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EXCEPTIONAL VALUES OF DERIVATIVES CONCERNING SHARED VALUE

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Abstract. Let $f(z)$ be a transcendental meromorphic function and $p(z)(\not\equiv 0)$ be a polynomial. If f and f' share 0, then $f'(z) - p(z)$ has infinitely many zeros. Moreover, one related normality criterion is obtained.

1. Introduction and Main Results

Let D be a domain in the plane C, f be a meromorphic function on D and a be finite complex number. Set

$$\overline{E}(a,f) = f^{-1}(\{a\}) \bigcap D = \{z \in D : f(z) = a\}$$

We say two meromorphic functions f and g share value a on D if $\overline{E}(a,f) = \overline{E}(a,g)$.

One of the central problems in the classical function theory is Hayman's Conjecture as follows.

In 1959, W.K. Hayman (see [**1**]) proved that if $f(z)$ is a transcendental meromorphic (entire) function and n is a positive integer satisfying $n \geq 4$ $(n \geq 3)$, then $(f^n)'$ assumes every finite non–zero value infinitely often. He conjectured in [**1**] that it remains true for $n = 2$ and $n = 3$. This conjecture was completely solved by W. Bergweiler and A. Eremenko [**17**], H.H. Chen and M.L. Fang [**2**], and L. Zalcman [**3**] independently in 1995.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function and n be a positive integer greater than 1, then $(f^n)'$ assumes every finite non–zero value infinitely often.*

Remark 1.2. The condition that n is a positive integer greater than 1 can not be omitted. For example, $f(z) = e^z + z$, but $f'(z) \neq 1$.

Afterwards, Wang [**4**] generalized Theorem 1.1 by allowing f to have only multiple zero points and pole points.

Theorem 1.3. *Let $f(z)$ be a transcendental meromorphic function. If f has only zeros of order at least 2 and poles of order at least 2, then f' assumes every finite non–zero value infinitely often.*

Theorem 1.4. *Let $f(z)$ be a meromorphic function and $c \in C \setminus \{0\}$. If $f(z)$ has only multiple zeros and poles, and $f(z) - c$ has no zeros, then f is constant.*

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Recently, shared–value theory has become one of active mathematical fields (see [**5**]–[**12**]). For a comprehensive collection of these results, we refer the reader to the monograph "Uniqueness theorems of meromorphic functions" by Yi–Yang [**13**].

Moreover, there are a lot of results on normal family concerning shared–values. For example, Schwick [**14**], Pang and Zalcman [**15**]–[**16**]. It seems, however, that an attempt has not yet been made to prove exceptional value of derivatives concerning shared-values. In a contrast with Theorem 1.1 and Theorem 1.3, we naturally ask the following question.

Question. Let $f(z)$ be a transcendental meromorphic function and a be a finite complex number. If f and f' share a in C, then does f' assume any finite value infinitely often?

Recently, the authors (see [**16**]) obtained the following result to the above question on the condition that $f(z)$ is an entire function.

Theorem 1.5. *Let $f(z)$ be a transcendental entire function and a be a finite value. If f and f' share a in C, then $f'(z)$ assumes every finite non–zero value infinitely often .*

Definition 1.6. Let $f(z)$ be a nonconstant meromorphic function and $z_0 \in C$, If $f(z_0) = z_0$, then z_0 is called a fixed point of $f(z)$.

Relate to fixed–point, we have the following result (see [**16**]).

Theorem 1.7. *Let $f(z)$ be a transcendental entire function and a be a finite value. If f and f' share a in C, then $f'(z)$ has infinitely many fixed points.*

Here we give a discussion of the case where f is meromorphic.

Theorem 1.8. *Let $f(z)$ be a transcendental meromorphic function and let $p(z)$ be a polynomial, $p(z) \not\equiv 0$. If f and f' share 0 in C, then $f'(z) - p(z)$ has infinitely many zeros.*

As the application of Theorem 1.8, we obtain the following result.

Corollary 1.9. *Let $f(z)$ be a transcendental meromorphic function. If f and f' share 0 in C, then $f'(z)$ has infinitely many fixed–points.*

Theorem 1.10. *Let $f(z)$ be a meromorphic function and $c \in C \setminus \{0\}$. If f and f' share 0 in C, and $f(z) - c$ has no zeros, then f is constant.*

Remark 1.11. The hypothesis that f and f' share 0 in C can not be weakened, as shown by the example $f(z) = c(z-1)^2/z$.

According to Bloch's Principle (see [**22**, pg. 250]), we have the following normal criterion relative to above theorems.

Theorem 1.12. *Let F be a family of meromorphic functions on the unit disc Δ and let $b(z)$ be a non–vanishing analytic function on Δ. If for any $f \in F$, f and f' share 0 on Δ, and $f'(z) \neq b(z)$, then F is normal on Δ.*

2. Some Lemmas

For the proof of our theorems, we need the following definitions and lemmas.

Definition 2.1. A meromorphic function f on C is called a normal function if there exists a positive number M such that

$$f^{\#}(z) \leq M$$

where, as usual, $f^{\#}(z) = |f'(z)|/(1+|f(z)|^2)$ denotes the spherical derivative.

From Definition 1.6, we obtain:

Lemma 2.2. *A normal meromorphic function has order at most* 2.

Definition 2.3. Let $f(z)$ be a nonconstant meromorphic function in the complex plane, we denote by $T_0(r,f)$ the Ahlfors–Shimizu characteristic function:

$$T_0(r,f) = \int_0^r \frac{A(t,f)}{t}\,dt,$$

where

$$A(t,f) = \frac{1}{\pi}\int\int_{|z|\leq t} |f^{\#}(z)|^2 dx\,dy.$$

We have the following relation between $T(r,f)$ and $T_0(r,f)$.

Lemma 2.4. $T(r,f) = T_0(r,f) + O(1)$.

Lemma 2.5 ([**16**])**.** *Let F be a family of meromorphic functions on the unit disc Δ, all of whose zeroes have multiplicity at least k, and suppose there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in F$. Then if F is not normal, for each α, $0 \leq \alpha \leq k$, there exist*

(a) *a number r, $0 < r < 1$;*

(b) *points z_n, $|z_n| < r$;*

(c) *functions $f_n \in F$; and*

(d) *positive numbers $\rho_n \to 0$;*

such that

$$\frac{f_n(z_n+\rho_n\xi)}{\rho_n^{\alpha}} = g_n(\xi) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a meromorphic function on C such that $g^{\#}(\xi) \leq g^{\#}(0) = kA+1$.

Lemma 2.6 ([**17**])**.** *Let $g(z)$ be a transcendental meromorphic function with finite order. If $g(z)$ has only finitely many critical values, then $g(z)$ has only finitely many asymptotic values.*

Lemma 2.7 ([**18**])**.** *Let $g(z)$ be a transcendental meromorphic function. Suppose that the set of finite critical and asymptotic values of $g(z)$ is bounded. Then there exists $R > 0$ such that if $|z| > R$ and $|g(z)| > R$, then*

$$|g'(z)| \geq \frac{|g(z)|\log|g(z)|}{16\pi|z|}.$$

Lemma 2.8 ([**21**]). *Let f be a transcendental meromorphic function and let p be a polynomial, $p \not\equiv 0$. Then at least one of the function f and $f' - p$ has infinitely many zeros.*

Lemma 2.9. *Let f be a transcendental meromorphic function with finite order and let $p(z)(\not\equiv 0)$ be a polynomial. If $\overline{E}(f, 0) = \overline{E}(f', 0)$, then $f'(z) - p(z)$ has infinitely many zeros.*

Proof. Suppose that $f'(z) - p(z)$ has only finitely many zeros. Since $f(z)$ is transcendental, by Lemma 2.8, we see that $f(z)$ has infinitely many zeros $z_1, z_2, \cdots$, and $\lim_{j\to\infty} z_j = \infty$.

Define $g(z) = f(z) - \int_0^z p(z)dz$, then $g'(z) = f'(z) - p(z)$. Noting that $g'(z)$ has only finitely many zeros, by Lemma 2.6 we know that $g(z)$ has only finitely many asymptotic values. Hence by Lemma 2.7 we deduce that

$$\frac{|z_j g'(z_j)|}{|g(z_j)|} \geq \frac{1}{16\pi} \log |g(z_j)| = \frac{1}{16\pi} \log |\int_0^{z_j} p(z)dz|.$$

Particularly, $\frac{|z_j g'(z_j)|}{|g(z_j)|} \to \infty$ as $j \to \infty$. On the other hand, we have from $\overline{E}(f, 0) = \overline{E}(f', 0)$ that

$$\frac{|z_j g'(z_j)|}{|g(z_j)|} = \frac{|z_j p(z_j)|}{|\int_0^{z_j} p(z)dz|} \to c_0,$$

as $j \to \infty$, where c_0 is a constant. This is a contradiction. Therefore, $f'(z) - p(z)$ has infinitely many zeros. □

Lemma 2.10. *Let f be a meromorphic function with finite order, and let b be non–zero numbers. If $\overline{E}(f, 0) = \overline{E}(f', 0)$, $f'(z) \neq b$. Then $f(z)$ is a constant.*

Proof. Since $f'(z) \neq b$, by Lemma 2.8 we obtain that $f(z)$ is not a transcendental meromorphic function.

Case 1. If f is a linear polynomial. It follows from $f'(z) \neq b$ that $f(z) = Bz + c$, where $B(\neq b)$ and c are complex numbers. Noting that $\overline{E}(f, 0) = \overline{E}(f', 0)$, we deduce that $f(z)$ is not a non-constant polynomial.

Case 2. Let

$$f(z) = \alpha + \frac{\beta}{\gamma} \tag{1}$$

where α, β and γ are polynomials, and β an γ satisfy $(\beta, \gamma) = 1$, and $\deg \beta < \deg \gamma$.

It follows from $f'(z) \neq b$ that $\alpha' \equiv b$. Thus $\alpha = bz + c$, here c is a complex number, and

$$f'(z) = b + \frac{\beta'\gamma - \gamma'\beta}{\gamma^2} . \tag{2}$$

Since $f'(z) \neq b$, we deduce from (2) that the zeros of $\beta'\gamma - \gamma'\beta$ are the zeros of γ^2. We denote them by $\omega_1, \omega_2, \cdots, \omega_m$, and denote the related orders by $l_1, l_2, \cdots, l_m$. Since β and γ are coprime, we see from (2) that ω_i is the zeros of γ with order $l_i + 1 (i = 1, 2, \cdots, m)$. Hence we have

$$\deg \gamma + \deg \beta - 1 = \deg(\beta'\gamma - \gamma'\beta) = \sum_{i=1}^{m} (l_i + 1) - m \leq \deg \gamma - m$$

which implies that $m = 1$ and $\deg \beta = 0$. Therefore,

$$f(z) \; = \; bz + c + \frac{A}{(z+d)^n}\,, \tag{3}$$

where $A(\neq 0)$, c and d are complex numbers, n is a positive integer.

As $\overline{E}(f, 0) = \overline{E}(f', 0)$, from (3) we deduce that $n = -1$. This is a contradiction. Thus the proof of the lemma is complete. □

Lemma 2.11. *Let f be a meromorphic function and s be a real number. Suppose that*

$$\limsup_{r\to\infty} \frac{T(r,\ f)}{(\log r)^2} \; = \; \infty,$$

then there exist sequence a_n, $a_n \to \infty$ such that the family

$$\Gamma \; = \; \left\{ f_n(z) := \frac{f(a_n z)}{a_n^s}\,, \quad n = 1, 2, \cdots \right\}$$

is not normal at 1.

Proof. Set

$$g(z) \; = \; \frac{f(z)}{z^s}\,, \tag{4}$$

where z^s denotes the branch of the $z^s = 1$ at $z = 1$. Since

$$\limsup_{r\to\infty} \frac{T(r,\ f)}{(\log r)^2} \; = \; \infty,$$

we obtain

$$\limsup_{r\to\infty} \frac{T(r,\ g(z))}{(\log r)^2} \; = \; \infty. \tag{5}$$

By Lemma 2.4 and (5), we get

$$\limsup_{r\to\infty} \frac{A(r,\ g(z))}{\log r} \; = \; \infty. \tag{6}$$

According to the definition of $A(r, g)$ and (6), we deduce

$$\limsup_{r\to\infty} \frac{|zg'(z)|}{1 + |g(z)|^2} \; = \; \infty.$$

It follows that there exist sequence a_n, $a_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{|a_n g'(a_n)|}{1 + |g(a_n)|^2} \; = \; \infty. \tag{7}$$

Define

$$F \; = \; \{g_n(z) := g(a_n z), \quad n = 1, 2, 3, \cdots\}.$$

By (7), we have $\lim_{n\to\infty} g_n^{\#}(1) = \infty$, and hence family
$F = \{g_n(z) := \frac{f(a_n z)}{a_n^s z^s}\,,\ n = 1, 2, 3, \cdots\}$ is not normal at 1. Therefore,

$$\Gamma \; = \; \left\{ f_n(z) := \frac{f(a_n z)}{a_n^s}, \quad n = 1, 2, 3, \cdots \right\}$$

is not normal at 1 either. □

3. Proof of the Theorems

Proof of Theorem 1.8. Set $p(z) \sim cz^k$, $(z \to \infty)$, where c is nonzero constant. We consider two cases.

Case 1. $f(z)$ is of finite order. Lemma 2.9 implies that the conclusion of Theorem 1.8 holds.

Case 2. $f(z)$ is of infinite order.

We have

$$\limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^2} = \infty.$$

Hence, using Lemma 2.11, we obtain that there exist sequence a_n, $a_n \to \infty$ such that the family

$$\Gamma = \left\{ g_n(z) := \frac{f(a_n z)}{a_n^{k+1}}, \quad n = 1, 2, \cdots \right\}$$

is not normal at 1.

We define the family F by

$$F = \left\{ f_n(z) := \frac{(k+1) f(a_n z^{\frac{1}{k+1}})}{c a_n^{k+1}}, \ n = 1, 2, \cdots, z \in D \right\}$$

where $D := \left\{ z : |z-1| < \frac{1}{2} \right\}$ and $z^{\frac{1}{k+1}}$ denotes the branch of the root that fixes 1. Thus, we have

$$f_n(1) = \frac{(k+1) f(a_n)}{c a_n^{k+1}}, \tag{8}$$

and

$$f_n'(1) = \frac{f'(a_n)}{c a_n^k}. \tag{9}$$

Therefore, we have from (8) and (9) that

$$f_n^{\#}(1) = \frac{\left|\frac{f'(a_n)}{c a_n^k}\right|}{1 + \left|\frac{(k+1) f(a_n)}{c a_n^{k+1}}\right|^2}.$$

Note that Γ is not normal at 1, by Marty's criterion we have that

$$g_n^{\#}(1) \to \infty, \quad n \to \infty$$

so we deduce that

$$f_n^{\#}(1) \to \infty, \quad n \to \infty.$$

And hence F is not normal at 1 either. By Lemma 2.5, we can find that there exist a sequence of complex numbers z_n $(z_n \to 1)$, a sequence of positive numbers ρ_n, $\rho_n \to 0$, and a sequence of functions $f_n \in F$ such that

$$h_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n} \to h(\xi) \tag{10}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C such that $h^{\#}(\xi) \le h^{\#}(0) = 2$.

We claim that $\overline{E}(h, 0) = \overline{E}(h', 0)$.

Clearly, It follows from $\overline{E}(f,0)=\overline{E}(f',0)$ that f has only zeros of order at least 2, hence h has only zeros of order at least 2 also. We have established $\overline{E}(h,0)\subseteq\overline{E}(h',0)$.

Suppose that w_0 is a point such that $h'(w_0)=0$. Since $h'(\xi)\not\equiv 0$, there exist w_n, $w_n\to w_0$, such that, for sufficiently large n,

$$h'_n(w_n)=f'_n(z_n+\rho_n w_n)=0.$$

Hence

$$h_n(w_n)=\frac{f_n(z_n+\rho_n w_n)}{\rho_n}=0.$$

Letting $n\to\infty$, we obtain $h(w_0)=0$. It follows that $\overline{E}(h',0)\subseteq\overline{E}(h,0)$. Thus $\overline{E}(h,0)=\overline{E}(h',0)$.

Now we suppose that $f'(z)-p(z)$ has only finitely many zeros. From the definition of F and (10), we have that for sufficiently large n,

$$h'_n(\xi)\ =\ \frac{f'[a_n(z_n+\rho_n\xi)^{\frac{1}{k+1}}](z_n+\rho_n\xi)^{\frac{1}{k+1}}}{ca_n^k(z_n+\rho_n\xi)}\neq\frac{p(a_n(z_n+\rho_n\xi)^{\frac{1}{k+1}})(z_n+\rho_n\xi)^{\frac{1}{k+1}}}{ca_n^k(z_n+\rho_n\xi)}. \tag{11}$$

On the other hand, we have

$$\frac{p(a_n z^{\frac{1}{k+1}})z^{\frac{1}{k+1}}}{ca_n^k z}\ \to\ 1 \tag{12}$$

as $n\to\infty$, uniformly for $z\in D_\delta:=|z-1|<\delta$, where δ is one sufficiently small constant. Again by (10), (11) and (12), we deduce that $h(\xi)\neq 1$, and hence by Lemma 2.10, we obtain a contradiction. The proof of the theorem is complete.

Proof of Theorem 1.12. Without loss of generality, we suppose that F is not normal at point $z_0=0$. By Lemma 2.5, we can find that there exist a sequence of complex numbers z_n, a sequence of positive numbers ρ_n, $\rho_n\to 0$, and a sequence of functions $f_n\in F$ such that

$$g_n(\xi)\ =\ \frac{f_n(z_n+\rho_n\xi)}{\rho_n}\to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C such that $g^{\#}(\xi)\le g^{\#}(0)=2$.

In the same way as in the proof of Theorem 1.8, we may also obtain that g is of order at most two and $\overline{E}(g,0)=\overline{E}(g',0)$. Thus, $g''(\xi)\not\equiv 0$.

Without loss of generality, we suppose that $z_n\to z_0$. Obviously, $b(z_0)\neq 0,\infty$. From Lemma 2.10, there exists ξ_0 such that $g'(\xi_0)=b(z_0)$. Moreover, there exists a positive number δ such that $g(\xi)$ is analytic on $D_{3\delta}(\xi:|\xi-\xi_0|<3\delta)$. Hence, $g'_n(\xi)$ are analytic on $D_{2\delta}(\xi:|\xi-\xi_0|<2\delta)$ for sufficiently large n.

As

$$g'_n(\xi)-b(z_n+\rho_n\xi)=f'_n(z_n+\rho_n\xi)-b(z_n+\rho_n\xi)\neq 0$$

and $g'_n(\xi)-b(z_n+\rho_n\xi)$ converges uniformly to $g'(\xi)-b(z_0)$ on $D_\delta(\xi:|\xi-\xi_0|<\delta)$. Therefore, by Hurwitz's theorem we deduce that $g'(\xi)-b(z_0)\equiv 0$ on D_δ, thus we have $g'(\xi)-b(z_0)\equiv 0$, for all $\xi\in C$. Hence we have $g''(\xi)\equiv 0$, which is a contradiction. The proof of the theorem is complete.

Proof of Theorem 1.10. Using Theorem 1.8 and Lemma 2.10, we immediately obtain Theorem 1.10.

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References

1. W.K. Hayman, *Research Problems in Function Theory*, Athlone Press University of London, 1967.
2. H.H. Chen and M.L. Fang, On the value distribution of $f^n f'$, Science in China, **25** (1995), 121–127.
3. L. Zalcman, *On Some Problems of Hayman*, Preprint (Bar–Ilan University), 1995.
4. Y.F. Wang and M.L. Fang, Picard values and normal families of meromorphic functions with multiple zeros, Acta Math. Sinica, New Series, **41** (1998), 743–748.
5. G. Frank and M. Reinders, *A unique range set for meromorphic functions with* 11 *elements*, Complex Variables, **37** (1998), 185–193.
6. H. Fujimoto, *On uniqueness of meromorphic functions sharing finite sets*, Amer. J. Math. **122** (2000), 1175–1203.
7. F. Gross, *On the distribution of values of meromorphic functions*, Trans. Amer. Math. Soc. **131** (1968), 199–214.
8. P. Li and C.C. Yang, *On the unique range set of meromorphic functions*, Proc. Amer. Math. Soc. **124** (1996), 177–185.
9. L. Liao and C.C. Yang, *On the cardinality of the unique range set for meromorphic and entire functions*, Indian J. Pure Appl. Math. **31** (2000), 431–440.
10. E. Mues and M. Reinders, *Meromorphic functions sharing one value and unique range sets*, Kodai Math. J. **18** (1995), 515–522.
11. H.X. Yi, *Uniqueness of meromorphic functions and a question of Gross*, Science in China (Ser A), **37** (1994), 802–813.
12. H.X. Yi, *On a question of Gross concerning uniqueness of entire functions*, Bull. Austral Math. Soc. **57** (1998), 343–349.
13. H.X. Yi, C.C. Yang, *Theory of the Uniqueness of Meromorphic Functions*, Science Press, Beijing, 1995.
14. W. Schwick, *Sharing values and normality*, Arch. Math. (Basel) **59** (1992), 52–54.
15. X. Pang and L. Zalcman, *Normality and shared values*, Ark. Mat. **38** (2000), 171–182.
16. X. Pang and L. Zalcman, *Normal families and shared values*, Bull. London Math. Soc. **32** (2000), 325–331.
17. W. Bergweiler and W. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoam. **11** (1995), 355–373.
18. P. Rippon and G. Stallard, *Iteration of a class of hyperbolic meromorphic functions*, Proc. Amer. Math. Soc. **115** (1999), No. 2, 353–362.
19. W. Bergweiler, *On the zeros of certain homogeneous differential polynomials*, Arch. Math. (Basel) **64** (1995), 199–202.
20. W. Bergweiler, *Nomality and exceptional values of derivatives*, Proc. Amer. Math. Soc. **129** (2000), 121–129.

21. X.H. Hua, *On a problem of Hayman*, Kodai Math. J. **13** (1990), 386–390.
22. Einar Hille, *Analytic Function Theory*, Vol. 2, Ginn, Boston, 1962.
23. L. Yang, *Value distribution theory*, Springer–Verlag, Berlin–Heidelberg, 1993.
24. Y. Gu, *Normal Families of Meromorphic Function*, Press Sichuan Education, 1991.
25. W.C. Lin and H.X. Yi, *Value distribution of entire function concerning shared–values*, Indian J. Pure Appl. Math.**34(4)**(2003), 535-541.

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