EXCEPTIONAL VALUES OF DERIVATIVES CONCERNING
SHARED VALUE

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Abstract. Let \( f(z) \) be a transcendental meromorphic function and \( p(z) \neq 0 \) be a polynomial. If \( f \) and \( f' \) share 0, then \( f'(z) - p(z) \) has infinitely many zeros. Moreover, one related normality criterion is obtained.

1. Introduction and Main Results

Let \( D \) be a domain in the plane \( \mathbb{C} \), \( f \) be a meromorphic function on \( D \) and \( a \) be finite complex number. Set

\[
\mathcal{E}(a, f) = f^{-1}(\{a\}) \cap D = \{z \in D : f(z) = a\}
\]

We say two meromorphic functions \( f \) and \( g \) share value \( a \) on \( D \) if \( \mathcal{E}(a, f) = \mathcal{E}(a, g) \).

One of the central problems in the classical function theory is Hayman’s Conjecture as follows.

In 1959, W.K. Hayman (see [1]) proved that if \( f(z) \) is a transcendental meromorphic (entire) function and \( n \) is a positive integer satisfying \( n \geq 4 \) \((n \geq 3)\), then \( (f^n)' \) assumes every finite non–zero value infinitely often. He conjectured in [1] that it remains true for \( n = 2 \) and \( n = 3 \). This conjecture was completely solved by W. Bergweiler and A. Eremenko [17], H.H. Chen and M.L. Fang [2], and L. Zalcman [3] independently in 1995.

**Theorem 1.1.** Let \( f(z) \) be a transcendental meromorphic function and \( n \) be a positive integer greater than 1, then \( (f^n)' \) assumes every finite non–zero value infinitely often.

**Remark 1.2.** The condition that \( n \) is a positive integer greater than 1 can not be omitted. For example, \( f(z) = e^z + z \), but \( f'(z) \neq 1 \).

Afterwards, Wang [4] generalized Theorem 1.1 by allowing \( f \) to have only multiple zero points and pole points.

**Theorem 1.3.** Let \( f(z) \) be a transcendental meromorphic function. If \( f \) has only zeros of order at least 2 and poles of order at least 2, then \( f' \) assumes every finite non–zero value infinitely often.

**Theorem 1.4.** Let \( f(z) \) be a meromorphic function and \( c \in \mathbb{C} \setminus \{0\} \). If \( f(z) \) has only multiple zeros and poles, and \( f(z) - c \) has no zeros, then \( f \) is constant.

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Recently, shared-value theory has become one of active mathematical fields (see [5]–[12]). For a comprehensive collection of these results, we refer the reader to the monograph “Uniqueness theorems of meromorphic functions” by Yi–Yang [13].

Moreover, there are a lot of results on normal family concerning shared-values. For example, Schwick [14], Pang and Zalcman [15]–[16]. It seems, however, that an attempt has not yet been made to prove exceptional value of derivatives concerning shared-values. In contrast with Theorem 1.1 and Theorem 1.3, we naturally ask the following question.

**Question.** Let $f(z)$ be a transcendental meromorphic function and $a$ be a finite complex number. If $f$ and $f'$ share $a$ in $C$, then does $f'$ assume any finite value infinitely often?

Recently, the authors (see [16]) obtained the following result to the above question on the condition that $f(z)$ is an entire function.

**Theorem 1.5.** Let $f(z)$ be a transcendental entire function and $a$ be a finite value. If $f$ and $f'$ share $a$ in $C$, then $f'(z)$ assumes every finite non-zero value infinitely often.

**Definition 1.6.** Let $f(z)$ be a nonconstant meromorphic function and $z_0 \in C$, if $f(z_0) = z_0$, then $z_0$ is called a fixed point of $f(z)$.

Relate to fixed-point, we have the following result (see [16]).

**Theorem 1.7.** Let $f(z)$ be a transcendental entire function and $a$ be a finite value. If $f$ and $f'$ share $a$ in $C$, then $f'(z)$ has infinitely many fixed points.

Here we give a discussion of the case where $f$ is meromorphic.

**Theorem 1.8.** Let $f(z)$ be a transcendental meromorphic function and let $p(z)$ be a polynomial, $p(z) \not\equiv 0$. If $f$ and $f'$ share $0$ in $C$, then $f'(z) - p(z)$ has infinitely many zeros.

As the application of Theorem 1.8, we obtain the following result.

**Corollary 1.9.** Let $f(z)$ be a transcendental meromorphic function. If $f$ and $f'$ share $0$ in $C$, then $f'(z)$ has infinitely many fixed points.

**Theorem 1.10.** Let $f(z)$ be a meromorphic function and $c \in C \setminus \{0\}$. If $f$ and $f'$ share $0$ in $C$, and $f(z) - c$ has no zeros, then $f$ is constant.

**Remark 1.11.** The hypothesis that $f$ and $f'$ share $0$ in $C$ can not be weakened, as shown by the example $f(z) = c(z - 1)^2/z$.

According to Bloch’s Principle (see [22, pg. 250]), we have the following normal criterion relative to above theorems.

**Theorem 1.12.** Let $F$ be a family of meromorphic functions on the unit disc $\Delta$ and let $b(z)$ be a non-vanishing analytic function on $\Delta$. If for any $f \in F$, $f$ and $f'$ share $0$ on $\Delta$, and $f'(z) \neq b(z)$, then $F$ is normal on $\Delta$. 
2. Some Lemmas

For the proof of our theorems, we need the following definitions and lemmas.

**Definition 2.1.** A meromorphic function $f$ on $C$ is called a normal function if there exists a positive number $M$ such that

$$f^\#(z) \leq M$$

where, as usual, $f^\#(z) = |f'(z)|/(1 + |f(z)|^2)$ denotes the spherical derivative.

From Definition 1.6, we obtain:

**Lemma 2.2.** A normal meromorphic function has order at most 2.

**Definition 2.3.** Let $f(z)$ be a nonconstant meromorphic function in the complex plane, we denote by $T_0(r, f)$ the Ahlfors–Shimizu characteristic function:

$$T_0(r, f) = \int_0^r A(t, f) \frac{dt}{t},$$

where

$$A(t, f) = \frac{1}{\pi} \int \int_{|z| \leq t} |f^\#(z)|^2 dx \, dy.$$

We have the following relation between $T(r, f)$ and $T_0(r, f)$.

**Lemma 2.4.** $T(r, f) = T_0(r, f) + O(1)$.

**Lemma 2.5** ([16]). Let $F$ be a family of meromorphic functions on the unit disc $\Delta$, all of whose zeroes have multiplicity at least $k$, and suppose there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in F$. Then if $F$ is not normal, for each $\alpha$, $0 \leq \alpha \leq k$, there exist
(a) a number $r$, $0 < r < 1$;
(b) points $z_n$, $|z_n| < r$;
(c) functions $f_n \in F$; and
(d) positive numbers $\rho_n \to 0$;

such that

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^\alpha} = g_n(\xi) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where $g$ is a meromorphic function on $C$ such that $g^\#(\xi) \leq g^\#(0) = kA + 1$.

**Lemma 2.6** ([17]). Let $g(z)$ be a transcendental meromorphic function with finite order. If $g(z)$ has only finitely many critical values, then $g(z)$ has only finitely many asymptotic values.

**Lemma 2.7** ([18]). Let $g(z)$ be a transcendental meromorphic function. Suppose that the set of finite critical and asymptotic values of $g(z)$ is bounded. Then there exists $R > 0$ such that if $|z| > R$ and $|g(z)| > R$, then

$$|g'(z)| \geq \frac{|g(z)| \log |g(z)|}{16\pi |z|}.$$
Lemma 2.8 ([21]). Let \( f \) be a transcendental meromorphic function and let \( p \) be a polynomial, \( p \neq 0 \). Then at least one of the function \( f \) and \( f' - p \) has infinitely many zeros.

Lemma 2.9. Let \( f \) be a transcendental meromorphic function with finite order and let \( p(z)(\neq 0) \) be a polynomial. If \( E(f, 0) = E(f', 0) \), then \( f' - p(z) \) has infinitely many zeros.

Proof. Suppose that \( f' - p \) has only finitely many zeros. Since \( f(z) \) is transcendental, by Lemma 2.8, we see that \( f(z) \) has infinitely many zeros \( z_1, z_2, \cdots \), and \( \lim_{j \to \infty} z_j = \infty \).

Define \( g(z) = f(z) - \int_0^z p(z)dz, \) then \( g'(z) = f'(z) - p(z) \). Noting that \( g'(z) \) has only finitely many zeros, by Lemma 2.6 we know that \( g(z) \) has only finitely many asymptotic values. Hence by Lemma 2.7 we deduce that

\[
\frac{|z_jg'(z_j)|}{|g(z_j)|} > \frac{1}{16\pi}\log |g(z_j)| = \frac{1}{16\pi}\int_0^{z_j} |p(z)|dz.
\]

Particularly, \( \frac{|z_jg'(z_j)|}{|g(z_j)|} \to \infty \) as \( j \to \infty \). On the other hand, we have from \( E(f, 0) = E(f', 0) \) that

\[
\frac{|z_jg'(z_j)|}{|g(z_j)|} = \frac{|z_jp(z_j)|}{|\int_0^{z_j} p(z)dz|} \to c_0,
\]

as \( j \to \infty \), where \( c_0 \) is a constant. This is a contradiction. Therefore, \( f'(z) - p(z) \) has infinitely many zeros. \( \square \)

Lemma 2.10. Let \( f \) be a meromorphic function with finite order, and let \( b \) be non-zero numbers. If \( E(f, 0) = E(f', 0) \), then \( f(z) \) is a constant.

Proof. Since \( f'(z) \neq b \), by Lemma 2.8 we obtain that \( f(z) \) is not a transcendental meromorphic function.

Case 1. If \( f \) is a linear polynomial. It follows from \( f'(z) \neq b \) that \( f(z) = Bz + c, \) where \( B(\neq b) \) and \( c \) are complex numbers. Noting that \( E(f, 0) = E(f', 0) \), we deduce that \( f(z) \) is not a non-constant polynomial.

Case 2. Let

\[
f(z) = \alpha + \frac{\beta}{\gamma}
\]

where \( \alpha, \beta \) and \( \gamma \) are polynomials, and \( \beta \) an \( \gamma \) satisfy \( (\beta, \gamma) = 1 \), and \( \deg \beta < \deg \gamma \).

It follows from \( f'(z) \neq b \) that \( \alpha' \equiv 0. \) Thus \( \alpha = b z + c; \) here \( c \) is a complex number, and

\[
f'(z) = b + \frac{\beta'\gamma - \gamma'\beta}{\gamma^2}.
\]

Since \( f'(z) \neq b \), we deduce from (2) that the zeros of \( \beta'\gamma - \gamma'\beta \) are the zeros of \( \gamma^2 \). We denote them by \( \omega_1, \omega_2, \cdots, \omega_m \), and denote the related orders by \( l_1, l_2, \cdots, l_m \). Since \( \beta \) and \( \gamma \) are coprime, we see from (2) that \( \omega_i \) is the zeros of \( \gamma \) with order \( l_i + 1(i = 1, 2, \cdots, m) \). Hence we have

\[
\deg \gamma + \deg \beta - 1 = \deg(\beta'\gamma - \gamma'\beta) = \sum_{i=1}^m (l_i + 1) - m \leq \deg \gamma - m
\]
which implies that $m = 1$ and $\deg \beta = 0$. Therefore,

$$f(z) = bz + c + \frac{A}{(z + d)^n},$$  

(3)

where $A \neq 0$, $c$ and $d$ are complex numbers, $n$ is a positive integer.

As $\overline{E}(f, 0) = \overline{E}(f', 0)$, from (3) we deduce that $n = -1$. This is a contradiction. Thus the proof of the lemma is complete. □

Lemma 2.11. Let $f$ be a meromorphic function and $s$ be a real number. Suppose that

$$\limsup_{r \to \infty} \frac{T(r, f)}{\log r^2} = \infty,$$

then there exist sequence $a_n$, $a_n \to \infty$ such that the family

$$\Gamma = \left\{ f_n(z) := \frac{f(a_n z)}{a_n^s}, \quad n = 1, 2, \cdots \right\}$$

is not normal at 1.

Proof. Set

$$g(z) = \frac{f(z)}{z^s},$$  

(4)

where $z^s$ denotes the branch of the $z^s = 1$ at $z = 1$. Since

$$\limsup_{r \to \infty} \frac{T(r, f)}{\log r^2} = \infty,$$

we obtain

$$\limsup_{r \to \infty} \frac{T(r, g(z))}{\log r^2} = \infty.$$  

(5)

By Lemma 2.4 and (5), we get

$$\limsup_{r \to \infty} \frac{A(r, g(z))}{\log r} = \infty.$$  

(6)

According to the definition of $A(r, g)$ and (6), we deduce

$$\limsup_{r \to \infty} \frac{|zg'(z)|}{|1 + |g(z)||^2} = \infty.$$  

It follows that there exist sequence $a_n$, $a_n \to \infty$ such that

$$\lim_{n \to \infty} \frac{|a_n g'(a_n)|}{1 + |g(a_n)|^2} = \infty.$$  

(7)

Define

$$F = \{ g_n(z) := g(a_n z), \quad n = 1, 2, 3, \cdots \}.$$  

By (7), we have $\lim_{n \to \infty} g_n^s(1) = \infty$, and hence family

$$F = \{ g_n(z) := \frac{f(a_n z)}{a_n^s}, \quad n = 1, 2, 3, \cdots \}$$

is not normal at 1. Therefore,

$$\Gamma = \left\{ f_n(z) := \frac{f(a_n z)}{a_n^s}, \quad n = 1, 2, 3, \cdots \right\}$$

is not normal at 1 either. □
3. Proof of the Theorems

Proof of Theorem 1.8. Set \( p(z) \sim cz^k, \ (z \to \infty) \), where \( c \) is nonzero constant. We consider two cases.

Case 1. \( f(z) \) is of finite order. Lemma 2.9 implies that the conclusion of Theorem 1.8 holds.

Case 2. \( f(z) \) is of infinite order. We have
\[
\limsup_{r \to \infty} \frac{T(r,f)}{(\log r)^2} = \infty.
\]
Hence, using Lemma 2.11, we obtain that there exist sequence \( a_n, a_n \to \infty \) such that the family
\[
\Gamma = \left\{ g_n(z) := \frac{f(a_n z)}{a_{n+1}^k}, \ n = 1, 2, \ldots \right\}
\]
is not normal at 1.
We define the family \( F \) by
\[
F = \left\{ f_n(z) := \frac{(k+1)f(a_n z)}{ca_{n+1}^k}, \ n = 1, 2, \ldots, z \in D \right\}
\]
where \( D := \{ z : |z - 1| < \frac{1}{2} \} \) and \( z^{1/k} \) denotes the branch of the root that fixes 1.
Thus, we have
\[
f_n(1) = \frac{(k+1)f(a_n)}{ca_{n+1}^k}, \ \text{(8)}
\]
and
\[
f'_n(1) = \frac{f'(a_n)}{ca_n^k}. \ \text{(9)}
\]
Therefore, we have from (8) and (9) that
\[
f_n^#(1) = \frac{|f(a_n)|}{ca_n^k} \frac{1}{1 + \frac{|f(a_n)|}{ca_n^k}^2}.
\]
Note that \( \Gamma \) is not normal at 1, by Marty’s criterion we have that
\[
g_n^#(1) \to \infty, \ n \to \infty
\]
so we deduce that
\[
f_n^#(1) \to \infty, \ n \to \infty.
\]
And hence \( F \) is not normal at 1 either. By Lemma 2.5, we can find that there exist a sequence of complex numbers \( z_n, \ (z_n \to 1) \), a sequence of positive numbers \( \rho_n, \ \rho_n \to 0 \), and a sequence of functions \( f_n \in F \) such that
\[
h_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n} \to h(\xi)
\]
locally uniformly with respect to the spherical metric, where \( g \) is a nonconstant meromorphic function on \( C \) such that \( h^#(\xi) \leq h^#(0) = 2 \).
We claim that \( E(h,0) = E(h',0) \).
Clearly, it follows from $E(f,0) = E(f',0)$ that $f$ has only zeros of order at least 2, hence $h$ has only zeros of order at least 2 also. We have established $E(h,0) \subseteq E(h',0)$.

Suppose that $w_0$ is a point such that $h'(w_0) = 0$. Since $h' (\xi) \neq 0$, there exist $w_n, w_n \to w_0$, such that, for sufficiently large $n$,

$$h'_n(w_n) = f'_n(z_n + \rho_n w_n) = 0.$$ 

Hence

$$h_n(w_n) = \frac{f_n(z_n + \rho_n w_n)}{\rho_n} = 0.$$ 

Letting $n \to \infty$, we obtain $h(w_0) = 0$. It follows that $E(h',0) \subseteq E(h,0)$. Thus $E(h,0) = E(h',0)$.

Now we suppose that $f'(z) - p(z)$ has only finitely many zeros. From the definition of $F$ and (10), we have that for sufficiently large $n$,

$$h'_n(\xi) = \frac{f'_n[a_n(z_n + \rho_n \xi)]}{ca_n^2(z_n + \rho_n \xi)} \neq \frac{p(a_n(z_n + \rho_n \xi))}{ca_n^2(z_n + \rho_n \xi)}.$$ 

On the other hand, we have

$$\frac{p(a_n z \frac{1}{\rho_n} z \frac{1}{\rho_n})}{ca_n^2 z} \to 1$$

as $n \to \infty$, uniformly for $z \in D_\delta := |z - 1| < \delta$, where $\delta$ is one sufficiently small constant. Again by (10), (11) and (12), we deduce that $h(\xi) \neq 1$, and hence by Lemma 2.10, we obtain a contradiction. The proof of the theorem is complete.

**Proof of Theorem 1.12.** Without loss of generality, we suppose that $F$ is not normal at point $z_0 = 0$. By Lemma 2.5, we can find that there exist a sequence of complex numbers $z_n$, a sequence of positive numbers $\rho_n$, $\rho_n \to 0$, and a sequence of functions $f_n \in F$ such that

$$g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where $g$ is a nonconstant meromorphic function on $C$ such that $g^#(\xi) \leq g^#(0) = 2$.

In the same way as in the proof of Theorem 1.8, we may also obtain that $g$ is of order at most two and $E(g,0) = E(g',0)$. Thus, $g'(\xi) \neq 0$.

Without loss of generality, we suppose that $z_n \to z_0$. Obviously, $b(z_0) \neq 0, \infty$. From Lemma 2.10, there exists $\xi_0$ such that $g'(\xi_0) = b(z_0)$. Moreover, there exists a positive number $\delta$ such that $g(\xi)$ is analytic on $D_3(\xi : |\xi - \xi_0| < 3\delta)$. Hence, $g'_n(\xi)$ are analytic on $D_{2\delta}(\xi : |\xi - \xi_0| < 2\delta)$ for sufficiently large $n$.

As

$$g'_n(\xi) - b(z_n + \rho_n \xi) = f'_n(z_n + \rho_n \xi) - b(z_n + \rho_n \xi) \neq 0$$

and $g'_n(\xi) - b(z_n + \rho_n \xi)$ converges uniformly to $g'(\xi) - b(\xi)$ on $D_{\delta}(\xi : |\xi - \xi_0| < \delta)$. Therefore, by Hurwitz’s theorem we deduce that $g'(\xi) - b(z_0) \equiv 0$ on $D_{\delta}$, thus we have $g'(\xi) - b(z_0) \equiv 0$, for all $\xi \in C$. Hence we have $g''(\xi) \equiv 0$, which is a contradiction. The proof of the theorem is complete.
Proof of Theorem 1.10. Using Theorem 1.8 and Lemma 2.10, we immediately obtain Theorem 1.10.

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