STATE-SPACE APPROACH TO 3D GENERALIZED THERMOVISCOELASTICITY UNDER GREEN-NAGDHI THEORY

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Abstract. The present paper is concerned at investigating the effects of viscosity on generalized thermoelastic interactions in a three-dimensional homogeneous isotropic semi-infinite medium whose surface is subjected to a thermal shock and is assumed to be traction free. The formulation is applied to the generalized thermoelasticity based on the G-N II and III theories. The normal mode analysis together with state-space approach is used to obtain the exact analytical expressions for the studied field variables. Numerical computations are done for a specific material and obtained numerical results are plotted in the form of two- and three-dimensional figures. Comparisons are performed with and without the effects of viscosity. A comparison is also made between G-N II and G-N III models to show the effects of viscosity.

1. Introduction

In order to eliminate the paradox of infinite speeds of propagation of thermal wave [2], Lord & Shulman [1] (L-S model) and Green & Lindsay [3] (G-L model) formulated two model of generalized thermoelasticity in the year 1967 and 1972 respectively. Later Green & Nagdhi [4-6] proposed three theories of generalized thermoelasticity. The first model (G-N I) is exactly the same as Biot’s theory [2]. The second and third model are labeled as G-N II and G-N III model. In G-N II and G-N III models, the thermal wave propagates with finite speeds which agrees with physical situations. An important feature of G-N II theory is that this theory does not accommodates dissipation of thermal energy whereas G-N III theory accommodates dissipation of thermal energy. Several works based on G-N II and G-N III models can be found in the references [7-15].

Due to the rapid development of polymer science and plastic industry as well as the large use of materials under high temperature in modern technology, the investigation and application in thermo-visco-elastic solid materials have become a significant task for researchers in the field of mechanics of solid body. Keeping this fact in mind, many researchers [16-27] investigated various types of problems on in linear thermo-visco-elastic and electro-magneto-thermo-visco-elastic solid body.

In this paper, we introduce a state-space approach to investigated the effects of viscosity on wave propagation in a homogeneous isotropic three-dimensional thermoelastic semi-infinite medium. The free surface of the medium is subjected to a thermal shock and is assumed to be traction free. The formulation is applied to the

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generalized thermoelasticity based on the G-N model II and III. The normal mode analysis [9, 28] together with state-space approach [28-32] is used to obtain the exact analytical solutions for the studied field variables. Numerical computations are done for a specific material and obtained numerical results are plotted in the form of two- and three-dimensional figures. Comparisons are performed with and without the effects viscosity. A comparison is also made between G-N II and G-N III model to show the effects of viscosity.

2. Governing Equations

Following [5, 6], the strain-displacement relations, the stress-strain-temperature relations, the equations of motion in absence of body forces and the heat conduction equation in absence of heat source for a homogeneous isotropic thermally conducting viscoelastic solid may be written as

\[ \sigma_{ij} = \lambda u_{k,k} \delta_{ij} + 2\mu e_{ij} - \gamma T \delta_{ij}, \]  

\[ \sigma_{ij,j} = \rho \ddot{u}_i, \]  

\[ (k^* + k \frac{\partial}{\partial t}) \nabla^2 T = \rho C_\text{E} \dddot{T} + \gamma T_0 \dddot{e}, \]

where

\[ e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \]

and

\[ \lambda = \lambda_\text{c} \left( 1 + \lambda_0 \frac{\partial}{\partial t} \right), \mu = \mu_\text{c} \left( 1 + \mu_0 \frac{\partial}{\partial t} \right), \gamma = \gamma_\text{c} \left( 1 + \gamma_0 \frac{\partial}{\partial t} \right), \]

\[ \gamma_\text{c} = (3\lambda_\text{c} + 2\mu_\text{c})\alpha_T, \gamma_0 = (3\lambda_0 \lambda_\text{c} + 2\mu_0 \mu_\text{c})\alpha_T / \gamma_\text{c}, \quad (i,j,k = x,y,z). \]

In the above equations, \( \lambda, \mu \) are Lame’s constants, \( \lambda_0, \mu_0, \gamma_0 \) are viscoelastic parameters, \( \rho \) is the constant mass density of the medium, \( C_\text{E} \) is the specific heat at constant strain, \( \sigma_{ij} \) are the stress components, \( e_{ij} \) are the strain components, \( u_i \) are displacement components, \( t \) is the time variable, \( x, y, z \) are the space variables, \( T \) is the absolute temperature, \( T_0 \) is the initial temperature of the medium assumed to be such that \( \frac{\partial T}{\partial t} \ll 1 \), \( \gamma = (3\lambda + 2\mu)\alpha_T \), a material constant, \( \alpha_T \) is the coefficient of linear thermal expansion, \( k^* \) is a material constant characteristic of the theory, \( k \) is the thermal conductivity.

The subscript comma denotes spatial derivatives and a superimposed dot represents differentiation with respect to time \( t \).

3. Formulation of the Problem

We consider a homogenous isotropic thermo-visco-elastic semi-infinite medium in the three-dimensional region: \( \Omega = \{(x,y,z) : 0 \leq x < \infty, -\infty < y < \infty, -\infty < z < \infty \} \). We assume the homogeneous initial conditions. We are using the orthogonal Cartesian coordinates system \( (x,y,z) \). The displacement components thus have
the form: \( u_i = (u, v, w) \). Hence the equations of motion and the expressions for normal stress components can be written from Eqs. (2) and (1) as:

\[
\rho \ddot{u} = \left[ (\lambda_c + 2\mu_c) + (\lambda_0 \lambda_c + 2\mu_0 \mu_c) \frac{\partial}{\partial t} \right] u_{xx} + \mu_c \left( 1 + \mu_0 \frac{\partial}{\partial t} \right) (u_{yy} + u_{zz}) \\
+ \left[ (\lambda_c + \mu_c) + (\lambda_0 \lambda_c + \mu_0 \mu_c) \frac{\partial}{\partial t} \right] (v_{xy} + w_{xz}) - \gamma_c \left( 1 + \gamma_0 \frac{\partial}{\partial t} \right) T_x, \tag{4}
\]

\[
\rho \ddot{v} = \left[ (\lambda_c + 2\mu_c) + (\lambda_0 \lambda_c + 2\mu_0 \mu_c) \frac{\partial}{\partial t} \right] v_{yy} + \mu_c \left( 1 + \mu_0 \frac{\partial}{\partial t} \right) (v_{zz} + v_{xz}) \\
+ \left[ (\lambda_c + \mu_c) + (\lambda_0 \lambda_c + \mu_0 \mu_c) \frac{\partial}{\partial t} \right] (w_{yz} + u_{yx}) - \gamma_c \left( 1 + \gamma_0 \frac{\partial}{\partial t} \right) T_y, \tag{5}
\]

\[
\rho \ddot{w} = \left[ (\lambda_c + 2\mu_c) + (\lambda_0 \lambda_c + 2\mu_0 \mu_c) \frac{\partial}{\partial t} \right] w_{zz} + \mu_c \left( 1 + \mu_0 \frac{\partial}{\partial t} \right) (w_{xx} + w_{yy}) \\
+ \left[ (\lambda_c + \mu_c) + (\lambda_0 \lambda_c + \mu_0 \mu_c) \frac{\partial}{\partial t} \right] (u_{xx} + v_{zy}) - \gamma_c \left( 1 + \gamma_0 \frac{\partial}{\partial t} \right) T_z, \tag{6}
\]

\[
\left( k^1 + k \frac{\partial}{\partial t} \right) (T_{xx} + T_{yy} + T_{zz}) = \rho C_T \ddot{T} + \gamma_c T_0 \left( 1 + \gamma_0 \frac{\partial}{\partial t} \right) \dot{\varepsilon}, \tag{7}
\]

\[
\sigma_{xx} = 2\mu_c \left( 1 + \mu_0 \frac{\partial}{\partial t} \right) u_{xx} + \lambda_c \left( 1 + \lambda_0 \frac{\partial}{\partial t} \right) e - \gamma_c \left( 1 + \gamma_0 \frac{\partial}{\partial t} \right) T, \tag{8}
\]

\[
\sigma_{yy} = 2\mu_c \left( 1 + \mu_0 \frac{\partial}{\partial t} \right) v_{yy} + \lambda_c \left( 1 + \lambda_0 \frac{\partial}{\partial t} \right) e - \gamma_c \left( 1 + \gamma_0 \frac{\partial}{\partial t} \right) T, \tag{9}
\]

\[
\sigma_{zz} = 2\mu_c \left( 1 + \mu_0 \frac{\partial}{\partial t} \right) w_{zz} + \lambda_c \left( 1 + \lambda_0 \frac{\partial}{\partial t} \right) e - \gamma_c \left( 1 + \gamma_0 \frac{\partial}{\partial t} \right) T, \tag{10}
\]

where

\[
e = e_{kk} = (u_{xx} + v_{yy} + w_{zz}). \tag{11}
\]

To make the above equations dimensionless, we introduce the following non-dimensional variables:

\[
(x', y', z') = \frac{1}{l} (x, y, z), \quad t' = \frac{c_1 l}{T}, \quad (u', v', w') = \frac{\lambda_0 + 2\mu_0}{\gamma_0 T_0} (u, v, w), \tag{12}
\]

\[
\theta = \frac{T}{T_0}, \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\gamma T_0}, \quad (\lambda_0', \mu_0', \gamma_0') = \frac{c_1}{l} (\lambda_0, \mu_0, \gamma_0), \tag{13}
\]

where \( l \) is assumed to be standard length.

Using the above parameters, Eqs. (4)-(11) become (suppressing the primes for convenience):

\[
\ddot{\bar{u}} = \beta \left( 1 + \mu_0 \frac{\partial}{\partial t} \right) \nabla^2 \bar{u} + \left[ (1 - \beta) + \{ \lambda_0 (1 - 2\beta) + \beta \mu_0 \} \frac{\partial}{\partial t} \right] e_x \\
- \left( 1 + \gamma_0 \frac{\partial}{\partial t} \right) \theta_x, \tag{12}
\]
\[\ddot{v} = \beta \left(1 + \mu_0 \frac{\partial}{\partial t}\right) \nabla^2 v + \left[(1 - \beta) + \{\lambda_0(1 - 2\beta) + \beta \mu_0\} \frac{\partial}{\partial t}\right] e, y - \left(1 + \gamma_0 \frac{\partial}{\partial t}\right) \theta, y, \] (13)

\[\ddot{w} = \beta \left(1 + \mu_0 \frac{\partial}{\partial t}\right) \nabla^2 w + \left[(1 - \beta) + \{\lambda_0(1 - 2\beta) + \beta \mu_0\} \frac{\partial}{\partial t}\right] e, z - \left(1 + \gamma_0 \frac{\partial}{\partial t}\right) \theta, z, \] (14)

\[C_T^2 \nabla^2 \theta + k_0 \nabla^2 \dot{\theta} = \ddot{\theta} + \varepsilon \left(1 + \gamma_0 \frac{\partial}{\partial t}\right) \ddot{e}, \] (15)

\[\sigma_{xx} = 2\beta \left(1 + \mu_0 \frac{\partial}{\partial t}\right) u, x + (1 - 2\beta) \left(1 + \lambda_0 \frac{\partial}{\partial t}\right) e - \left(1 + \gamma_0 \frac{\partial}{\partial t}\right) \theta, \] (16)

\[\sigma_{yy} = 2\beta \left(1 + \mu_0 \frac{\partial}{\partial t}\right) v, y + (1 - 2\beta) \left(1 + \lambda_0 \frac{\partial}{\partial t}\right) e - \left(1 + \gamma_0 \frac{\partial}{\partial t}\right) \theta, \] (17)

\[\sigma_{zz} = 2\beta \left(1 + \mu_0 \frac{\partial}{\partial t}\right) w, z + (1 - 2\beta) \left(1 + \lambda_0 \frac{\partial}{\partial t}\right) e - \left(1 + \gamma_0 \frac{\partial}{\partial t}\right) \theta, \] (18)

\[e = (u, x + v, y + w, z), \] (19)

Differentiating Eqs. (12)-(14) with respect to \(x, y, z\) respectively and then adding we obtain

\[\left[1 + \{\lambda_0 - 2\beta(\lambda_0 - \mu_0)\} \frac{\partial}{\partial t}\right] \nabla^2 e - \left(1 + \gamma_0 \frac{\partial}{\partial t}\right) \nabla^2 \theta = \ddot{e}. \] (20)

We define

\[\sigma = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \] (21)

to be the mean stress [9].

Adding Eqs. (16)-(18) and then using Eqs. (19) and (21), we get after some manipulations

\[\sigma = \left[1 - \frac{4\beta}{3} + \left(1 - 2\beta\right) \lambda_0 + \frac{2\beta}{3} \mu_0\right] \frac{\partial}{\partial t} e - \left(1 + \gamma_0 \frac{\partial}{\partial t}\right) \theta. \] (22)

4. Normal Mode Analysis

The solution of the field variables can be decomposed in terms of normal modes [9, 28] as:

\[[u, v, w, e, \theta, \sigma, \sigma_{ij}] (x, y, z, t) = [u^*, v^*, w^*, e^*, \theta^*, \sigma^*, \sigma_{ij}^*] (x) \exp [\omega t + i(ay + bz)], \] (23)
where \( u^*(x) \) etc. are the amplitude of the function \( u(x,y,t) \) etc., \( i = \sqrt{-1} \), \( \omega \) (complex) is the angular frequency and \( a, b \) are the wave number in the \( y \) and \( z \) direction respectively.

Using the normal modes (23) in Eqs. (15), (20), (22) and then eliminating \( e^*(x) \) from the resulting equations we get the following equations, namely:

\[
D^2 \theta^* = C_1 \theta^* + C_2 \sigma^*, \tag{24}
\]
\[
D^2 e^* = D_1 \theta^* + D_2 \sigma^*, \tag{25}
\]

where

\[
D \equiv \frac{d}{dx}, \quad C_1 = \left[ a^2 + b^2 + \frac{\omega^2(1 + \gamma_0 \omega)^2}{\alpha (C_T^2 + k_0 \omega)} \right], \quad C_2 = \frac{\epsilon \omega^2(1 + \gamma_0 \omega)}{\alpha (C_T^2 + k_0 \omega)} ,
\]
\[
D_1 = \left[ \frac{\omega^2(1 + \gamma_0 \omega)}{\alpha (C_T^2 + k_0 \omega)} - \frac{\omega^2(d - \alpha)(1 + \gamma_0 \omega)(\alpha + \epsilon(1 + \gamma_0 \omega)^2)}{d \alpha (C_T^2 + k_0 \omega)} \right],
\]
\[
D_2 = \left[ a^2 + b^2 + \frac{\omega^2(d - \alpha)(1 + \gamma_0 \omega)^2}{d \alpha (C_T^2 + k_0 \omega)} \right],
\]
\[
d = [1 + \omega \{ \lambda_0 - 2 \beta(\lambda_0 - \mu_0) \}], \quad \alpha = \left[ 1 - \frac{4\beta}{3} + \omega \left( \lambda_0(1 - 2\beta) + \frac{2\beta}{3} \mu_0 \right) \right].
\]

We choose \( \theta^*(x) \) and \( \sigma^*(x) \) as the state variables in the \( x \)-direction and write Eqs. (24) and (25) in a vector-matrix differential equation as:

\[
D^2 \vec{V}(x) = A \vec{V}(x), \tag{26}
\]

where

\[
\vec{V}(x) = \begin{pmatrix} \theta^* \\ \sigma^* \end{pmatrix}, \quad A = \begin{pmatrix} C_1 & C_2 \\ D_1 & D_2 \end{pmatrix}.
\]

The initial conditions of the problem are assumed to be homogeneous. Now, we consider the following boundary conditions in non-dimensional form:

\[
\dot{q}_n + \nu \theta(0,y,z,t) = r(y,z,t), \tag{27}
\]
\[
\sigma(0,y,z,t) = \sigma_{xx}(0,y,z,t) = \sigma_{yy}(0,y,z,t) = \sigma_{zz}(0,y,z,t) = 0, \tag{28}
\]

where \( q_n \) denotes the normal component of the heat flux, \( \nu \) is the Biot’s number and \( r(y,z,t) \) represents the intensity of the applied heat sources at \( x = 0 \).

The regularity condition is that all the field variables are bounded for \( x \to +\infty \). In order to use the boundary condition (27), we need to consider the following non-dimensional Fourier’s law of heat conduction of G-N model [5], namely

\[
\dot{q}_n = -\frac{\partial \theta}{\partial n}, \tag{29}
\]

Substituting from Eq. (29) in Eq. (27) and then using the normal modes (23), we get:

\[
\nu \theta^*(0) - D \theta^*(0) = r^*, \tag{30}
\]
\[
\sigma^*(0) = \sigma_{xx}^*(0) = \sigma_{yy}^*(0) = \sigma_{zz}^*(0) = 0. \tag{31}
\]
5. State Space Approach

The formal solution of the vector-matrix differential equation (26) can be written as (see [27-30] for details)

\[ \vec{V}(x) = \exp \left[ -\sqrt{\mathbf{A}} x \right] \vec{V}(0) \]  \hspace{1cm} (32)

where

\[ \vec{V}(0) = \begin{pmatrix} \theta^*(0) \\ \sigma^*(0) \end{pmatrix} = \begin{pmatrix} \theta^*_0 \\ 0 \end{pmatrix}, \]

and the constant \( \theta^*_0 \) is to be determined using (30).

We have omitted the positive exponential part in (32) to obtain a bounded solution for large \( x \). Now, our first task is to obtain the matrix form of \( \exp \left[ -\sqrt{\mathbf{A}} x \right] \).

The characteristic equation of \( \mathbf{A} \) is given by

\[ \lambda^2 - (C_1 + D_2)\lambda + (C_1 D_2 - C_2 D_1) = 0. \]  \hspace{1cm} (33)

Let \( \lambda_1 \) and \( \lambda_2 \) be the roots (distinct) of the Eq. (33), where

\[ \lambda_j = (C_1 + D_2) + (-1)^j \sqrt{(C_1 - D_2)^2 + 4C_2 D_1}, \quad j = 1, 2. \]

Following Simmons [33], the spectral decomposition of the matrix \( \mathbf{A} \) can be written as

\[ \mathbf{A} = \lambda_1 E + \lambda_2 F, \]  \hspace{1cm} (34)

where \( E \) and \( F \) are called the projectors of \( \mathbf{A} \) satisfying the following relations (see [33] for details):

\[ E + F = I, \quad EF = FE = O, \quad E^2 = E, \quad F^2 = F. \]  \hspace{1cm} (35)

Since \( \sqrt{\mathbf{A}} \) and \( \mathbf{A} \) has the same projectors ([33]) and if \( p_1, p_2 \) are the eigenvalues of \( \sqrt{\mathbf{A}} \) then \( p_j = \sqrt{\lambda_j}, \quad j = 1, 2. \) Hence the spectral decomposition of the matrix \( \sqrt{\mathbf{A}} \) is given by

\[ \sqrt{\mathbf{A}} = p_1 E + p_2 F, \]  \hspace{1cm} (36)

where

\[ E = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} C_1 - \lambda_2 \\ D_1 \end{pmatrix}, \quad F = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} C_1 - \lambda_1 \\ D_1 \end{pmatrix}. \]  \hspace{1cm} (37)

and \( E, F \) satisfy Eq. (35).

Thus, we find

\[ \mathbf{A}^* = \sqrt{\mathbf{A}} = \frac{1}{p_1 + p_2} \begin{pmatrix} C_1 + p_1 p_2 \\ D_1 \end{pmatrix}. \]  \hspace{1cm} (38)

With help of Eq. (38), the Taylor series expansion of the matrix exponential in Eq. (32) may take the following form

\[ \exp \left[ -\sqrt{\mathbf{A}} x \right] = \exp \left[ -\mathbf{A}^* x \right] = \sum_{n=0}^{\infty} \frac{[-\mathbf{A}^* x]^n}{n!}. \]  \hspace{1cm} (39)
Using the Cayley-Hamilton theorem, we can write the second and higher orders of the matrix $A^*$ in terms of $I$ and $A^*$. Thus, Eq. (39) reduces to

$$\exp [-A^* x] = a_0(x)I + a_1(x)A^*,$$

(40)

where the coefficients $a_0$ and $a_1$ depends on $x$ only.

By Cayley-Hamilton theorem, the characteristic roots $p_1$ and $p_2$ of the matrix $A^*$ must satisfy Eq. (40) and thus we get

$$e^{-p_1x} = a_0(x) + a_1(x)p_1,$$

(41)

$$e^{-p_2x} = a_0(x) + a_1(x)p_2.$$

(42)

By solving the above system of Eqs. (41) and (42), we obtain

$$a_0(x) = \frac{p_1 e^{-p_2 x} - p_2 e^{-p_1 x}}{(p_1 - p_2)}, \quad a_1(x) = \frac{e^{-p_1 x} - e^{-p_2 x}}{(p_1 - p_2)}.$$  

(43)

From Eqs. (40) and (43), we get

$$\exp [-\sqrt{A} x] = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

(44)

where

$$A_{11} = \frac{(\lambda_1 - C_1)e^{-\sqrt{C_1} x} - (\lambda_2 - C_1)e^{-\sqrt{C_2} x}}{(\lambda_1 - \lambda_2)}, \quad A_{12} = \frac{C_2(e^{-\sqrt{C_1} x} - e^{-\sqrt{C_2} x})}{(\lambda_1 - \lambda_2)},$$

(45)

$$A_{21} = \frac{D_1(e^{-\sqrt{C_1} x} - e^{-\sqrt{C_2} x})}{(\lambda_1 - \lambda_2)}, \quad A_{22} = \frac{(\lambda_1 - D_2)e^{-\sqrt{C_2} x} - (\lambda_2 - D_2)e^{-\sqrt{C_1} x}}{(\lambda_1 - \lambda_2)}.$$  

(46)

Thus the solution of Eq. (26) can be written as

$$\vec{V}(x) = A_{jk} \vec{V}(0), \quad j, k = 1, 2.$$  

(47)

From Eqs. (45)-(47), the solutions for $\theta^*(x)$ and $\sigma^*(x)$ can be obtained as

$$\theta^*(x) = \theta_1 e^{-\sqrt{C_1} x} - \theta_2 e^{-\sqrt{C_2} x},$$

(48)

$$\sigma^*(x) = \frac{D_1 \theta_0^*}{(\lambda_1 - \lambda_2)} \left[ e^{-\sqrt{C_1} x} - e^{-\sqrt{C_2} x} \right].$$

(49)

where

$$\theta_1 = \theta_0^* \left[ \frac{C_1 - \lambda_2}{\lambda_1 - \lambda_2} \right], \quad \theta_2 = \theta_0^* \left[ \frac{C_1 - \lambda_1}{\lambda_1 - \lambda_2} \right].$$

The boundary condition (30) gives the value of the constant $\theta_0^*$ as

$$\theta_0^* = \frac{r^*(\lambda_1 - \lambda_2)}{\nu(\lambda_1 - \lambda_2) + \sqrt{\lambda_1}(C_1 - \lambda_2) - \sqrt{\lambda_2}(C_1 - \lambda_1)).$$

Using Eqs. (23), (48) and (49) in Eq. (22), the cubical dilatational $e^*(x)$ is obtained as

$$e^*(x) = e_1 e^{-\sqrt{C_1} x} - e_2 e^{-\sqrt{C_2} x},$$

(50)

where

$$e_1 = \frac{[\theta_0^* D_1 + (C_1 - \lambda_2)(1 + \gamma_0 \omega)]}{\alpha(\lambda_1 - \lambda_2)}, \quad e_2 = \frac{[\theta_0^* D_1 + (C_1 - \lambda_1)(1 + \gamma_0 \omega)]}{\alpha(\lambda_1 - \lambda_2)}.$$
Substituting from Eqs. (23), (48) and (50) into the Eq. (12), we get

\[(D^2 - \lambda_n^2) u^*(x) = \sum_{j=1}^{2} (-1)^{j-1} u_j (\lambda_j^2 - \lambda_n^2) e^{-\sqrt{\lambda_j^2} x}, \tag{51}\]

where

\[\lambda_n^2 = \left[ a^2 + b^2 + \frac{\omega^2}{\beta(1 + \mu_0 \omega)} \right], \]

\[u_j = \frac{\sqrt{\lambda_j} \left[ e_j \{(1 - \beta) + \omega(\lambda_0(1 - 2\beta) + \mu_0 \beta)\} - (1 + \gamma_0 \omega)\theta_j \right]}{\beta(1 + \mu_0 \omega)(\lambda_j^2 - \lambda_n^2)} , \quad j = 1, 2.\]

The general solution of Eq. (51) can be written as

\[u^*(x) = u_0 e^{-\lambda_n x} + \sum_{j=1}^{2} (-1)^{j-1} u_j e^{-\sqrt{\lambda_j^2} x}, \tag{52}\]

where \(\lambda_1^2 \neq \lambda_2^2 \neq \lambda_n^2\) and \(u_0\) is a constant to be determined from the boundary conditions (31).

By using Eqs. (23), (48), (50) and (52) in Eq. (16), we find the solution for the stress component \(\sigma_{xx}^*(x)\) as

\[\sigma_{xx}^*(x) = \sum_{j=1}^{2} (-1)^{j-1} \sigma_j e^{-\sqrt{\lambda_j^2} x} + \sigma_3 e^{-\lambda_n x}, \tag{53}\]

where

\[\sigma_j = -2\beta u_j \sqrt{\lambda_j(1 + \mu_0 \omega)} + e_j(1 - \lambda(1 + \lambda_0 \omega) - \theta_j(1 + \gamma_0 \omega), \quad j = 1, 2,
\]

\[\sigma_3 = -2\beta \lambda_n u_0 (1 + \mu_0 \omega).\]

With the help of the boundary condition (31) and Eq. (53), we find the constant \(u_0\) as

\[u_0 = \frac{(\sigma_1 - \sigma_2)}{2\beta \lambda_n (1 + \mu_0 \omega)}.
\]

6. Numerical Example and Discussions

Since we have \(\omega = \omega_0 + i\epsilon\), where \(i\) is the imaginary unit, \(e^{\omega t} = e^{\omega_0 t} (\cos \epsilon t + i \sin \epsilon t)\) and for small values of time, we can take \(\omega = \omega_0\) (real). For the discussions of the nature of dependence of all the physical variables on viscosity, we shall compute them numerically for a particular model. For this purpose, we choose the following numerical values of the relevant parameters for copper like material:

\[\lambda_e = 7.76 \times 10^{10} \text{ N/m}^2, \quad \mu_e = 3.86 \times 10^{10} \text{ N/m}^2, \quad \lambda_0 = 0.06 \text{ s}, \quad \mu_0 = 0.09 \text{ s},\]

\[\alpha = 1.78 \times 10^{-5} \text{ K}^{-1}, \quad C_E = 383.1 \text{ m}^2/\text{K}, \quad \rho = 8954 \text{ kg/m}^3, \quad T_0 = 293 \text{ K},\]

\[\epsilon = 0.0168, \quad \beta = 0.25, \quad \omega = 3, \quad a = 1.2, \quad b = 1.3, \quad \nu = 50, \quad r^* = 100, \quad C_T = 2.\]

Using the above numerical values, the variations of the temperature distribution \(\theta\), the mean stress \(\sigma\), the displacement component \(u\) and the stress component \(\sigma_{xx}\) along \(x\) axis at two different plane \(y = z = 0.0\) and \(y = z = 0.4\) for a particular time instant \(t = 0.25\) have been shown for (i) generalized thermoviscoelastic solid (GTVE) by solid line at \(y = z = 0.0, t = 0.25\), (ii) generalized thermoviscoelastic...
solid (GTVE) by solid-dot line at \( y = z = 0.4, t = 0.25 \), (iii) generalized thermoelastic solid (GTE) by solid-dot (bold) line at \( y = z = 0.0, t = 0.25 \) and (iv) generalized thermoelastic solid (GTE) by dashed line at \( y = z = 0.4, t = 0.25 \). These variations are shown in Figs. 1–4.

From Figs. 1–4 it is clear that \( y \) and \( z \) have decreasing effect on \( \theta \), \( \sigma \), \( u \) and \( \sigma_{xx} \) for both GTVE and GTE model for fixed \( t \). Also it is depicted that the numerical values of \( \sigma \), \( u \) and \( \sigma_{xx} \) are greater in GTVE model than GTE model for fixed \( x, y, z \) and \( t \) but Fig. 1 shows that the viscosity has no significant effect on \( \theta \). The maximum value of all the physical quantities attain in the case of GTVE at the plane \( y = z = 0.0 \).

Figs. 5–8 show the distributions of \( \theta \), \( \sigma \), \( u \) and \( \sigma_{xx} \) for the GTVE model at \( y = z = 0.0 \) and \( y = z = 0.4 \) for two different time instants \( t = 0.2 \) and \( t = 0.3 \). It is clear from all these figures that \( t \) has increasing effects on all the physical quantities.

Figs. 9–12 display the temperature \( \theta \), the mean stress \( \sigma \), the displacement \( u \) and the stress \( \sigma_{xx} \) distributions at \( y = z = 0.4 \) with wide range of \( 0 \leq x \leq 4.0 \) and \( 0.1 \leq t \leq 0.5 \). From these figures it can be noted that the speed of the wave propagation of all the physical quantities are finite and coincide with the physical behavior of elastic materials. Also we can see from all the figures that the boundary conditions (30) and (31) are satisfied.

It is clear from Figs. 1–12 that all the distributions considered have a non-zero value only in a bounded region of the space \( \Omega \). Outside of this region all the values vanish identically and this means that the region has not felt thermal disturbance yet. Behavior of all the physical variables at \( y = z = 0.0 \) and \( y = z = 0.4 \) are likely to be similar but only differences are lie in the magnitudes.

Figs. 13–16 are plotted to show the effects of the viscosity parameter on \( \theta \), \( \sigma \), \( u \) and \( \sigma_{xx} \) with different positions of the displacement \( x \) at \((y, z, t) = (0.4, 0.4, 0.25)\). In all these figures, solid line (-) represent G-N III model with viscosity (G-N III WV), solid star line (-*) represent G-N II model with viscosity (G-N II WV), solid line with ‘o’(-o) represent G-N III model without viscosity (G-N III WOV) and dashed line (- -) represent G-N II model without viscosity (G-N II WOV). These figures exhibit that the presence of viscosity effects increase the values of all the studied field variables in both the theories (G-N II and G-N III). We can also notice that the magnitudes of all the physical quantities are greater for G-N III model than G-N II model. Figs. 14 and 15 show that the mean stress \( \sigma^{*}(x) \) and the normal stress \( \sigma_{xx}^{*}(x) \) starts with zero value at the boundary \( x = 0 \) which agree with the boundary condition (31) of our problem. Figs. 13–16 depict that the behavior of all the four curves predicted by G-N II and G-N III model in the presence and absence of the viscosity parameter are same in nature. Finally all the curves in Figs. 13–16 converges to zero for some large \( x \) which agree with the characteristic of the generalized thermoelasticity.
Fig. 1 Temperature distribution $\theta$ vs. $x$ at $t = 0.25$.

Fig. 2 Mean stress $\sigma$ distribution vs. $x$ at $t = 0.25$.

Fig. 3 Displacement distribution $u$ vs. $x$ at $t = 0.25$. 
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Fig. 4 Stress distribution $\sigma_{xx}$ vs. $x$ at $t = 0.25$.

Fig. 5 Temperature distribution $\theta$ vs. $x$ for two time instants.

Fig. 6 Mean stress $\sigma$ distribution vs. $x$ for two time instants.
Fig. 7 The displacement distribution $u$ vs. distance $x$ for two time instants.

Fig. 8 The stress distribution $\sigma_{xx}$ vs. $x$ for two time instants.

Fig. 9 Temperature distribution vs. $x$ and $t$ at $y = z = 0.4$. 
Fig. 10 Mean stress $\sigma$ distribution vs. $x$ and $t$ at $y = z = 0.4$.

Fig. 11 Displacement $u$ distribution vs. $x$ and $t$ at $y = z = 0.4$.

Fig. 12 Stress distribution $\sigma_{xx}$ vs. $x$ and $t$ at $y = z = 0.4$. 
Fig. 13 Temperature distribution $\theta$ vs. $x$ for G-N II and III theory.

Fig. 14 Mean stress $\sigma$ distribution vs. $x$ for G-N II and III theory.

Fig. 15 The displacement distribution $u$ vs. distance $x$ for G-N II and III theory.
Fig. 16 The stress distribution $\sigma_{xx}$ vs. $x$ for G-N II and III theory.

7. Concluding Remark

Many researchers in the field of generalized thermoelasticity have applied state-space approach only for one-dimensional thermoelastic problem and very few of them can successfully applied for two-diemsional case. In the present paper, we apply state-space approach for solving a three-dimensional generalized thermoelastic problem for the first time. We tried to implement such a very useful technique which may be applied to solve a three-dimensional generalized thermoviscoelastic problem.

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