A REMARK ON THE DUAL OF BANACH-VALUED TENT SPACES

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Abstract. We identify the dual of the atomic vector-valued tent space \( \Upsilon^1_{q}(X) \), with an appropriate space of measures with values in \( X^* \), where \( X \) is a real Banach space.

1. Introduction

Much work has been done on the theory of tent spaces, originally studied in [4] and [5]. In those papers, the authors developed a successful theory closely related to that of Hardy spaces on the upper half-space, and very appropriate for the study of maximal functions, square functions and other classical operators in harmonic analysis.

In this paper we restrict our attention to Banach-valued versions of atomic tent spaces, to be precise, those formed by series of \((1,q)\)-atoms where \( 1 < q < \infty \), with an adequate notion of convergence.

Our aim in the present note is to give a description of the dual of our atomic tent space \( \Upsilon^1_{q}(X) \), in terms of certain kind of measures with values in \( X^* \), namely, the family of vector-valued measures \( C_{q'}(X^*) \), where \( q' \) is the conjugate exponent of \( q \). The role of this family of measures is very similar to that played by the family of functions \( f \) such that \( C_{q'}(f) \in L^\infty \), studied in [5] but adjusted to our general context.

The proofs given in this paper follow classical ideas and patterns (cf. [2], [8]) with appropriate modifications when it is required. Moreover, no geometrical property on the Banach space \( X \) is assumed to set up the duality.

2. Preliminaries

Let us denote by \( \Omega \) a space of homogeneous type, that is, a topological space endowed with a Borel measure \( \mu \) and a quasi-distance \( d \), such that the family of balls \( B_r(x) \) is a local basis for each \( x \in \Omega \), and moreover, there exists an absolute constant \( C \) satisfying the doubling condition

\[
\mu(B_{2r}(x)) \leq C \mu(B_r(x)).
\]

From this condition we can assume that \( 0 < \mu(B) < \infty \) for every ball \( B \), which implies that the measure \( \mu \) is \( \sigma \)-finite.

The letter \( X \) will always denote a real Banach space and \( L^p_X(\Omega, \mu) \), \( 1 \leq p < \infty \), will be the space of \( X \)-valued Borel measurable functions \( f \) defined on \( \Omega \) such that

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\[ \|f\|_X \in L^p_q(\Omega, \mu). \] By \( L^p_{X,loc}(\Omega, \mu) \) will denote the space of functions \( f \in L^p_{X}(K, \mu) \) for each compact subset \( K \) of \( \Omega \). If there is no place for confusion we will simply write \( L^p_X \) instead of \( L^p_{X}(\Omega, \mu) \), \( L^p_{X,loc} \) instead of \( L^p_{X,loc}(\Omega, \mu) \) and \( L^p \) for \( L^p_{\mathbb{R}}(\Omega, \mu) \).

For \( 1 \leq q < \infty \), \( V^q_{X,loc}(\mu) \) will be the space of vector measures \( \nu \) defined on \( \mathcal{B} = \mathcal{B}(\Omega) \), the family of Borel sets of \( \Omega \), with values in \( X \) and finite \( q \)-variation, that is,

\[
\|\nu\|_{V^q_{X,loc}(\mu)} = \sup \left( \sum_{A \in \pi} \|\mu (A)\|_{X}^{q-1} \right)^{1/q} < \infty, \tag{1}
\]

where the supremum is taken over all finite measurable partitions \( \pi \) of \( \Omega \) (with the convention \( \lambda^{n-1} \) equals to 0 or \( \infty \) provided \( \lambda = 0 \) or \( \lambda > 0 \)). We remark that according to [3], Lemma 1, \( \|\nu (A)\|_{X} \) can be replaced by \( |\nu| (A) \) in (1), where \( |\nu| \) denotes the variation of \( \nu \).

If the context is clear, we simply write \( V^q_{X} \). Sometimes, it will be useful to consider a measurable subset \( A \) of \( \Omega \), in which case we will write \( V^q_{X}(A) \).

When \( q > 1 \), it is well known that the dual space of \( L^q_{X} \) is isometrically isomorphic to \( V^q_{X,loc}(\mu) \), being \( q^* \) the conjugate exponent for \( q \) (see [6], [7]).

Now, we shall restrict our attention to the case \( \Omega = \mathbb{R}^{n+1}_+ \), provided with the usual topology and the doubling measure \( d\mu = \frac{dxdt}{t} \). The set \( \mathcal{B}_0 \) will denote the family of bounded Borel sets \( A \subset \mathbb{R}^{n+1}_+ \), hence the \( \sigma \)-algebra generated by \( \mathcal{B}_0 \) coincides with \( \mathcal{B} \). When necessary, we will write \( |\cdot| \) for the Lebesgue measure on the \( n \)-dimensional euclidean space.

**Definition 1.** For \( 1 < q < \infty \), the set \( \mathcal{C}_q(X) \) will denote the space of countably additive vector measures \( \nu : \mathcal{B}_0 \rightarrow X \) satisfying the following conditions:

1. \( \nu \in V^q_{X}(T_\varepsilon(B)) \) for every ball \( B \subset \mathbb{R}^n \) and each \( \varepsilon > 0 \), where

\[
T_\varepsilon(B) = \{(x,t) \in \mathbb{R}^{n+1}_+ : B_t(x) \subset B, \ t > \varepsilon \}.
\]

2. There exists an absolute constant \( C > 0 \) such that for every ball \( B \subset \mathbb{R}^n \) and each \( \varepsilon > 0 \)

\[
\frac{1}{|B|^{1/q}} \|\nu\|_{V^q_{X}(T_\varepsilon(B))} \leq C. \tag{2}
\]

The infimum of the constants \( C \) such that (2) holds is denoted by \( \|\nu\|_{\mathcal{C}_q(X)} \), which is a norm on \( \mathcal{C}_q(X) \).

We notice that every truncated tent \( T_\varepsilon(B) \) is contained in some open ball of \( \mathbb{R}^{n+1}_+ \) whose distance to \( \mathbb{R}^n \) is positive. Moreover, \( \nu \) is absolutely continuous with respect to \( \mu \), i.e., \( \lim_{\mu(A)\rightarrow 0} \nu(A) = 0 \) and thus \( \nu \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^{n+1}_+ \).

Another useful remark is the fact that the countably additive vector measure \( \nu \in \mathcal{C}_q(X) \) if and only if \( |\nu| \in \mathcal{C}_q(\mathbb{R}) \). In such case \( \|\nu\|_{\mathcal{C}_q(X)} = \|\nu\|_{\mathcal{C}_q(\mathbb{R})} \). Moreover, since the scalar measure \( |\nu| \) is absolutely continuous with respect to \( \mu \), we can find a non-negative function \( \eta \in L^1_{loc}(\mathbb{R}^{n+1}_+, d\mu) \) such that for each \( E \in \mathcal{B}_0 \) we have

\[
|\nu|(E) = \int_E \eta d\mu. \tag{3}
\]
An elementary example of an element in $C^q(X)$ can be given as follows (cf. [5]).

Let us consider the space of functions $C^q(X) = \left\{ f \in L^q_{X,\text{loc}}(\mathbb{R}^{n+1}, \frac{dxdt}{t}) : \|f\|_{C^q(X)} < \infty \right\}$

where

$$\|f\|_{C^q(X)} = \sup_B \left( \frac{1}{|B|} \int_{T(B)} \|f(x,t)\|_X^q \frac{dxdt}{t} \right)^{1/q} < \infty$$

and the supremum is taken over all balls $B$ on $\mathbb{R}^n$.

Clearly, $\|\cdot\|_{C^q(X)}$ defines a norm in $C^q(X)$, and if we consider the measure $d\nu(x,t) = f(x,t) dxdt$ with $f \in C^q(X)$, it turns out that for every ball $B$ and each $\varepsilon > 0$

$$\|\nu\|_{V^q_{X}(T_\varepsilon(B))} = \|f\|_{L^q_X(T_\varepsilon(B), \frac{dxdt}{t})} \leq \|f\|_{C^q(X)} |B|^{1/q},$$

that is, $\nu \in C^q(X)$ and $\|\nu\|_{C^q(X)} \leq \|f\|_{C^q(X)}$.

Of course, without any geometric prescription for $X$, we have only the inclusion $C^q(X) \subset C^q(X)$. Now, we define our vector-valued tent spaces.

**Definition 2.** Given $1 < q < \infty$, the tent space $T^1_q(X)$ will be the set of measurable functions $f : \mathbb{R}^{n+1} \to X$ such that $A_q(\|f\|_X) \in L^1(\mathbb{R}^n)$, provided with the norm $\|f\|_{T^1_q(X)} = \|A_q(\|f\|_X)\|_1$.

As usual,

$$A_q(\|f\|_X)(x) = \left( \int_{\Gamma(x)} \|f(y,t)\|_X^q \frac{dydt}{t^{n+1}} \right)^{1/q},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < t\}$.

As in the scalar case, we can obtain (see [5], p. 306) for any compact set $K \subset \mathbb{R}^{n+1}_+$

$$\left( \int_K \|f(y,t)\|_X^q \frac{dydt}{t} \right)^{1/q} \leq C(K, q, n) \|A_q(\|f\|_X)\|_1,$$

estimate that allows us to prove that $\left(T^1_q(X), \|\cdot\|_{T^1_q(X)}\right)$ is a Banach space.

**Definition 3.** A measurable function $a : \mathbb{R}^{n+1}_+ \to X$ is called an $(X,q)$-atom, or simply an atom in $X$ if the following conditions are satisfied:

1. $\text{supp} \ a \subset T(B)$, where $B$ is a ball in $\mathbb{R}^n$.

2. $\left( \int_{T(B)} \|a(x,t)\|_X^q \frac{dxdt}{t} \right)^{1/q} \leq |B|^\frac{1}{q} - 1$.
We remark the fact that every \((X,q)\)-atom belongs to 
\[ L^1_X \left( \mathbb{R}^{n+1}_+, \frac{dxdt}{t} \right) \].
This can be easily seen taking into account that the support of \(a\) is contained in \( T_\varepsilon (B) \) for some \( \varepsilon > 0 \) and therefore, there exists a positive constant \( C \) depending on \( n \) and \( q \) such that
\[
\hat{T}_\varepsilon (B) \left\| a(x,t) \right\|_X d\mu(x,t) = \hat{T}_\varepsilon (B) \left\| a(x,t) \right\|_X d\mu(x,t) \leq C \left( \frac{r_B}{\varepsilon} \right)^{1/q'} < \infty,
\]
where \( r_B \) denotes the radius of \( B \). This observation suggests that the appropriate norm to consider in our atomic vector-valued tent space must be the \( T^1_{q'}(X) \) norm.

**Definition 4.** The \( X \)-valued tent space \( \Upsilon^1_q(X) \), will be the set of functions \( f \in T^1_q(X) \) such that
\[
f = \sum_{j=1}^{\infty} \lambda_j a_j \quad (5)
\]
where \( a_j \) is a \((X,q)\)-atom for every \( j \) and \( (\lambda_j)_{j=1}^{\infty} \) is a sequence of real numbers such that \( \sum_{j=1}^{\infty} |\lambda_j| < \infty \). The convergence in (5) is considered in the norm \( \|\cdot\|_{T^1_q(X)} \).

If we define for \( f \in \Upsilon^1_q(X) \)
\[
\|f\|_{T^1_q(X)} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \right\}
\]
where the infimum is taken over all representations of \( f \) as a sum of atoms, then we obtain a norm.

As in [5], p. 312, it can be seen that if \( a \) is a \((X,q)\)-atom supported on the tent \( T(B) \), then \( A_q(\|a\|_X) \) is supported on \( B \) and \( \|a\|_{T^1_q} \leq 1 \). This implies that the series in (5) is absolutely convergent in the norm \( \|\cdot\|_{T^1_q(X)} \) and for \( f \in \Upsilon^1_q(X) \) we have that \( \|f\|_{T^1_q(X)} \leq \|f\|_{T^1_q(X)} \) that is, \( \Upsilon^1_q(X) \) is continuously included in \( T^1_q(X) \).

Moreover, \( \left( \Upsilon^1_q(X), \|\cdot\|_{T^1_q(X)} \right) \) is a Banach space, fact that can be proved taking into account that the \((X,q)\)-atoms live in the unit ball of \( T^1_q(X) \) and using Lemma 1.1 in [1].

**3. The Duality Result**

Now, we compute the dual of the vector-valued tent space \( \Upsilon^1_q(X) \). Our first result is the following.

**Proposition 5.** Let \( 1 < q < \infty \). Given \( \Lambda \in (\Upsilon^1_q(X))^* \), there exists a unique measure \( \nu \in C_{q'}(X^*) \) such that
\[
\Lambda(f) = \int_{\mathbb{R}^{n+1}_+} fd\nu
\]
for every function \( f \in L^q_X \left( \mathbb{R}^{n+1}_+, \frac{dxdt}{t} \right) \) supported on a tent \( T(B) \).
Proof. Let \( f \in L^q_X (\mathbb{R}^{n+1}_+, \frac{dxdt}{t}) \) supported on \( T(B) \), hence it is supported on \( T_\varepsilon(B) \) for some \( \varepsilon > 0 \). Clearly, the function \( a(x,t) = |B|^\frac{1}{q} \| f \|_{L^q_X(T(B), \frac{dxdt}{t})}^q \) \( f(x,t) \) is an atom in \( \mathcal{T}^1_q(X) \) and so

\[
|\Lambda(f)| \leq \| \Lambda \|_{(\mathcal{T}^1_q(X))^*} \| f \|_{L^q_X(T(B), \frac{dxdt}{t})} |B|^{1-\frac{1}{q}}.
\]

Since \( \Lambda \) is linear and continuous on \( L^q_X (T_\varepsilon(B), \frac{dxdt}{t}) \), we can find a unique vector measure \( \nu_B^* \in V^q_X(T_\varepsilon(B)) \) such that

\[
\Lambda(f) = \int f \, d\nu_B^*
\]

for every \( f \in L^q_X (T_\varepsilon(B), \frac{dxdt}{t}) \), and

\[
\| \Lambda \|_{(L^q_X(T_\varepsilon(B), \frac{dxdt}{t}))^*} = \| \nu_B^* \|_{V^q_X(T_\varepsilon(B))} \leq \| \Lambda \|_{(\mathcal{T}^1_q(X))^*} |B|^{1-\frac{1}{q}}. \tag{6}
\]

Next, we can decompose \( \mathbb{R}^{n+1}_+ = \bigcup_{k=1}^{\infty} T_{\frac{1}{\sqrt{k}}} (B_k) \), where \( B_k \subset B_{k+1} \) and \( n_k \uparrow \infty \), and applying the same idea as above, we can find measures \( \nu_{B_k} \in V^q_X(T_{\frac{1}{\sqrt{k}}} (B_k)) \) representing \( \Lambda \) on \( L^q_X (T_{\frac{1}{\sqrt{k}}} (B_k), \frac{dxdt}{t}) \) such that \( \nu_{B_{k+1}} \) restricted to \( T_{\frac{1}{\sqrt{k}}} (B_k) \) coincides with \( \nu_{B_k} \).

Defining \( \nu \) on \( B_0 \) by means of

\[
\nu_{T_{\frac{1}{\sqrt{k}}} (B_k)} = \nu_{B_k}
\]

we can represent

\[
\Lambda(f) = \int f \, d\nu
\]

for every \( f \in L^q_X ([\mathbb{R}^{n+1}_+, \frac{dxdt}{t}) \) compactly supported on a tent and by (6)

\[
\| \nu \|_{\mathcal{E}'_q(X^*)} \leq \| \Lambda \|_{(\mathcal{T}^1_q(X))^*}. \tag{7}
\]

This concludes the proof. \( \square \)

Corollary 6. The space \( (\mathcal{T}^1_q(X))^* \) is continuously included in \( \mathcal{E}'_q(X^*) \).

Proof. This is a consequence of the fact that the space of functions in \( L^q_X ([\mathbb{R}^{n+1}_+, \frac{dxdt}{t}) \) compactly supported on tents is dense in \( \mathcal{T}^1_q(X) \), together with the estimate (7). \( \square \)

Proposition 7. Let \( 1 < q < \infty \). Every measure \( \nu \in \mathcal{E}'_q(X^*) \) induces a linear functional

\[
\Lambda(f) = \int_{\mathbb{R}^{n+1}_+} f \, d\nu \tag{8}
\]

where \( f \in L^q_X ([\mathbb{R}^{n+1}_+, \frac{dxdt}{t}) \) is compactly supported on a tent, that can be continuously extended to \( \mathcal{T}^1_q(X) \) and \( \| \Lambda \|_{(\mathcal{T}^1_q(X))^*} \leq C \| \nu \|_{\mathcal{E}'_q(X^*)} \) for some positive constant \( C \).

Proof. It is immediate that the linear functional (8) is well defined since \( |\Lambda(f)| \leq \| f \|_{L^q_X([\mathbb{R}^{n+1}_+, \frac{dxdt}{t})} \| \nu \|_{\mathcal{E}'_q(T_\varepsilon(B))} \) for some ball \( B \) and some positive \( \eta \).
If $a$ is an $(X,q)$-atom in $X$, supported on $T(B)$, then it is supported in $T_{\varepsilon}(B)$ for some $\varepsilon > 0$ and thus we have

$$|\Lambda(a)| = \left| \int_{T(B)} a(x,t) \, d\nu(x,t) \right|$$

$$\leq \|a\|_{L^q_{\text{loc}}(\mathbb{R}^{n+1}, A_q \, dx \, dt)} \|\nu\|_{V_{q'}^{	ext{loc}}(T(B))}$$

$$\leq \|a\|_{L^q_{\text{loc}}(\mathbb{R}^{n+1}, A_q \, dx \, dt)} \|B\|^{-\frac{1}{q'}} \|\nu\|_{\mathcal{E}_{q'}(X^*)}$$

$$\leq \|\nu\|_{\mathcal{E}_{q'}(X^*)}.$$

Now, if we were able to show the existence of a constant $C$ such that

$$|\Lambda(f)| \leq C \|f\|_{T^1_2(X)}$$

(9)

for every $f \in L^q_{\text{loc}}(\mathbb{R}^{n+1}, dx \, dt)$ compactly supported on a tent, we would have that if $f = \sum \lambda_j a_j$ is an atomic representation for $f$, then

$$\Lambda(f) = \sum \lambda_j \Lambda(a_j),$$

(10)

and therefore

$$|\Lambda(f)| \leq \sum \lambda_j \left| \int a_j \, d\nu \right|$$

$$\leq C \sum |\lambda_j| \|\nu\|_{\mathcal{E}_{q'}(X^*)}$$

$$\leq C \|f\|_{T^1_2(X)} \|\nu\|_{\mathcal{E}_{q'}(X^*)},$$

(11)

obtaining in this way the desired result (notice that estimate (9) implies the continuity of $\Lambda$ on $T^1_2(X)$, however we need the more precise estimate (11)).

To prove (9) we first notice that since $\nu \in \mathcal{E}_{q'}(X^*)$, it follows from (3) that $\eta \in L^q_{\text{loc}}(\mathbb{R}^{n+1}, d\mu)$, and for every truncated tent $T_{\varepsilon}(B)$

$$\|\eta\|_{L^q_{\text{loc}}(T_{\varepsilon}(B), d\mu)} = \|\nu\|_{V_{q'}^\infty(T_{\varepsilon}(B))} \leq \|\nu\|_{\mathcal{E}_{q'}(X^*)} |B|^{1/q'},$$

which implies that $C_{q'}(\eta) \in L^\infty(\mathbb{R}^n)$, where $C_{q'}(\eta)(x) = \sup_{x \in B} \left( \frac{1}{|B|} \int_{T(B)} |\eta|^{q'} \, dx \, dt \right)^{1/q'}$.

Thus, if $f$ is supported on $T(B)$, we will have

$$|\Lambda(f)| \leq \int_{T(B)} \|f(x,t)\|_{X} \eta(x,t) \, dx \, dt \leq C \int_{\mathbb{R}^n} A_q(\|f\|_X)(x) C_{q'}(\eta)(x) \, dx$$

(12)

(see [5], (4.1), p. 313) and thus, the right hand side of (12) can be estimated by

$$C \|A_q(\|f\|_X)\|_1 \|\nu\|_{\mathcal{E}_{q'}(X^*)} = C \|f\|_{T^1_2(X)} \|\nu\|_{\mathcal{E}_{q'}(X^*)}$$

as we wanted to show.

This finishes the proof of the proposition. \qed

We collect our previous results in the following theorem.
Theorem 8. Let $X$ be a real Banach space and $1 < q < \infty$. Then $(\Upsilon_q^1(X))^* \cong \mathcal{C}_{q'}(X^*)$ with equivalent norms.

References


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