ON THE DIOPHANTINE EQUATION $x! = py^\alpha$

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Abstract. In this note we investigate the Diophantine equation

$$x! = py^\alpha$$

where $x$ and $y$ are two integer variables and $p$ and $\alpha$ are two integer constants. We prove that the equation always has a finite number of solutions. In particular, if $p$ is an odd prime and $\alpha > 3$ then the equation has no solution.

1. Background and Main Results

A proof of the simplicity of the alternate group $A_n$, $n > 4$, given in Herstein [2], uses the fact that the degree of $A_n$, $n!/2$, is not a perfect square number. That is the equation $x! = 2y^2$ has no solution $x > 4$.

In this note, we consider the general equation $x! = py^\alpha$ of two unknowns $x$ and $y$. We will prove the following main results:

**Theorem 1.1.** For any $p, \alpha \in \mathbb{N}$, $\alpha > 1$, the equation $x! = py^\alpha$ has at most $2p+1$ solutions.

**Theorem 1.2.** Let $p$ be an odd prime and $\alpha > 3$. Then the equation $x! = py^\alpha$ has no solution.

The proofs of these results will employ the following well–known theorem by Chebyshev. This theorem was first conjectured by Bertrand in 1845 and first proved by Chebyshev in 1850 (see [3, 1]).

**Theorem 1.3** (Chebyshev). For any $n \geq 2$, there exists a prime $p$ such that $n < p < 2n$.

2. Proof of the Main Results

We first prove the Theorem 1.1.

**Proof.** We show that the equation $x! = py^\alpha$, $\alpha > 1$, has no solution $x \geq 2p + 2$.

Suppose that $x! = py^\alpha$ for some $\alpha > 1$ and $x \geq 2p + 2$. Since $[x/2] \geq p + 1 \geq 2$, by Chebyshev theorem, there exists a prime $q$ such that $[x/2] < q < 2[x/2] \leq x$.

It follows that $q \mid x! = py^\alpha$. Since $q > [x/2] > p$, $q \mid q\alpha$, and so $q \mid y$. Since $\alpha > 1$, we have $q^2 \mid y\alpha$. Therefore, $q^2 \mid x!$, and so $x \geq 2q$.

We have $2[x/2] < 2q \leq x$. This implies $x$ is an odd number. Let $x = 2x' + 1$ then $2x' < 2q \leq 2x' + 1$, this is impossible.

Therefore, the equation $x! = py^\alpha$, $\alpha > 1$, only has solutions $x \leq 2p + 1$. □
Using Theorem 1.1, it is now possible to determine all solutions for the equation when $p \leq 5$.

**Corollary 2.1.**

1. For any $\alpha > 1$, the equation $x! = y^\alpha$ only has a trivial solution $x = y = 1$.
2. For any $\alpha > 1$, the equation $x! = 2y^\alpha$ only has a trivial solution $x = 2, y = 1$.
3. For any $\alpha > 1$, $\alpha \neq 3$, the equation $x! = 3y^\alpha$ has one solution.
4. For any $\alpha > 1$, the equation $x! = 4y^\alpha$ has no solution.
5. For any $\alpha > 2$, the equation $x! = 5y^\alpha$ has one solution $x = 6, y = 12$. For any $\alpha > 2$, the equation $x! = 5y^\alpha$ has no solution.

**Proof.** From the Proof of Theorem 1.1, in the equation $x! = py^\alpha$, $\alpha > 1$, we only look for solution $x \leq 2p + 1$.

1. If $p = 1$ The equation $x! = y^\alpha$, $\alpha > 1$, only has solution $x \leq 3$. $x$ cannot be 2 or 3, so $x = y = 1$ is the only solution.
2. If $p = 2$ The equation $x! = 2y^\alpha$, $\alpha > 1$, only has solution $2 \leq x \leq 5$. If $x = 2$ then $y = 1$. If $x \geq 3$ then $3 \mid x! = 2y^\alpha$, and so $3 \mid y$. Thus $9 \mid x!$, and so $x$ cannot be 3, 4 or 5.
3. If $p = 3$ The equation $x! = 3y^\alpha$, $\alpha > 1$, only has solution $3 \leq x \leq 7$. $x$ cannot be 3. If $x = 4$ then $y^\alpha = 8$, so $y = 2$ and $\alpha = 3$. If $x \geq 5$ then $5 \mid x! = 3y^\alpha$, and so $25 \mid x!$; this cannot happen when $x \leq 7$.
4. If $p = 4$ The equation $x! = 4y^\alpha$, $\alpha > 1$, only has solution $3 \leq x \leq 9$. $x$ cannot be 3 or 4. If $x \geq 5$ then similarly as above it implies that $25 \mid x!$. This cannot happen when $x \leq 9$.
5. If $p = 5$ The equation $x! = 5y^\alpha$, $\alpha > 1$, only has solution $5 \leq x \leq 11$. $x$ cannot be 5. If $x = 6$ then $y^\alpha = 6 \times 4 \times 3 \times 2 = 2^4 3^2$, it only happens when $\alpha = 2$ and $y = 12$. If $x \geq 7$ then similar argument as above shows $49 \mid x!$; this is impossible for $x \leq 11$.

We are now in position to prove Theorem 1.2.

**Proof.** Assume $x! = py^\alpha$ where $p$ is an odd prime and $\alpha > 3$.

It follows from Theorem 1.1 that $p < x \leq 2p + 1$. And from Corollary 2.1, $p$ cannot be less than or equal to 5. So $p \geq 7$.

Since $\frac{p - 1}{2} \geq 3$, Chebyshev theorem says that there exists a prime $s$ such that $\frac{p - 1}{2} < s < p - 1$. We have $s \mid x! = py^\alpha$, it implies $s \mid y$. Since $\alpha > 3$, $s^4 \mid y^\alpha$ and so $s^4 \mid x!$. Since $s$ is a prime and $s > \frac{p - 1}{2} \geq 3$, it follows that $x \geq 4s$. Since $s \geq \frac{p + 1}{2}$, we have $x \geq 2p + 2$. This is a contradiction.

**3. Open Questions**

Theorem 1.2 says that the equation $x! = py^\alpha$ has no solution when $p$ is an odd prime and $\alpha > 3$. For $p = 2$, Corollary 2.1 shows that the equation has only one solution. What happen in the case $\alpha = 2$ or 3?

To conclude this note, we list two open questions
Question 1. The equation $x! = 5y^2$ has solution $x = 6, y = 12$. The equation $x! = 7y^2$ has solution $x = 10, y = 720$. For finite or infinite number of primes $p$, does the equation $x! = py^2$ has solution?

Question 2. The equation $x! = 3y^3$ has solution $x = 4, y = 2$. For finite or infinite number of primes $p$, does the equation $x! = py^3$ has solution?

References


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