiplication R-module. Let K and N be nonzero submodules of M and $0 \neq m \in M$.*

(1) $Rm = (Rm)_v\,M$.

(2) $([K : M]\,m)_v = K_v(Rm)_v$.

(3) $(N_v)_v = N_v$.

(4) $([K : M]\,N)_v = (K_vN_v)_v$.

(5) $N^{-1} = (N_v)^{-1} = N_v^{-1}$.

Proof. (1) Since M is a faithful multiplication module over an integral domain, M is a D_1-module (that is, every nonzero cyclic submodule is invertible), [18, page 402]. So Rm is invertible and hence it is a v-submodule of M.

(2) Since Rm is invertible, $Rm = (Rm)_v M$. Hence $(Rm)_v = [Rm : M]$ is an invertible ideal of R, and hence

$$([K : M]\, m)_v = [K : M]_v\, (Rm)_v = K_v (Rm)_v.$$

(3)

$$(N_v)_v = ([N : M]_v)_v = [N : M]_v = N_v.$$

(4) We have

$$\begin{aligned} ([K : M]\, N)_v &= [[K : M]\, N : M]_v = ([K : M]\, [N : M])_v \\ = ([K : M]_v\, [N : M])_v &= (K_v\, [N : M])_v = (K_v\, [N : M]_v)_v = (K_v N_v)_v\,. \end{aligned}$$

(5)

$$N^{-1} = [N : M]^{-1} = ([N : M]_v)^{-1} = (N_v)^{-1}\,.$$

Also,

$$N^{-1} = [N : M]^{-1} = [N : M]_v^{-1} = N_v^{-1}.$$

□

Compare the following lemma with [21, Lemma 1.2].

Lemma 6. *Let R be an integral domain and M a faithful multiplication R-module. Let N be a nonzero submodule of M. Then N_v is invertible if and only if $([K : M]\, N)^{-1} = ([N : M]\, K)^{-1} = K^{-1}N^{-1}$ for all submodules K of M.*

Proof. We first prove that if $([K : M]\, N)^{-1} = K^{-1}N^{-1}$ then $([K : M]\, N_v)^{-1} = K^{-1}N^{-1}$. This follows from the fact that

$$\begin{aligned} ([K : M]\, N)^{-1} &= ([K : M]\, N : M])^{-1} = ([K : M]\, [N : M])^{-1} \\ &= ([K : M]_v\, [N : M])^{-1} = ([K : M]_v\, [N : M]_v)^{-1} \\ &= ([K : M]\, [N : M]_v)^{-1} = ([K : M]\, N_v)^{-1}\,. \end{aligned}$$

Suppose now N_v is invertible. Then $N_v N_v^{-1} = R$ and by Lemma 5, $N_v N^{-1} = R$. Let K be any nonzero submodule of M and let $x \in ([K : M]\, N)^{-1}$. Then $x\, [K : M]\, N \subseteq M$. But $[K : M]\, N = [N : M]\, K$. Thus $x\, [N : M]\, K \subseteq M$, and hence $x\, [N : M] \subseteq K^{-1}$. This gives $x\, [N : M]_v = (x\, [N : M])_v \subseteq K_v^{-1} = K^{-1}$ and this implies that $x \in K^{-1}\, [N : M]_v^{-1} = K^{-1}\, [N : M]^{-1} = K^{-1}N^{-1}$. So $([K : M]\, N)^{-1} \subseteq K^{-1}N^{-1}$. For the reverse inclusion, let $x \in K^{-1}N^{-1}$. Since N_v is invertible,

$$xN_v \subseteq K^{-1}N_v N^{-1} = K^{-1}N_v N_v^{-1} = K^{-1},$$

and hence $xN_v K \subseteq M$. This shows that $xN_v\, [K : M] = [xN_v K : M] \subseteq R$ and hence $x \in ([K : M]\, N_v)^{-1} = ([K : M]\, N)^{-1}$. So $K^{-1}N^{-1} \subseteq ([K : M]\, N)^{-1}$ and hence $([K : M]\, N)^{-1} = K^{-1}N^{-1}$. Conversely, suppose that for all nonzero submodules K and N of M, $([K : M]\, N)^{-1} = K^{-1}N^{-1}$. Put $K = [N : M]^{-1} M$. Then $[K : M] = \left[[N : M]^{-1} M : M\right] = [N : M]^{-1} = N^{-1}$. It follows that

$$K^{-1} = [K : M]^{-1} = \left(N^{-1}\right)^{-1} = N_v.$$

So $([K:M]\,N_v)^{-1} = K^{-1}N^{-1}$ implies $\left(N_vN^{-1}\right)^{-1} = N_vN^{-1}$. Now, since $N^{-1} = N_v^{-1}$, $N_vN^{-1} \subseteq R$. So $R \supseteq \left(N_vN^{-1}\right)^{-1}$. But $N_vN^{-1} = \left(N_vN^{-1}\right)^{-1}$. Thus $R \supseteq N_vN^{-1}$ and hence $N_vN^{-1} = R$. This shows that N_v is invertible. $\square$

Recalling that an invertible submodule is a v-submodule we make the following statement.

Corollary 7. *Let R be an integral domain and M a faithful multiplication R-module. Let N be a nonzero submodule of M. Then N is invertible if and only if N is a v-submodule and for all nonzero submodules K of M, $([K:M]\,N)^{-1} = K^{-1}N^{-1}$.*

An immediate consequence of Lemma 6 is the following.

Corollary 8. *Let R be an integral domain and M a faithful multiplication R-module. Then M is a generalized Dedekind module if and only if N_v is invertible for all nonzero submodules N of M.*

The next result shows that an integral domain R is a generalized Dedekind domain if and only if there exists a faithful multiplication generalized Dedekind R-module.

Proposition 9. *Let R be an integral domain and M a faithful multiplication R-module. Then R is a generalized Dedekind domain if and only if M is a generalized Dedekind module.*

Proof. Let R be a generalized Dedekind domain and N a nonzero submodule of M. Then $[N:M]$ is a nonzero ideal of R, and hence $[N:M]_v$ is invertible. But $N_v = [N:M]_v$. Thus N_v is invertible and hence M is a generalized Dedekind module. Conversely, let M be a generalized Dedekind module and I a nonzero ideal of R. Hence IM is a nonzero submodule of M and hence $(IM)_v$ is invertible. But $I_v = (IM)_v$. Thus I_v is invertible and R is a generalized Dedekind domain. $\square$

The author defined in [4] an R-module M is a *generalized GCD module (GGCD module)* if the set of invertible submodules of M is closed under intersection. It is shown, [4, Theorem 16] that a faithful multiplication module over an integral domain is a GGCD module if and only if every v-submodule of finite type is invertible.

As a consequence of the above result we give the following. Compare with [21, Corollary 1.5].

Corollary 10. *Let R be an integral domain and M a faithful multiplication R-module. If M is a generalized Dedekind module then M is a GGCD module.*

Proof. By Proposition 9, R is a generalized Dedekind domain and by [21, Corollary 1.5], R is a GGCD domain. The result follows by [4, Proposition 15]. $\square$

Before we give the next result which shows a relationship between integrally closed domains and integrally closed modules, we need a lemma.

Lemma 11. *Let R be an integral domain and M a faithful multiplication R-module. Let N be a nonzero submodule of M and I a nonzero ideal of R.*

(1) N is v-invertible if and only if $[N:M]$ is v-invertible.

(2) I is v-invertible if and only if IM is v-invertible.

Proof. (1) Let N be v-invertible. Then $\left(NN^{-1}\right)_v = R$. It follows by Lemma 5 that

$$\begin{aligned} R &= (N[N:M]^{-1})_v = \left(N_v\,[N:M]^{-1}\right)_v = \left([N:M]_v\,[N:M]^{-1}\right)_v \\ &= \left([N:M]\,[N:M]^{-1}\right)_v, \end{aligned}$$

and hence $[N:M]$ is v-invertible. Conversely, suppose $[N:M]$ is v-invertible. Then

$$\begin{aligned} R &= \left([N:M]\,[N:M]^{-1}\right)_v = \left([N:M]_v\,[N:M]^{-1}\right)_v = \left(N_v\,[N:M]^{-1}\right)_v \\ &= (NN^{-1})_v, \end{aligned}$$

and N is v-invertible.

(2) Let I be v-invertible. Then

$$R = \left(II^{-1}\right)_v = \left(I_vI^{-1}\right)_v = \left((IM)_v\,(IM)^{-1}\right)_v = \left(IM\,(IM)^{-1}\right)_v,$$

and this shows that IM is v-invertible. Conversely, suppose IM is v-invertible. Then

$$R = \left(IM\,(IM)^{-1}\right)_v = \left((IM)_v\,(IM)^{-1}\right)_v = \left(I_vI^{-1}\right)_v = \left(II^{-1}\right)_v,$$

and hence I is v-invertible. □

Proposition 12. *Let R be an integral domain and M a faithful multiplication R-module. Then R is a completely integrally closed domain if and only if M is a completely integrally closed module.*

Proof. Suppose R is integrally closed and N a nonzero submodule of M. Then $[N:M]$ is a nonzero ideal of R. It follows that $[N:M]$ is v-invertible and by Lemma 11, N is v-invertible. Hence M is completely integrally closed. Conversely, suppose M is completely integrally closed and I a nonzero ideal of R. Then IM is a nonzero submodule of M. It follows that IM is v-invertible and by Lemma 11, I is v-invertible. Hence R is completely integrally closed. □

The next result shows that generalized Dedekind modules are in fact completely integrally closed modules.

Proposition 13. *Let R be an integral domain and M a faithful multiplication R-module. If M is a generalized Dedekind module then M is completely integrally closed.*

Proof. Suppose N is a nonzero submodule of M. Then N_v is invertible and hence $N_vN^{-1} = R$. It follows by Lemma 5 that

$$R = R_v = \left(N_vN^{-1}\right)_v = \left(NN^{-1}\right)_v.$$

So N is v-invertible and hence M is completely integrally closed. Alternatively, if M is a generalized Dedekind module, then Proposition 9 says that R is a generalized Dedekind domain. It follows by [21, Proposition 1.8] that R is completely integraly closed and by Proposition 12, M is completely integrally closed. □

The next result leads to an interesting characterization of generalized Dedekind modules.

Theorem 14. *Let R be an integral domain and M a faithful multiplication R-module. Then M is a generalized Dedekind module if and only if M is completely integrally closed and for all nonzero submodules K and N of M, $([K:M]\,N)_v = K_v N_v$.*

Proof. Suppose M is a generalized Dedekind module. By Proposition 13, M is completely integrally closed. Let K and N be nonzero submodules of M. Then $([K:M]\,N)^{-1} = K^{-1}N^{-1}$, and hence

$$([K:M]\,N)_v = \left(([K:M]\,N)^{-1}\right)^{-1} = \left(K^{-1}N^{-1}\right)^{-1} = (K^{-1})^{-1}(N^{-1})^{-1} = K_v N_v.$$

Note that since M is a generalized Dedekind module, R is a generalized Dedekind domain, and hence

$$\begin{aligned}\left(K^{-1}N^{-1}\right)^{-1} &= \left([K:M]^{-1}[N:M]^{-1}\right)^{-1} = ([K:M]^{-1})^{-1}([N:M]^{-1})^{-1} \\ &= (K^{-1})^{-1}(N^{-1})^{-1}.\end{aligned}$$

Conversely, suppose M is completely integrally closed and for all nonzero submodules K and N of M, $([K:M]\,N)_v = K_v N_v$. Now, since K is v-invertible, $\left(KK^{-1}\right)_v = R$ and hence

$$R = \left(KK^{-1}\right)_v = [(K[K:M]^{-1})_v = K_v[K:M]_v^{-1} = K_v K_v^{-1} = K_v K^{-1},$$

and hence K_v is invertible. This shows that M is a generalized Dedekind module. □

The next result shows that an integral domain R is a $*$-module if and only if there exists a faithful multiplication $*$-module over R.

Proposition 15. *Let R be an integral domain and M a faithful multiplication R-module. Then R is a $*$-domain if and only if M is a $*$-module.*

Proof. Let R be a $*$-domain. Let K and N be nonzero finitely generated submodules of M. Then $[K:M]$ and $[N:M]$ are nonzero finitely generated ideals of R. It follows that

$$\begin{aligned}([K:M]\,N)^{-1} &= [[K:M]\,N:M]^{-1} = ([K:M]\,[N:M])^{-1} \\ &= [K:M]^{-1}\,[N:M]^{-1} = \left([K:M]_v\,[N:M]_v\right)^{-1},\end{aligned}$$

But $[K:M]^{-1}\,[N:M]^{-1} = K^{-1}N^{-1}$ and $\left([K:M]_v\,[N:M]_v\right)^{-1} = (K_vN_v)^{-1}$. Hence $([K:M]\,N)^{-1} = K^{-1}N^{-1} = (K_vN_v)^{-1}$and this implies that M is a $*$-module. Conversely, suppose I and J are nonzero finitely generated ideals of R. Then IM and JM are nonzero finitely generated submodules of M. Hence

$$(IJM)^{-1} = (IM)^{-1}\,(JM)^{-1} = \left((IM)_v\,(JM)_v\right)^{-1}.$$

Since $I^{-1} = (IM)^{-1}, I_v = (IM)_v, J^{-1} = (JM)^{-1}$ and $J_v = (JM)_v$, we get that $(IJ)^{-1} = I^{-1}J^{-1} = (I_vJ_v)^{-1}$, and hence R is a $*$-domain. □

The next result gives a nice characterization to GGCD modules. Compare with [21, Corollary 1.7].

Proposition 16. *Let R be an integral domain and M a faithful multiplication R-module. Then M is a GGCD module if and only if M is a $*$-module and K^{-1} is of finite type for all finitely generated submodules K of M.*

Proof. Let M be a GGCD module. Let K be a finitely generated submodule of M. It follows by [4, Theorem 16] that K_v is invertible. Hence $K^{-1} = K_v^{-1}$ is invertible and hence K^{-1} is of finite type. Next, since M is a GGCD module, it follows by [4, Proposition 15] that R is a GGCD domain and by [21, Corollary 1.7], R is a $*$-domain. Proposition 15 shows that M is a $*$-module. Conversely, suppose M is a $*$-module and K a finitely generated submodule of M. To prove that K_v is invertible. By the assumption K^{-1} is of finite type. Let $K^{-1} = [K : M]^{-1} = N_v$ for some finitely generated submodule N of M. Hence

$$\left(K_v K^{-1}\right)^{-1} = \left(K K^{-1}\right)^{-1} = (K N_v)^{-1} = \left(K\left[N : M\right]_v\right)^{-1} = \left(K\left[N : M\right]\right)^{-1}.$$

But M is a $*$-module. Thus $(K[N : M])^{-1} = K^{-1}[N : M]^{-1}$. Since $[N : M]^{-1} = [N : M]_v^{-1} = N_v^{-1} = \left(K^{-1}\right)^{-1} = K_v$, we get that $(K[N : M])^{-1} = K^{-1}K_v$, and hence $\left(K_v K^{-1}\right)^{-1} = K_v K^{-1}$. It follows that $K_v K^{-1} = R$ and this shows that K_v is invertible. Hence M is a GGCD module. □

The next result gives another characterization of generalized Dedekind modules. Compare with [21, Proposition 1.8].

Proposition 17. *Let R be an integral domain and M a faithful multiplication R-module. Then M is a generalized Dedekind module if and only if M is a $*$-module and for every submodule K of M, K_v is of finite type.*

Proof. Obviously, generalized Dedekind modules are $*$-modules. Let K be a submodule of M. Then K_v is invertible and hence K_v is of finite type, [1, Proposition 2.1]. Conversely, let M be a $*$-module and K_v is of finite type for all submodules K of M. Let N be a submodule of M. Then

$$\left(\left[K : M\right] N\right)^{-1} = \left(K_v N_v\right)^{-1}.$$

Now, since each of K_v and N_v is of finite type, we get that

$$\left(K_v N_v\right)^{-1} = K_v^{-1} N_v^{-1} = K^{-1} N^{-1}.$$

So $([K : M] N)^{-1} = K^{-1}N^{-1}$, and hence M is a generalized Dedekind module. □

The next result gives a relationship between v-multiplication domains and v-multiplication modules.

Proposition 18. *Let R be an integral domain and M a faithful multiplication R-module. Then M is a v-multiplication module if and only if R is a v-multiplication domain.*

Proof. Suppose M is a v-multiplication module. Let I and J be proper ideals of R such that $I_v \subseteq J_v$. Then IM and JM are proper submodules of M with $(IM)_v \subseteq (JM)_v$. There exists an ideal A of R such that $(IM)_v = (AJM)_v$. It follows that $I_v = (AJ)_v$ and this shows that R is a v-multiplication domain. Conversely, let R be a v-multiplication domain and K and N proper submodules of M with $K_v \subseteq N_v$. Then $[K : M]$ and $[N : M]$ are proper ideals of R with $[K : M]_v \subseteq [N : M]_v$. There exists an ideal I of R such that

$$[K : M]_v = (I\left[N : M\right])_v = [IN : M]_v.$$

It follows that $K_v = (IN)_v$, and hence M is a v-multiplication module. □

We enclose this section by a further property of v-multiplication modules.

Theorem 19. *Let R be an integral domain and M a faithful multiplication R-module. Then M is v-multiplication if and only if M is completely integrally closed.*

Proof. Let M be completely integrally closed module. Let K and N be proper submodules of M such that $K_v \subseteq N_v$. Let $C = [K : M]\, N^{-1}$. Then C is an ideal of R. For, since $K_v \subseteq N_v, [K : M]_v \subseteq [N : M]_v$ and hence

$$[N : M]^{-1} = [N : M]_v^{-1} \subseteq [K : M]_v^{-1} = [K : M]^{-1}.$$

It follows that

$$[K : M]\, N^{-1} = [K : M]\, [N : M]^{-1} \subseteq [K : M]\, [K : M]^{-1} \subseteq R.$$

Now,

$$\begin{aligned}(CN)_v &= \left([K : M]\, N^{-1}N\right)_v = \left([K : M]\left(NN^{-1}\right)_v\right)_v \\ &= \left([K : M]\, R\right)_v = [K : M]_v = K_v,\end{aligned}$$

and hence M is v-multiplication. Conversely, suppose M is v-multiplication and N a nonzero submodule of M. We need to show that N is v-invertible. Let $0 \neq m \in N$. Hence $(Rm)_v \subseteq N_v$ and hence there exists an ideal I of R such that $(Rm)_v = (IN)_v$. Since M is a faithful multiplication module over an integral domain, M is a D_1-module, [18] and hence Rm is invertible. It follows by Lemma 5 that $Rm = (Rm)_v M = (IN)_v\, M$. Hence

$$[Rm : M] = (IN)_v = [IN : M]_v = (I\,[N : M])_v\,.$$

Now, since Rm is invertible, $[Rm : M]$ is an invertible ideal of R and hence

$$\begin{aligned}R &= [Rm : M]\,[Rm : M]^{-1} = (I\,[N : M])_v\,[Rm : M]^{-1} \\ &= (I\,[N : M])_v\,[Rm : M]_v^{-1} = (I\,[N : M]\,[Rm : M]^{-1})_v.\end{aligned}$$

Note that $I\,[Rm : M]^{-1}\,[N : M]_v \subseteq R$ and hence

$$I\,[Rm : M]^{-1} \subseteq [N : M]_v^{-1} = [N : M]^{-1}.$$

Therefore,

$$R = \left(I\,[Rm : M]^{-1}\,[N : M]\right)_v \subseteq \left([N : M]^{-1}\,[N : M]\right)_v \subseteq R,$$

so that $R = \left([N : M]\,[N : M]^{-1}\right)_v$. This show that $[N : M]$ is v-invertible and by Lemma 11, N is v-invertible. This completes the proof of the theorem. $\square$

4. Trace Properties for Multiplication Modules.

We investigated trace properties of faithful multiplication modules in [2, Section 4]. In this section, we continue our study and give several properties of faithful multiplication modules over integral domains that go under the general heading of trace properties. We start by the following result.

Theorem 20. *Let R be an integral domain and M a faithful multiplication R-module.*

(1) R is an RTP domain if and only if M is an RTP module.

(2) R is a TPP domain if and only if M is a TPP module.

Proof. (1) Suppose R is an RTP domain. Let N be a nonzero submodule of M. Then $[N:M]$ is a nonzero ideal of R. Hence either $[N:M]\,[N:M]^{-1} = R$ from which we get that $NN^{-1} = [N:M]\,M\,[N:M]^{-1} = M$, or $[N:M]\,[N:M]^{-1} = \sqrt{[N:M]\,[N:M]^{-1}}$. From the second case we get that

$$\begin{aligned} NN^{-1} &= [N:M]\,[N:M]^{-1}\,M = \sqrt{[N:M]\,[N:M]^{-1}}M \\ &= \sqrt{[N:M]\,N^{-1}}M = \sqrt{[NN^{-1}:M]}M = \mathrm{rad}_M\left(NN^{-1}\right), \end{aligned}$$

see [12, Theorem 2.12]. So M is an RTP module. Conversely, suppose M is an RTP module. Let I be a nonzero ideal of R. Then IM is a nonzero submodule of M. It follows that either $IM\,(IM)^{-1} = M$ and this implies that $II^{-1}M = M$, and hence $II^{-1} = R$, or $IM\,(IM)^{-1} = \mathrm{rad}_M\left(IM\,(IM)^{-1}\right)$. By [12, Theorem 2.12] we get that

$$\begin{aligned} \left(II^{-1}\right)M = (IM)\,I^{-1} &= (IM)\,(IM)^{-1} = \mathrm{rad}_M\left(IM\,(IM)^{-1}\right) \\ &= \sqrt{\left[IM\,(IM)^{-1}:M\right]}M = \sqrt{II^{-1}}M, \end{aligned}$$

and hence $II^{-1} = \sqrt{II^{-1}}$. This shows that R is an RTP domain.

(2) Suppose R is a TPP domain. Let Q be a primary submodule of M. Then $[Q:M]$ is a primary ideal of R. We discuss two cases:

Case 1: $[Q:M]$ is an invertible ideal of R. It follows by [18, Remark 3.2] that $Q = [Q:M]\,M$ is an invertible submodule of M.

Case 2: $[Q:M]\,[Q:M]^{-1}$ is a prime ideal of R. It follows that $QQ^{-1} = [Q:M]\,M\,[Q:M]^{-1}$ is a prime submodule of M, [12, Corollary 2.11]. Hence M is a TPP module. Conversely, let M be a TPP module and $Q^{'}$ a primary ideal of R. Since M is a finitely generated faithful multiplication module, $Q^{'}M$ is a primary submodule of M. We have two cases:

Case 1: $Q^{'}M$ is invertible and by [1, Proposition 2.1] and [18, Lemma 3.3], $Q^{'} = \left[Q^{'}M:M\right]$ is an invertible ideal of R.

Case 2: $\left(Q^{'}M\right)\left(Q^{'}M\right)^{-1}$ is a prime submodule of M. Hence

$$Q^{'}Q^{'-1} = \left[\left(Q^{'}Q^{'-1}\right)M:M\right] = \left[\left(Q^{'}M\right)\left(Q^{'}M\right)^{-1}:M\right]$$

is a prime ideal of R. Hence R is a TPP domain. □

A submodule Q of an R-module M is called *P-Primary* if Q is a primary submodule of M and $\mathrm{rad}_M Q = P$. It is showm, [2, Lemma 4] that if M is a finitely generated multiplication R-module then Q is a P-Primary submodule of M, if and only if $[Q:M]$ is a $[P:M]$-Primary ideal of R. Assuming further that M is faithful, then Q is a P-Primary ideal of R if and only if QM is a PM-Primary submodule of M. We note further properties of RTP modules. We first give a lemma which it may compared with [14, Lemmas 4.2.14 and 4.2.15].

Lemma 21. *Let R be an integral domain and M a faithful multiplication R-module.*

(1) Let N be a submodule of M such that N contains a P-primary submodule Q and not contained in P. Then $N^{-1} \subseteq \left[Q :_{Q(R)} Q\right]$.

(2) Let N be a submodule of M and Q a P-primary submodule of M such that $Q \subset N \subseteq QQ^{-1}$ and N not contained in P, then

$$N^{-1} = \left(QQ^{-1}\right)^{-1} = \left[QQ^{-1} :_{Q(R)} QQ^{-1}\right] = \left[Q :_{Q(R)} Q\right].$$

Proof. (1) Since Q is P-primary, it follows by [2, Lemma 4] that $[Q : M]$ is a $[P : M]$-primary ideal of R. Also, $[N : M]$ contains $[Q : M]$ and is not contained in $[P : M]$. It follows by [14, Theorem 2.4.14] that

$$N^{-1} = [N : M]^{-1} \subseteq \Big[[Q : M]_{:Q(R)} [Q : M]\Big] = \left[Q :_{Q(R)} Q\right].$$

(2) $[Q : M]$ is a $[P : M]$-primary ideal of R, [2, Lemma 4]. Also, $[Q : M] \subset [N : M] \subseteq [Q : M]\,[Q : M]^{-1}$ and $[N : M]$ is not contained in $[P : M]$. It follows by [14, Lemma 4.2.15] that

$$\begin{aligned} N^{-1} &= [N : M]^{-1} = \Big([Q : M]\,[Q : M]^{-1}\Big)^{-1} = \left(QQ^{-1}\right)^{-1} \\ &= [[Q : M][Q : M]^{-1} :_{Q(R)} [Q : M][Q : M]^{-1} \\ &= \left[QQ^{-1} :_{Q(R)} QQ^{-1}\right] = \left[Q :_{Q(R)} Q\right]. \end{aligned}$$

□

Compare the next theorem with [14, Theorem 4.2.16].

Theorem 22. *Let R be an integral domain and M a faithful multiplication R-module. If M is an RTP module then for each nonzero nonmaximal prime submodule P of R, P is divisorial and $P^{-1} = \left[P :_{Q(R)} P\right]$.*

Proof. By Theorem 20, R is an RTP domain. Let P be a nonzero nonmaximal prime submodule of M. Then $[P : M]$ is a nonzero nonmaximal prime ideal of R, see [12, Theorem 2.5 and Corollary 2.11]. It follows by [14, Theorem 4.2.16] that $[P : M] = [P : M]_v$ from which one gets that $P = [P : M]\,M = [P : M]_v\,M = P_vM$, that is, P is divisorial and $[P : M]^{-1} = \left[[P : M] :_{Q(R)} [P : M]\right]$, which implies that $P^{-1} = \left[P :_{Q(R)} P\right]$, [2] and [3]. Alternatively, let $\left[P :_{Q(R)} P\right] = R$. Let N be a submodule of M containing P. By Lemma 21 we have that $N^{-1} \subseteq \left[P :_{Q(R)} P\right] = R \subseteq N^{-1}$, so that $N^{-1} = R$. Thus $NN^{-1} = N$ is a radical submodule of M since M is an RTP module. Since P is a nonmaximal prime submodule of M, let $m \in M/P$ so that $N = [Rm : M] + P \neq M$. It is easily verified that N is a non radical submodule of M, a contradiction. Hence $\left[P :_{Q(R)} P\right] \neq R$, and this implies that P cannot be invertible. We claim that $PP^{-1} = P$. Suppose not. Then there exists $m \in PP^{-1} \setminus P$ and let $N = [Rm : M]\,m + P \neq M$. By assumption NN^{-1} is a radical submodule of M which implies that $m \in NN^{-1}$. It follows by Lemma 21 that

$$N^{-1} = \left(PP^{-1}\right)^{-1} = \left[PP^{-1} :_{Q(R)} PP^{-1}\right] = \left[P :_{Q(R)} P\right].$$

Thus $Rm = Rp + u\,[Rm : M]\,m$ for some $u \in N^{-1}$ and $p \in P$. Hence $m = rp + usm$ for some $r \in R$ and $s \in [Rm : M]$. Since $sM \subseteq Rm$, we infer that $rp = (1 - us)\,m \in P$. As $m \notin P$,

$$1 - us \in [P : M] \subseteq [N : M] \subseteq [P : M]^{-1}\,[P : M].$$

It follows that $[P : M]^{-1}\,[P : M] = R$. Hence $[P : M]$ is invertible and hence P is an invertible submodule of M, a contradiction. Hence $P^{-1} = \left[P :_{Q(R)} P\right]$. Since $P^{-1} \neq R$, $P_v M$ is a proper submodule of M and hence

$$P^{-1} = P_v^{-1} \supseteq \left[P_v :_{Q(R)} P_v\right] = \left[P_v M :_{Q(R)} P_v M\right] = \left[P :_{Q(R)} P\right] = P^{-1}.$$

So we have that

$$P_v^{-1} = \left[P_v :_{Q(R)} P_v\right] = \left[P :_{Q(R)} P\right] = P^{-1}.$$

If $P_v M \supset P$, let $m \in P_v M / P$ and we get a contradiction as in the previous argument. Therefore P is a divisorial submodule of M, and this finishes the proof of the theorem. □

The next theorem gives several properties of TPP modules. It should be compared with [14, Theorem 4.2.17, Lemma 4.2.18, Theorem 4.2.20, Corollary 4.2.21 and Theorem 4.2.22].

Theorem 23. *Let R be an integral domain and M a TPP faithful multiplication R-module.*

(1) If Q is a P-primary submodule of M then either $QQ^{-1} = P$ or $QQ^{-1} = M$ and P is a maximal submodule of M.

(2) If N is a submodule of M such that $N^{-1} = \left[N :_{Q(R)} N\right]$, then for each prime submodule P of M that is minimal over N, $N_{[P:M]} = P_{[P:M]}$.

(3) M_P is a TPP module for each prime ideal P of R.

(4) If Q is a submodule of M, then either $QQ^{-1} = rad_M Q$ or Q is invertible and $rad_M Q$ is maximal.

(5) Each nonmaximal prime submodule P of M is divisorial and $P^{-1} = \left[P :_{Q(R)} P\right]$.

Proof. (1) By Theorem 20, R is a TPP domain. Since Q is P-primary, it follows from [2, Lemma 4] that $[Q : M]$ is a $[P : M]$-primary ideal of R. Hence by [14, Theorem 4.2.17] we have either $[Q : M]\,[Q : M]^{-1} = [P : M]$ from which we get that

$$QQ^{-1} = [Q : M]^{-1}\,M\,[Q : M] = [P : M]\,M = P,$$

or $[Q : M]$ is invertible and $[P : M]$ is a maximal ideal of R. Since M is finitely generated faithful multiplication R-module, $Q = [Q : M]\,M$ is invertible and $P = [P : M]\,M$ is maximal, [18, Remark 3.2] and [12, Theorem 2.5].

(2) Let $Q = N_{[P:M]} \cap M$. Then $Q^{-1} \subseteq N^{-1}$. Since M is a TPP module, $P \subseteq QQ^{-1}$. (For, if $m \in P$, then $[Rm : M]^n \subseteq Q \subseteq QQ^{-1}$ for some positive integer n because Q is P-primary. Since M is a TPP module, QQ^{-1} is a prime submodule of M or $QQ^{-1} = M$. This implies that $m \in QQ^{-1}$.). Thus

$$N_{[P:M]} \subseteq P_{[P:M]} \subseteq \left(QQ^{-1}\right)_{[P:M]} \subseteq \left(NN^{-1}\right)_{[P:M]} = N_{[P:M]},$$

so that $N_{[P:M]} = P_{[P:M]}$. Alternatively, R is a TPP domain. Assume N is a submodule of M such that $N^{-1} = \left[N :_{Q(R)} N\right]$. It follows by [2, Lemma 1] that

$[N:M]^{-1}=\left[[N:M]:_{Q(R)}[N:M]\right]$. Let P be a prime submodule of M minimal over N. Then $[P:M]$ is prime ideal of R minimal over $[N:M]$. It follows by [14, Lemma 4.2.18] that $[N:M]_{[P:M]}=[P:M]_{[P:M]}$, and hence

$$N_{[P:M]}=([N:M]\,M)_{[P:M]}=([P:M]\,M)_{[P:M]}=P_{[P:M]}.$$

(3) Again by Theorem 20, R is a TPP domain. It follows by [14, Theorem 4.2.20] that R_P is a TPP domain for each prime ideal P of R. Since M is a finitely generated faithful multiplication R-module, $M_P\cong R_P$, and the result follows.

(4) R is a TPP domain. Let Q be a primary submodule of M. Then $[Q:M]$ is a primary ideal of R. It follows by [14, Corollary 4.2.21] that either $[Q:M]\,[Q:M]^{-1}=\sqrt{[Q:M]}$ and by [12, Theorem 12] we get that

$$QQ^{-1}=[Q:M]\,M\,[Q:M]^{-1}=\sqrt{[Q:M]}M=\mathrm{rad}_M Q,$$

or $[Q:M]$ is invertible and $\sqrt{[Q:M]}$ is a maximal ideal of R. From the last case we get that $Q=[Q:M]\,M$ is an invertible submodule of M, and by [12, Theorem 2.5] $\mathrm{rad}_M Q=\sqrt{[Q:M]}M$ a maximal submodule of M.

(5) Let P be a nonmaximal prime ideal of M. Then $[P:M]$ is a nonmaximal prime ideal of R, [12, Theorem 2.5 and Corollary 2.11]. Since R is a TPP domain, $[P:M]=[P:M]_v$. Hence

$$P=[P:M]\,M=[P:M]_v\,M=P_vM,$$

and $[P:M]^{-1}=\Big[[P:M]_{:Q(R)}\,[P:M]\Big]$, from which one gets that $P^{-1}=\left[P_{:Q(R)}P\right]$. This completes the proof of the theorem. □

There are some classes of modules for which RTP and TPP are equivalent. We close our work by the following result which gives one such case. Compare with [14, Theorem 4.2.19].

Proposition 24. *Let R be an integral domain and M a one dimensional faithful multiplication R-module. Then M is an RTP module if and only if M is a TPP module.*

Proof. As M is TPP module, Theorem 20 says that R is a TPP domain. Since M is a one dimensional module, it follows by [12, Theorem 2.5 and Corollary 2.11] that R is a one dimensional domain. Let N be a nonzero invertible submodule of M. Let $L=NN^{-1}$ (and hence $[L:M]=[N:M][N:M]^{-1}$). It follows that

$$L^{-1}=[L:M]^{-1}=\left[\left[L:M\right]:_{Q(R)}[L:M]\right]=\left[L:_{Q(R)}L\right].$$

Suppose that $\{P_\alpha\}$ is the set of minimal prime submodules of M that is contained in L. Then $\{[P_\alpha:M]\}$ is the set of minimal prime ideals of R that is contained in $[L:M]$. It follows by [14, Theorem 4.2.19] that

$$\begin{aligned}[L:M] &= \bigcap_\alpha [N:M]_{[P_\alpha:M]}=\bigcap_\alpha [P_\alpha:M]_{[P_\alpha:M]}\\ &= \bigcap_\alpha \Big([P_\alpha:M]_{[P_\alpha:M]}\cap R\Big)=\bigcap [P_\alpha:M]\,,\end{aligned}$$

and by [12, Corollary 1.7] we get that

$$L = [L : M]\, M = \left(\bigcap_{\alpha} [P_\alpha : M] \right) M = \bigcap_{\alpha} [P_\alpha : M]\, M = \bigcap_{\alpha} P_\alpha .$$

Hence $L = NN^{-1}$ is a radical submodule of M. The other direction follows by Theorem 23(1). □

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