TOPOLOGY OF QUASI–PROJECTIVE VARIETIES AND LEFSCHETZ THEORY

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Abstract. In 1924, Lefschetz published a fundamental work on the topology of non–singular complex projective varieties. Since then, several generalizations (in different directions) were proved by a number of mathematicians. We try to give here a short exposition of this field of Mathematics covering almost 80 years history.

Introduction

In the general study of the topology of algebraic varieties, Lefschetz proved in \cite{20} two fundamental results on the topology of non–singular complex projective varieties (i.e., non-singular irreducible closed algebraic subsets of some complex projective space \(\mathbb{P}^n\)), namely: the Hyperplane Section Theorem (HST) and the so–called Second Lefschetz Theorem (SLT). These results describe how the topology of such a variety is connected to the topology of a generic hyperplane section. More precisely, the HST asserts that, if \(X\) is such a variety and \(L\) a generic hyperplane of \(\mathbb{P}^n\), then the natural map (between homology groups)

\[ H_q(L \cap X) \rightarrow H_q(X) \]

is bijective for \(q \leq d - 2\) and surjective for \(q = d - 1\), where \(d\) is the dimension of \(X\). The SLT expresses the kernel of the natural map

\[ H_{d-1}(L \cap X) \rightarrow H_{d-1}(X) \]

in terms of “vanishing cycles” which appear in a pencil of hyperplanes with generic axis and having \(L\) as a member.

The original proofs of Lefschetz are incomplete. The first complete proofs are due to Wallace \cite{26}. The details of Wallace’s presentation still remain, nevertheless, rather complicated. A simpler presentation was given later by Lamotke \cite{18}. Both the proofs of \cite{26} and \cite{18} follow Lefschetz’s methodology.

In an unpublished lecture given at Princeton (as cited in \cite{2}), Thom suggested an alternative approach to prove Lefschetz’s theorems: the use of Morse theory. Using Thom’s remarks, Andreotti–Frankel \cite{1} and Bott \cite{3} then gave a new proof of the HST. Note that \cite{3} also contains a homotopical version of the HST (see also \cite{22}). Later, in \cite{2}, Andreotti–Frankel also gave a new proof of the SLT (again, a proof based on Morse theory).

In \cite{29}, Zariski gave a version of the HST for the fundamental groups of the complements of projective hypersurfaces. Its proof is incomplete. The first complete proof is due to Ham–Lê \cite{15}.

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In [15], Hamm–Lê in fact extended the theorem of Zariski to higher homotopy groups with the same bound as in the HST.

Later, in [16] and [12, 13], Hamm–Lê and Goresky–MacPherson gave a general homotopical version of the HST for singular quasi–projective varieties. The influence of the singularities is measured by the rectified homotopical depth rhd of Grothendieck (in [16]) or with the help of an invariant involving the number of equations needed to define locally the variety (in [12, 13]). The HST of [16] and the one of [12, 13] are equivalent (cf. [14] and [17]).

In [11], the author introduced a more general invariant to measure the effect of the singularities, namely: the global rectified homotopical depth grhd, and also proved a general homotopical version of the HST for singular quasi-projective varieties generalizing, in some direction, the version of Hamm–Lê and Goresky–MacPherson.

The fact that the grhd is more general than the rhd appears as a special case of a general conjecture of Grothendieck [14] (completely independent of the results mentioned above) proved by the author in [10].

In [5], Chéniot showed that, in the particular case of the complements of projective algebraic sets, the results of [16], [12, 13] and [11] could be improved.

In [25] (see also [28]), van Kampen gave a presentation by generators and relations of the fundamental group of the complement of an algebraic curve in $\mathbb{P}^2$. The generators are loops in a generic line, around its intersection points with the curve. The relations are obtained by considering a generic pencil containing this line: each loop must be identified with its transforms by monodromy around the exceptional members of the pencil. The proof of van Kampen is incomplete. The first complete proof is due to Chéniot [4].

In [21], Libgober gave a high–dimensional analogue of the van Kampen theorem for higher–homotopy groups of the complements of hypersurfaces with isolated singularities in $\mathbb{C}^n$ behaving well at infinity. Libgober also gave a way to deduce the projective case from the affine case. The projective version of his theorem is effectively deduced in [8]. It describes the homotopy group $\pi_{n-1}(\mathbb{P}^n \setminus H, +)$ (with $n \geq 3$), where $H$ is a projective hypersurface of $\mathbb{P}^n$ having only isolated singularities (notice that $\pi_{n-1}(\mathbb{P}^n \setminus H, +)$ is the first homotopy group not preserved by hyperplane section). This group is given as a quotient of the $(n-1)$–st homotopy group of a general hyperplane section, using a generic pencil containing this hyperplane. But, in contrast with the van Kampen theorem, in order to obtain the subgroup by which the quotient is taken it is no more enough to identify each homotopy $(n-1)$–cell of the general hyperplane section to its transforms by monodromy around the exceptional members of the pencil. One must also take into account some “degeneration operators” associated with these exceptional hyperplanes.

In [6], Chéniot observed that the SLT did not apply unchanged to quasi–projective varieties (even in the very special case of the complements of hypersurfaces), and proposed a modified statement in the non–singular quasi-projective case. In this statement, the kernel of the natural map concerned by the SLT is described in terms of some “homological variation operators”. These operators are defined with the help of the monodromies around the exceptional hyperplanes of a generic pencil containing the given general hyperplane; but a crucial point here is that these operators take into account the relative cycles of the general hyperplane section modulo...
the section by the axis of the pencil. This version of the SLT may be considered as a high-dimensional generalized homological form of the van Kampen theorem.

In [8], Chéniot–Libgober defined a homotopical version of Chéniot's homological variation operators, in the special case of the complements of projective hypersurfaces with isolated singularities, and showed that these “homotopical operators” lead to a high-dimensional van Kampen theorem equivalent to the projective version of [21]. More precisely, they proved that the homotopical variation operators are sufficient to describe entirely the subgroup by which the quotient is taken in the projective version of Libgober’s theorem mentioned above. Chéniot–Libgober’s theorem is a homotopical analogue of [6], in the special case of the complements of projective hypersurfaces with isolated singularities, and thus may also be viewed as a homotopical version of the SLT for this particular case.

But the homotopical variation operators, as well as the degeneration operators used in [21], are not so easy to define. The definitions go through the homology of universal covers, and the link between homotopy and homology depends strongly on the special topology of the complement of hypersurfaces with isolated singularities. In [7], Chéniot and the author gave a very short and purely homotopical definition of the homotopical variation operators. Moreover, this new definition makes sense in the general setting of singular quasi–projective varieties, and thus opens the way to obtain more general van Kampen theorems (a conjecture is stated in [7] in the special case of non–singular quasi-projective varieties).

The aim of this paper is to collect, in a short exposition, the results mentioned above which cover an important domain of Mathematics.

Notation

Throughout the paper, homology groups are singular homology groups with integer coefficients. We shall note the homology class in a space $A$ of an (absolute) cycle $z$ by $[z]_A$ and the homology class in $A$ modulo a subspace $B$ of a relative cycle $z'$ by $[z']_{A,B}$. If there is no ambiguity, we shall omit the subscripts. If $(A,B)$ is a pointed pair with base point $* \in B$, we shall denote by $F^q(A,B,*)$ the set of relative homotopy $q$–cells of $A$ modulo $B$ based at $*$. These are maps from the $q$-cube $I^q$ to $A$ with the face $x_q = 0$ sent into $B$ and all other faces sent to $*$ (as in [24, §15]). We designate by $F^q(A,*)$ the set of absolute homotopy $q$–cells of $A$ based at $*$, that is maps from $I^q$ to $A$ sending the boundary $\partial I^q$ of $I^q$ to $*$. Given $f \in F^q(A,B,*)$ (resp. $F^q(A,*)$), the homotopy class of $f$ in $A$ modulo $B$ based at $*$ (resp. in $A$ based at $*$) will be denoted by $\langle f \rangle_{A,B,*}$ (resp. $\langle f \rangle_{A,*}$). Again, if there is no ambiguity, we shall omit the subscripts.

1. The Hyperplane Section Theorem

In Section 1.1, we recall the classical HST of Lefschetz on non–singular complex projective varieties. In Section 1.2, we recall Zariski’s theorem on the fundamental groups of the complements of projective hypersurfaces. In Section 1.3, we discuss about the generalizations of these theorems to singular quasi–projective varieties.

1.1. The original theorem of Lefschetz.

Throughout this Section 1.1, the notation is as follows.
Notation 1.1. Let $X$ be a non-empty $d$-dimensional non-singular irreducible closed algebraic subset of the complex projective space $\mathbb{P}^n$, with $n \geq 1$, and $L$ be a projective hyperplane of $\mathbb{P}^n$ transverse to $X$ (the choice of such a hyperplane is generic).

The classical HST is as follows.

Theorem 1.2 (cf. [20]). With Notation 1.1, the relative homology group $H_q(X, L \cap X)$ vanishes for $q \leq d - 1$.

Equivalently, the natural map $H_q(L \cap X) \to H_q(X)$ is an isomorphism for $q \leq d - 2$ and an epimorphism for $q = d - 1$.

Lefschetz proves this theorem by embedding $L$ in a pencil of hyperplanes with generic axis (here, this means with an axis transverse to $X$), and examining the degenerations of the intersections of $X$ with the hyperplanes of the pencil. The topological part of its proof is incomplete. The first complete proof, using the modern technics of singular homology theory, is due to Wallace [26]. The details of Wallace’s presentation still remain, nevertheless, rather complicated. A simpler presentation was given later by Lamotke [18]. Both the proofs of Wallace and Lamotke use the method of generic pencils of [20].

As already mentioned in the introduction, in an unpublished lecture given at the Institute for Advanced Study in Princeton in 1957, Thom described the geometric situation of the Lefschetz theorems in terms of Morse theory. Using Thom’s remarks, Andreotti–Frankel [1] and Bott [3] then gave a new proof of the HST, a proof which does not involve the use of a pencil. Moreover, [3] also contains the following homotopical extension of the HST (see also [22]).

Theorem 1.3 (cf. [3], [22]). With Notation 1.1, the pair $(X, L \cap X)$ is $(d - 1)$-connected.

Equivalently, for every base point $*$ in $L \cap X$, the natural map $\pi_q(L \cap X, *) \to \pi_q(X, *)$, between homotopy groups, is bijective for $0 \leq q \leq d - 2$ and surjective for $q = d - 1$. Of course, if $d = 1$ the first assertion is empty, and if $d = 0$ the two assertions are empty (as the conclusion of Theorem 1.3). Observe that, if $d \geq 1$, then $L \cap X \neq \emptyset$.

Remarks 1.4.

(i) Theorems 1.2 and 1.3 are in fact true for any hyperplane $L$ of $\mathbb{P}^n$.

(ii) Theorem 1.3 implies Theorem 1.2 by the Whitehead theorem (cf. [23, Chapter 7]) and some elementary properties of homology groups.

1.2. The Zariski theorem.

In this Section 1.2, the notation is as follows.

Notation 1.5. Let $H$ be a (closed) algebraic hypersurface of $\mathbb{P}^n$, with $n \geq 1$. Take a Whitney stratification $\mathcal{S}$ of $\mathbb{P}^n$ such that $\mathbb{P}^n \setminus H$ is a stratum (cf. [27], [19]), and consider a projective hyperplane $L$ of $\mathbb{P}^n$ transverse to (the strata of) $\mathcal{S}$ (the choice of such a hyperplane is generic).

In [29], Zariski stated the following result.
Theorem 1.6 (cf. [29]). With Notation 1.5, if \( n \geq 3 \) then the natural map
\[
\pi_1(L \cap (\mathbb{P}^n \setminus H), *) \to \pi_1(\mathbb{P}^n \setminus H, *)
\]
is an isomorphism for any base point \( * \) in \( L \cap (\mathbb{P}^n \setminus H) \).

The proof of Zariski is incomplete. The first complete proof is due to Hamm–Lê [15]. It uses Morse theory. Later, Chêniot [4] gave another proof, without Morse theory, closer to the ideas indicated by Zariski in [29].

Theorem 1.6 can be generalized to higher homotopy groups with the same bound as in the HST. This was observed by Hamm–Lê [15]. More precisely:

Theorem 1.7 (cf. [15]). With Notation 1.5, the pair \( (\mathbb{P}^n \setminus H, L \cap (\mathbb{P}^n \setminus H)) \) is \((n - 1)\)-connected.

The proof uses Morse theory. Note that Hamm–Lê prove in fact a stronger local result which implies Theorem 1.7.

1.3. Generalizations to quasi–projective varieties.

In this Section 1.3, the notation is the following.

Notation 1.8. Let \( X := Y \setminus Z \), where \( Y \) is a non–empty closed algebraic subset of \( \mathbb{P}^n \), with \( n \geq 1 \), and \( Z \) a proper closed algebraic subset of \( Y \) (such an \( X \) underlies a reduced embedded quasi–projective variety). Take a Whitney stratification \( S \) of \( Y \) such that \( Z \) is a union of strata (cf. [27], [19]), and consider a projective hyperplane \( L \) of \( \mathbb{P}^n \) transverse to \( S \) (the choice of such a hyperplane is generic). Denote by \( d \) the least dimension of the irreducible components of \( Y \) not contained in \( Z \).

(a) Non–singular case.

In [16] and [12, 13], Hamm–Lê and Goresky–MacPherson showed that Theorems 1.3 and 1.7 can be generalized to non–singular quasi–projective varieties as follows.

Theorem 1.9 (cf. [16] and [12, 13]). With Notation 1.8, if \( X \) is non–singular then the pair \( (X, L \cap X) \) is \((d - 1)\)-connected.

In the particular case where \( Y = \mathbb{P}^n \), Chêniot [5] showed that the degree of connectivity of the pair \( (\mathbb{P}^n \setminus Z, L \cap (\mathbb{P}^n \setminus Z)) \), given by Hamm–Lê and Goresky–MacPherson from Theorem 1.9 above, can be improved as follows.

Theorem 1.10 (cf. [5]). With Notation 1.8, if \( Y = \mathbb{P}^n \) then the pair \( (\mathbb{P}^n \setminus Z, L \cap (\mathbb{P}^n \setminus Z)) \) is \((n + c - 2)\)-connected, where \( c \) is the least of the codimensions of the irreducible components of \( Z \).

Theorems 1.9 and 1.10 are also true for homology groups.

(b) Singular case.

If \( X \) is singular, in general the pair \( (X, L \cap X) \) is not \((d - 1)\)-connected but only \((\delta - 1)\)-connected for some \( \delta \leq d \) depending on the singularities of \( X \). The influence of the singularities can be measured by some invariants that we define now.
Definition 1.11 (cf. [14] and [17]). Let $A$ be a complex analytic space.

(i) Let $B$ be a locally closed complex analytic subspace of $A$. One says that the \textit{rectified homotopical depth} $\text{rhd}_B(A)$ of $A$ along $B$ is greater or equal to an integer $\delta$, if for every point $b \in B$ there is a fundamental system of neighbourhoods $(U_\alpha)_\alpha$ of $b$ in $A$ such that, for every $\alpha$, the pair $(U_\alpha, U_\alpha \setminus B)$ is $(\delta - 1 - \dim_b B)$–connected.

(ii) One says that the \textit{rectified homotopical depth} $\text{rhd}(A)$ of $A$ is greater or equal to $\delta$, if for every locally closed complex analytic subspace $B$ of $A$, the rectified homotopical depth of $A$ along $B$ is greater or equal to $\delta$.

Of course, the rectified homotopical depth $\text{rhd}_B(A)$ of $A$ along $B$ is defined as the supremum of the set of integers $\delta$ as in the definition (i) above (this supremum can be $+\infty$). Similarly for $\text{rhd}(A)$ with definition (ii).

Examples 1.12 (cf. [17]).

(i) If $A$ is a non–empty non-singular complex analytic space of pure dimension $d$, then $\text{rhd}(A) = d$.

(ii) If a complex analytic space $A$ is locally a non–empty complete intersection of pure dimension $d$, then $\text{rhd}(A) = d$.

The general version of the HST for singular quasi–projective varieties proved by Hamm–Lê and Goresky–MacPherson is as follows.

Theorem 1.13 (cf. [16] and [12, 13]). Let $\delta$ be an integer. With Notation 1.8, if $\text{rhd}(X) \geq \delta$ then the pair $(X, L \cap X)$ is $(\delta - 1)$–connected.

In fact, in [12, 13], Goresky–MacPherson do not use the rectified homotopical depth. They measure the effect of the singularities with the help of an invariant involving the number of equations needed to define locally the variety. But the HST of [12, 13] and the one of [16] (above) are equivalent (cf. [14] and [17]).

Note also that Goresky–MacPherson as well as Hamm–Lê prove in fact theorems analogue to Theorem 1.13 in some much more general situations (cf. [16] and [12, 13]), but we shall not consider them in this paper in which the study is limited to varieties embedded in a projective space and to sections by hyperplanes.

A more general approach to measure the influence of the singularities is the \textit{global} rectified homotopical depth which is defined as follows.

Definition 1.14 (cf. [11]). Let $A$ be a complex analytic space, and $B$ a locally closed complex analytic subspace of $A$. One says that the \textit{global rectified homotopical depth} $\text{grhd}_B(A)$ of $A$ along $B$ is greater or equal to an integer $\delta$, if the pair $(A, A \setminus B)$ is $(\delta - 1 - \dim B)$–connected.

Of course, the global rectified homotopical depth $\text{grhd}_B(A)$ of $A$ along $B$ is defined as the supremum of the set of integers $\delta$ as in the definition above (this supremum can be $+\infty$).
2. The Second Lefschetz Theorem

2.1. The original theorem of Lefschetz.

(i) If \( \lambda_1, \ldots, \lambda_p \), with \( p \geq 2 \), are some points in \( \mathbb{P}^1 \), then the global rectified homotopical depth of \( \mathbb{P}^1 \) along \( \{\lambda_1, \ldots, \lambda_p\} \) is 2.

(ii) Let \( A \) and \( B \) be two closed complex analytic subspaces of a compact complex analytic manifold \( M \). Assume that there are Whitney stratifications (cf. [27], [19]) \( S_A \) and \( S_B \) of \( A \) and \( B \), respectively, such that the strata of \( S_A \) intersect transversally the strata of \( S_B \). Then, grhd\(_{A \cap B}(A) \geq 2 \dim M - \dim B \).

The link between rhd\(_B(A)\) and grhd\(_B(A)\), when \( B \) is closed, is as follows.

**Theorem 1.16 (cf. [10]).** Let \( A \) be a complex analytic space, \( B \) a closed complex analytic subspace of \( A \), and \( \delta \) an integer. If \( \text{rhd}_B(A) \geq \delta \), then \( \text{grhd}_B(A) \geq \delta \).

Theorem 1.16 is a special case of a general conjecture of Grothendieck [14] proved by the author in [10].

The global rectified homotopical depth was introduced in [11] as a measure of singularities to prove the following general version of the HST for singular quasi-projective varieties.

**Theorem 1.17 (cf. [11]).** Let \( \delta \) be an integer \( \leq d \). With Notation 1.8, assume that for every projective \((n-r)\)–plane \( K \) of \( \mathbb{P}^n \), with \( 1 \leq r \leq \delta - 1 \) (when \( \delta \geq 2 \)), and every finite subsets \( C \subset X \) and \( D \subset K \cap X \), one has:

\[
\text{grhd}_C(X) \geq \delta \quad \text{and} \quad \text{grhd}_D(K \cap X) \geq \delta - r.
\]

Then, the pair \((X, L \cap X)\) is \((\delta - 1)\)–connected.

**Theorem 2.** The Second Lefschetz Theorem

In Section 2.1, we recall the classical SLT stated in [20]. In Section 2.2, we examine the non–singular quasi-projective version of [6], and we give the statement of the van Kampen theorem on curves. In Section 2.3, we present the high–dimensional van Kampen theorem of Chéniot–Libgober [8] on the complements of projective hypersurfaces with isolated singularities. Finally, in Section 2.4, we present the “generalized homotopical variation” of [7].

2.1. The original theorem of Lefschetz.

We use Notation 1.1. By Theorem 1.2, \( H_q(X, L \cap X) = 0 \) for \( q \leq d - 1 \). The classical SLT describes, when \( n \geq 2 \), the group \( H_d(X, L \cap X) \) in terms of the sections of \( X \) by the hyperplanes of a generic pencil having \( L \) as a member.

Throughout this Section 2.1, we work under the following hypotheses.

**Hypotheses 2.1.** We work with Notation 1.1. Moreover, we assume \( n \geq 2 \). Let \( P \) be a pencil of hyperplanes of \( \mathbb{P}^n \) having \( L \) as a member and the axis \( M \) of which is transverse to \( X \) (the choice of such an axis is generic inside \( L \)). All the elements of \( P \) are transverse to \( X \) with the exception of a finite number of them \( (L_i)_i \), called exceptional hyperplanes, for which, nevertheless, there are only a finite number
of points of non–transversality, all of them situated outside of \( M \) (cf. [26]). If necessary, one may take the liberty of considering some ordinary members of \( \mathcal{P} \), different from \( \mathcal{L} \), as exceptional ones. We parametrize the elements of \( \mathcal{P} \) by the complex projective line \( \mathbb{P}^1 \) as usual. Let \( \lambda \) be the parameter of \( \mathcal{L} \) and, for each \( i \), let \( \lambda_i \) be the parameter of \( \mathcal{L}_i \). For each \( i \), take the image \( \Gamma_i \) of a simple real–analytic arc in \( \mathbb{P}^1 \) joining \( \lambda \) to \( \lambda_i \) such that \( \Gamma_i \cap \Gamma_{i'} = \lambda \) for \( i \neq i' \), and note

\[
X_{\Gamma_i} := \bigcup_{\mu \in \Gamma_i} X \cap \mathcal{P}(\mu),
\]

where \( \mathcal{P}(\mu) \) is the member of \( \mathcal{P} \) with parameter \( \mu \).

The classical SLT given in [20] (with a complete proof in [26]) is the following result.

**Theorem 2.2** (cf. [20]). Assume Hypotheses 2.1 are satisfied. We have the two following assertions.

(i) There is a natural epimorphism

\[
\bigoplus_{i} H_d(X_{\Gamma_i}, \mathcal{L} \cap X) \twoheadrightarrow H_d(X, \mathcal{L} \cap X)
\]

which is the sum of homomorphisms induced by inclusion.

(ii) If moreover \( M \) is so chosen that each \( \mathcal{L}_i \cap X \) has only one quadratic singularity (the choice of such a \( M \) is generic inside \( \mathcal{L} \)), then each \( H_d(X_{\Gamma_i}, \mathcal{L} \cap X) \) is infinite cyclic generated by a hemispherical homology class \([\Delta]\).

By a hemispherical homology class, we mean the image of a generator of the infinite cyclic group \( H_d(B^d, S^{d-1}) \) under the homomorphism induced by some continuous map \( (B^d, S^{d-1}) \to (X_{\Gamma_i}, \mathcal{L} \cap X) \), where \( B^d \) is the closed \( d \)-ball and \( S^{d-1} \) its boundary.

The \( \Delta_i \) are called the Lefschetz thimbles.

Assertion (i) of Theorem 2.2 implies that the kernel of the natural epimorphism

\[
H_{d-1}(\mathcal{L} \cap X) \to H_{d-1}(X)
\]

is the sum of the kernels of the natural homomorphisms

\[
H_{d-1}(\mathcal{L} \cap X) \to H_{d-1}(X_{\Gamma_i}),
\]

and assertion (ii) shows that, if \( M \) is such that each \( \mathcal{L}_i \cap X \) has only one quadratic singularity, the kernel of the natural map \( H_{d-1}(\mathcal{L} \cap X) \to H_{d-1}(X_{\Gamma_i}) \) is generated by the homology class \([\delta_i]\) of the boundary \( \delta_i \) of \( \Delta_i \).

So, we have the following result.

**Theorem 2.3.** Under Hypotheses 2.1, if \( M \) is such that each \( \mathcal{L}_i \cap X \) has only one quadratic singularity, then the kernel of the natural epimorphism \( H_{d-1}(\mathcal{L} \cap X) \to H_{d-1}(X) \) is generated by the family \( ([\delta_i])_i \).

The \( \delta_i \) are called the vanishing cycles.

In [2], Andreotti-Frankel call Second Lefschetz Theorem this Theorem 2.3 and give a proof using Morse theory.

Combined with Theorem 1.2, Theorem 2.3 implies the following result.

**Theorem 2.4.** Under Hypotheses 2.1, if \( M \) is such that each \( \mathcal{L}_i \cap X \) has only one quadratic singularity, then there is a natural isomorphism

\[
H_{d-1}(\mathcal{L} \cap X)/([\delta_i])_i \xrightarrow{\sim} H_{d-1}(X),
\]

where \( ([\delta_i])_i \) is the subgroup of \( H_{d-1}(\mathcal{L} \cap X) \) generated by the family \( ([\delta_i])_i \).
2.2. Non–singular quasi–projective versions of Theorem 2.2 (i) and a homological generalization of the van Kampen theorem on curves.

The SLT (Theorem 2.2) does not generalize, without change, to quasi–projective varieties (even in the very special case of the complements of hypersurfaces). This was observed by Chéniot in [6] in which a counter–example can be found. This counter–example shows that already part (i) of the theorem becomes false. Chéniot, nevertheless, proved in [6] a weak generalization of this part (i) for non–singular quasi–projective varieties (cf. Theorem 2.6 below). In [7], Chéniot and the author proved a homotopical version of Theorem 2.6 (cf. Theorem 2.8).

In [6], Chéniot also gave an expression of the kernel of the first non–bijective natural map occurring in the homological version of the non–singular quasi–projective HST. This kernel is described in terms of some “homological variation operators” (cf. Theorem 2.12). These operators are defined with the help of the monodromies around the exceptional hyperplanes of a generic pencil containing the given general hyperplane. They take into account the relative cycles of the general hyperplane section modulo the section by the axis of the pencil. Combined with a homological version of Theorem 1.9, Theorem 2.12 leads to a high–dimensional generalized homological form (cf. Theorem 2.13) of the van Kampen theorem on curves. The classical van Kampen theorem on curves will be recalled in Section (d) below (cf. Theorem 2.15).

Throughout Section 2.2, we shall assume that the following hypotheses are satisfied.

**Hypotheses 2.5.** We work with Notation 1.8. Moreover, we assume $n \geq 2$. Let $P$ be a pencil of hyperplanes of $\mathbb{P}^n$ having $L$ as a member and the axis $M$ of which is transverse to $S$ (the choice of such an axis is generic inside $L$). All the members of $P$ are transverse to $S$ with the exception of a finite number of them ($L_i$), called *exceptional hyperplanes*, for which, nevertheless, there are only a finite number of points of non–transversality, all of them situated outside of $M$ (cf. [5]). If necessary, one may take the liberty of considering some ordinary members of $P$, different from $L$, as exceptional ones. We parametrize the members of $P$ by the complex projective line $\mathbb{P}^1$ as usual. Let $\lambda$ be the parameter of $L$ and, for each $i$, let $\lambda_i$ be the parameter of $L_i$. For each $i$, take a small closed semi–analytic disk $D_i \subset \mathbb{P}^1$ with centre $\lambda_i$ together with a point $\gamma_i$ in its boundary. Choose the $D_i$ mutually disjoint. Take also the image $\Gamma_i$ of a simple real–analytic arc in $\mathbb{P}^1$ joining $\lambda$ to $\gamma_i$ and such that:

(i) $\Gamma_i \cap D_i = \gamma_i$;
(ii) $\Gamma_i \cap \Gamma_i' = \lambda$ if $i \neq i'$;
(iii) $\Gamma_i \cap D_i' = \emptyset$ if $i \neq i'$. Set

$$K_i := \Gamma_i \cup D_i.$$

For any subset $E \subset \mathbb{P}^1$, note

$$X_E := \bigcup_{\mu \in E} X \cap P(\mu),$$

where $P(\mu)$ is the member of $P$ with parameter $\mu$. 
(a) Non–singular quasi–projective versions of Theorem 2.2 (i).

The following result is a weak generalization of Theorem 2.2 (i) to non–singular quasi–projective varieties.

**Theorem 2.6** (cf. [6]). Under Hypotheses 2.5, if \( X \) is non–singular then there is a natural epimorphism

\[
\bigoplus_i H_d(X_{K_i}, \mathcal{L} \cap X) \to H_d(X, \mathcal{L} \cap X)
\]

which is the sum of homomorphisms induced by inclusion.

So, if \( X \) is non–singular, the kernel of the natural epimorphism \( H_{d-1}(\mathcal{L} \cap X) \to H_{d-1}(X) \) is the sum of the kernels of the natural homomorphisms \( H_{d-1}(\mathcal{L} \cap X) \to H_{d-1}(X_{K_i}) \).

**Remark 2.7** (cf. [6]). Under the hypotheses of Theorem 2.6 (i.e., if \( X \) is non–singular), for each \( i \), the kernels of the natural homomorphisms \( H_{d-1}(\mathcal{L} \cap X) \to H_{d-1}(X_{K_i}) \) and \( H_{d-1}(\mathcal{L} \cap X) \to H_{d-1}(\partial X_{K_i}) \) are equal, where \( \partial X_{K_i} \) is the boundary of \( K_i \).

Theorem 2.6 can be “extended” to homotopy groups as follows. We note

\[ K := \bigcup_i K_i. \]

**Theorem 2.8** (cf. [7]). Under Hypotheses 2.5, if \( X \) is non–singular and \( d \geq 2 \), then for every base point \( * \) in \( \mathcal{L} \cap X \) (which is non–empty) there is a natural epimorphism

\[
\pi_d(X_{K_i}, \mathcal{L} \cap X, *) \to \pi_d(X, \mathcal{L} \cap X, *).
\]

So, if \( X \) is non–singular and \( d \geq 2 \), for every base point \( * \) in \( \mathcal{L} \cap X \), the kernel of the natural epimorphism \( \pi_{d-1}(\mathcal{L} \cap X, *) \to \pi_{d-1}(X, *) \) is equal to the kernel of the natural homomorphism \( \pi_{d-1}(\mathcal{L} \cap X, *) \to \pi_{d-1}(X_{K_i}, *) \).

In the special case \( Y = \mathbb{P}^n \), so that \( X = \mathbb{P}^n \setminus Z \), and provided \( Z \neq \emptyset \), the conclusions of Theorems 2.6 and 2.8 are valid with \( n + c - 1 \) instead of \( d \), where \( c \) is the least of the codimensions of the irreducible components of \( Z \) (cf. [6] and [7]).

(b) Monodromies.

For each \( i \), consider a loop \( \omega_i \) in the boundary \( \partial K_i \) of \( K_i \) starting from \( \lambda \), running along \( \Gamma_i \) up to \( \gamma_i \), going once counter–clockwise around the boundary of \( D_i \) and coming along \( \Gamma_i \) back to \( \lambda \).

**Lemma 2.9** (cf. [6]). For each \( i \), there is an isotopy

\[ H : (\mathcal{L} \cap X) \times I \to X_{\partial K_i} \]

such that:

(i) \( H(x, 0) = x \), for every \( x \in \mathcal{L} \cap X \);

(ii) \( H(x, t) \in X_{\omega_i(t)} \) for every \( x \in \mathcal{L} \cap X \) and every \( t \in I \);

(iii) for every \( t \in I \), the map \( \mathcal{L} \cap X \to X_{\omega_i(t)} \), defined by \( x \mapsto H(x, t) \), is a homeomorphism;

(iv) \( H(x, t) = x \), for every \( x \in \mathcal{M} \cap X \) and every \( t \in I \).
If $X$ is non-singular then, for each $i$, the kernel of the natural map $H_{d-1}(L \cap X) \to H_{d-1}(X)$ is equal to the image of $\text{var}_{i,d-1}$.

So, combined with Theorem 2.6, Lemma 2.11 implies the following result.

**Theorem 2.12 (cf. [6]).** Under Hypotheses 2.5, if $X$ is non-singular then the kernel of the natural map $H_{d-1}(L \cap X) \to H_{d-1}(X)$ is the sum $\sum_i \text{Im} \text{var}_{i,d-1}$ of the images of the homological variation operators ($\text{var}_{i,d-1}$).

Then, combined with a homological version of Theorem 1.9, Theorem 2.12 implies the following result.

**Theorem 2.13 (cf. [6]).** Under Hypotheses 2.5, if $X$ is non-singular then there is a natural isomorphism

$$H_{d-1}(L \cap X)/\sum_i \text{Im} \text{var}_{i,d-1} \cong H_{d-1}(X).$$
In the special case $Y = \mathbb{P}^n$, so that $X = \mathbb{P}^n \setminus Z$, the conclusion of Theorem 2.13 is valid with $n + c - 2$ instead of $d - 1$, where $c$ is the least of the codimensions of the irreducible components of $Z$ (cf. [6]).

Theorem 2.13 may be considered as a non–singular quasi–projective version of the SLT and at the same time as a high–dimensional generalized homological form of the van Kampen theorem on curves which we recall in the following section.

(d) The van Kampen theorem on curves.

In this section (d), we shall assume that the following hypotheses are satisfied.

Hypotheses 2.14. Recall that we work under Hypotheses 2.5. Moreover, we assume here that $n = 2, Y = \mathbb{P}^2$ and $Z = C$, where $C$ is a reduced closed algebraic curve with degree $k$.

The axis $M$ of $\mathbb{P}$ is reduced to a single point • which is not situated on $C$. There is a natural presentation of $\pi_1(L \cap (\mathbb{P}^2 \setminus C), \bullet)$ by $k$ generators $g_1, \ldots, g_k$ and the relation $g_1 \cdots g_k = 1$. For each $i$, let $h_i$ be a geometric monodromy of $L \cap (\mathbb{P}^2 \setminus C)$ relative to • above $\omega_i$ (cf. Section 2.2 (b)); the latter induces a homomorphism $h_i : \pi_1(L \cap (\mathbb{P}^2 \setminus C), \bullet) \to \pi_1(L \cap (\mathbb{P}^2 \setminus C), \bullet)$.

The theorem of van Kampen can be stated as follows.

Theorem 2.15 (cf. [25]). Under Hypotheses 2.14, the fundamental group $\pi_1(\mathbb{P}^2 \setminus C, \bullet)$ is presented by the generators $g_1, \ldots, g_k$ and the relations $g_1 \cdots g_k = 1$ and $h_i(g_j) = g_j$ for all $i$ and $j$.

The proof of van Kampen is incomplete. The first complete proof is due to Chénot [4].

2.3. A (homotopical) generalization of the van Kampen theorem to the complements of projective hypersurfaces with isolated singularities.

In [21], Libgober gave a high–dimensional analogue of the van Kampen theorem for higher-homotopy groups of the complements of hypersurfaces with isolated singularities in $\mathbb{C}^n$, with $n \geq 3$, behaving well at infinity. Libgober also gave a way to deduce the projective case from the affine case. The projective version of his theorem is effectively deduced in [8]. Roughly speaking, it expresses, under Hypotheses 2.16 below, the homotopy group $\pi_{n-1}(\mathbb{P}^n \setminus H, \ast)$ as a quotient of $\pi_{n-1}(L \cap (\mathbb{P}^n \setminus H), \ast)$ (notice that $\pi_{n-1}(\mathbb{P}^n \setminus H, \ast)$ is the first homotopy group not preserved by hyperplane section), using the pencil $P$. But, in contrast with the van Kampen theorem, in order to obtain the subgroup by which the quotient is taken, it is no more enough to identify each homotopy $(n - 1)$–cell of $L \cap (\mathbb{P}^n \setminus H)$ to its transforms by monodromy around the exceptional members of $P$. One must also take into account some “degeneration operators” associated with these exceptional hyperplanes.

In [8], Chéniot–Libgober defined a homotopical version of Chéniot’s homological variation operators, in the special case of the complements of projective hypersurfaces with isolated singularities, and showed that these “homotopical operators” lead to a high-dimensional van Kampen theorem (cf. Theorem 2.17) equivalent to the projective version of [21]. More precisely, they proved that the homotopical variation operators are sufficient to describe entirely the subgroup by which the quotient is taken in the projective version of Libgober’s theorem mentioned above.
Chéniot–Libgober’s theorem provides a homotopical analogue of Theorem 2.13, in the special case of the complements of projective hypersurfaces with isolated singularities, and thus may also be viewed as a homotopical version of the SLT for this particular case. This section is devoted to this result.

Throughout this Section 2.3, we work under the following hypotheses.

**Hypotheses 2.16.** We work under Hypotheses 2.5. Moreover, we assume that $Y = \mathbb{P}^n$, with $n \geq 3$, and $Z = H$, where $H$ is a (closed) algebraic hypersurface of $\mathbb{P}^n$, with degree $k$, having only isolated singularities. We also assume that $S$ is the Whitney stratification the strata of which are: $\mathbb{P}^n \setminus H$, the singular part $H_{\text{sing}}$ of $H$, and the non–singular part $H \setminus H_{\text{sing}}$ of $H$. Finally, we fix a base point $\ast$ in $\mathcal{M} \cap X$ (which is non–empty).

The homotopical variation operators of Chéniot–Libgober [8] are defined as follows.

By taking homogeneous coordinates $(x_1 : \cdots : x_{n+1})$ on $\mathbb{P}^n$ so chosen that $\mathcal{M}$ is defined by the equations

$$x_n = x_{n+1} = 0,$$

we can assume that the parametrization of $\mathcal{P}$ is as follows: given homogeneous coordinates on $\mathbb{P}^1$, for each $\mu = (a : b) \in \mathbb{P}^1$, the hyperplane $\mathcal{P}(\mu)$ of $\mathcal{P}$ with parameter $\mu$ is defined by the equation

$$ax_n + bx_{n+1} = 0.$$

Now, assume that $H$ is given by the equation

$$f(x_1, \ldots, x_{n+1}) = 0,$$

where $f$ is a homogeneous reduced polynomial of degree $k$, and in $\mathbb{P}^{n+1}$ with homogeneous coordinates $(x_0 : x_1 : \cdots : x_{n+1})$ consider the hypersurface $H'$ defined by the equation

$$x_0^k + f(x_1, \ldots, x_{n+1}) = 0.$$

So, if

$$\text{Proj} : \mathbb{P}^{n+1} \setminus (1 : 0 : \cdots : 0) \to \mathbb{P}^n$$

is the projection defined by

$$(x_0 : x_1 : \cdots : x_{n+1}) \mapsto (x_1 : \cdots : x_{n+1}),$$

a classical argument shows that its restriction to $H'$ (which is well–defined because $(1 : 0 : \cdots : 0) \notin H'$), denoted by

$$p := \text{Proj}_{H'} : H' \to \mathbb{P}^n,$$

is a holomorphic $k$–fold covering of $\mathbb{P}^n$ totally ramified along $H$.

Moreover, if

$$j : \mathbb{P}^n \to \mathbb{P}^{n+1}$$

is the embedding defined by

$$(x_1 : \cdots : x_{n+1}) \mapsto (0 : x_1 : \cdots : x_{n+1}),$$

it is not difficult to see that the singular points of $H'$ are simply the images by $j$ of the singular points of $H$. In particular, we have that $H' \setminus j(H)$ is non–singular. On the other hand, we also have:

$$H' \cap j(\mathbb{P}^n) = j(H) = p^{-1}(H).$$
Now, denote by $\mathcal{M}'$ the projective $(n - 1)$–plane of $\mathbb{P}^{n+1}$ defined in $\mathbb{P}^{n+1}$ by the same equations as $\mathcal{M}$ in $\mathbb{P}^n$ (i.e., $x_n = x_{n+1} = 0$), and consider the pencil $\mathcal{P}'$ of hyperplanes of $\mathbb{P}^{n+1}$ with axis $\mathcal{M}'$. We parametrize the members of $\mathcal{P}'$ as follows: with the same homogeneous coordinates on $\mathbb{P}^3$ as above, for each $\mu = (a : b) \in \mathbb{P}^1$, the hyperplane $\mathcal{P}'(\mu)$ of $\mathcal{P}'$ with parameter $\mu$ is defined in $\mathbb{P}^{n+1}$ by the same equation as $\mathcal{P}(\mu) \in \mathcal{P}$ in $\mathbb{P}^n$ (i.e., $ax_n + bx_{n+1} = 0$). One then has the following properties:

(i) $p^{-1}(\mathcal{P}(\mu)) = \mathcal{P}'(\mu) \cap H'$ and $p^{-1}(\mathcal{M}) = \mathcal{M}' \cap H'$;
(ii) $\mathcal{P}'(\mu) \cap j(\mathcal{P}) = j(\mathcal{P}(\mu))$ and $\mathcal{M}' \cap j(\mathcal{P}) = j(\mathcal{M})$;
(iii) $\mathcal{M}'$ is transverse to the Whitney stratification $\mathcal{S}'$ of $H'$ defined by:

$$\mathcal{S}' := \{ j(H_{\text{sing}}), j(H \setminus H_{\text{sing}}), H' \setminus j(H) \};$$

(iv) all the members of $\mathcal{P}'$ are transverse to $\mathcal{S}'$ with the exception of the hyperplanes $\mathcal{P}'(\lambda_i)$ for which, nevertheless, there are only a finite number of points of non-transversality, all of them situated outside of $\mathcal{M}'$.

Set

$$X' := H' \setminus j(H) \quad \text{and} \quad \mathcal{L}' := \mathcal{P}'(\lambda).$$

The foregoing shows that the restriction of $p$ to $\mathcal{L}' \cap X'$,

$$p_1 : \mathcal{L}' \cap X' \to \mathcal{L} \cap X,$$

is a holomorphic (unramified) $k$–fold covering. So, by the homotopy sequence of weak fibrations, for every point $\bullet \in p^{-1}(*)$, the map $p_1$ induces an isomorphism

$$\pi_q(\mathcal{L}' \cap X', \bullet) \sim \pi_q(\mathcal{L} \cap X, \bullet)$$

for all $q \geq 2$. Thus, in particular, for every $\bullet \in p^{-1}(*)$, there is a homomorphism

$$\psi : \pi_{n-1}(\mathcal{L} \cap X, \bullet) \to H_{n-1}(\mathcal{L}' \cap X')$$

defined by the composition

$$\pi_{n-1}(\mathcal{L} \cap X, \bullet) \to \pi_{n-1}(\mathcal{L}' \cap X', \bullet) \to H_{n-1}(\mathcal{L}' \cap X'),$$

where the left arrow is the inverse of isomorphism (1) and the right one is the Hurewicz homomorphism.

On the other hand, since $p^{-1}(\mathcal{M} \cap X) = \mathcal{M}' \cap X'$, a well–known result on weak fibrations shows that, for every $\bullet \in p^{-1}(*)$, the map $p_1$ also induces an isomorphism

$$\pi_{n-1}(\mathcal{L}' \cap X', \mathcal{M}' \cap X', \bullet) \sim \pi_{n-1}(\mathcal{L} \cap X, \mathcal{M} \cap X, \bullet).$$

Thus, for every $\bullet \in p^{-1}(*)$, there is also a homomorphism

$$\phi : \pi_{n-1}(\mathcal{L} \cap X, \mathcal{M} \cap X, \bullet) \to H_{n-1}(\mathcal{L}' \cap X', \mathcal{M}' \cap X'),$$

defined (similarly to $\psi$) by the composition of the inverse of isomorphism (2) with the relative Hurewicz homomorphism.

Now, since $H$ is a hypersurface of $\mathbb{P}^n$ having only isolated singularities, $\mathcal{L} \cap H$ is a non–singular hypersurface of $\mathcal{L} \simeq \mathbb{P}^{n-1}$; and, since $n - 1 \geq 2$, one deduces (cf. [21]) that

$$\pi_1(\mathcal{L} \cap X, \bullet) \simeq \mathbb{Z}/k\mathbb{Z} \quad \text{and} \quad \pi_q(\mathcal{L} \cap X, \bullet) = 0$$

for $2 \leq q \leq n - 2$ (this range may be empty).
Knowing that $L' \cap X'$ is pathwise connected (this is not trivial), the first assertion implies that the holomorphic $k$–fold covering

$$p_1 : L' \cap X' \to L \cap X,$$

is a universal covering. Combined with the second assertion and (1), this shows that $L' \cap X'$ is $(n-2)$–connected. The Hurewicz Isomorphism Theorem then implies that, for every $\bullet \in p^{-1}(\ast)$, the Hurewicz homomorphism

$$\pi_{n-1}(L' \cap X', \bullet) \to H_{n-1}(L' \cap X'),$$

and consequently $\psi$, is an isomorphism.

Thus, for each $i$, and for every $\bullet \in p^{-1}(\ast)$, there is a homomorphism

$$\mathcal{VAR}_{i,n} : \pi_{n-1}(L \cap X, \mathcal{M} \cap X, \ast) \to \pi_{n-1}(L \cap X, \ast)$$

defined by the composition

$$\psi^{-1} \circ \text{var}_{i,n-1} \circ \phi$$

where, of course, both $\psi$ and $\phi$ are the homomorphisms associated to $\bullet$, and where $\text{var}_{i,n-1} : H_{n-1}(L' \cap X', \mathcal{M}' \cap X') \to H_{n-1}(L' \cap X')$ is defined from $\omega_i$ in the same way as $\text{var}_{i,n-1}$ (cf. Section 2.2 (c)) but for the pencil $P'$ cutting $X'$ instead of the pencil $P$ cutting $X$.

One shows easily that homomorphism $\mathcal{VAR}_{i,n-1}$ does not in fact depend on the choice of the base point $\bullet \in p^{-1}(\ast)$.

Homomorphism $\mathcal{VAR}_{i,n-1}$ is called a homotopical variation operator associated to $\langle \omega_i \rangle$.

These operators were used in [8] to prove the following high–dimensional van Kampen theorem.

**Theorem 2.17** (cf. [8]). Under Hypotheses 2.16, there is a natural isomorphism

$$\pi_{n-1}(L \cap X, \ast) \cong \sum_i \text{Im } \mathcal{VAR}_{i,n-1} \to \pi_{n-1}(X, \ast).$$

As mentioned above, Theorem 2.17 is equivalent to the projective version of [21, Theorem 2.4]. It provides a homotopical analogue of Theorem 2.13 in the special case of the complements of projective hypersurfaces with isolated singularities, and thus may also be viewed as a homotopical version of the SLT for this special case. Two proofs of this theorem are proposed in [8]: one from [6] and the other from [21].

### 2.4. Generalized homotopical variation operators, a new description of Theorem 2.17, and a conjecture generalizing the van Kampen theorem to non–singular quasi–projective varieties.

As we have seen in Section 2.3, the homotopical variation operators of Chéniot–Libgober, which lead to Theorem 2.17, are not so easy to define. The definition go through the homology of universal covers, and the link between homotopy and homology depends strongly on the special topology of the complement of hypersurfaces with isolated singularities. The same also occurs with the degeneration operators of [21]. In [7], Chéniot and the author gave a very short and purely homotopical definition of the homotopical variation operators. Moreover, this new definition makes sense in the general setting of singular quasi–projective varieties,
and thus opens the way to obtain more general van Kampen theorems. In particular, [7] gives a conjecture extending Theorem 2.17 to non-singular quasi-projective varieties (cf. Conjecture 2.21 below).

(a) Generalized homotopical variation operators of [7].

In this section (a), we work under Hypotheses 2.5. We assume further that \( M \cap X \neq \emptyset \) (observe that this condition is equivalent to \( \dim X \geq 2 \)).

Fix an index \( i \), and consider a geometric monodromy \( h \) of \( L \cap X \) relative to \( M \cap X \) above \( \omega_i \) (cf. Section 2.2 (b)). Denote by \( q \) an integer \( \geq 1 \), and consider a base point \(*\) in \( M \cap X \). Since \( h \) leaves \( M \cap X \) pointwise fixed, if \( f \in F^q(L \cap X, M \cap X, *) \), then the map \( f \perp (h \circ f) \) defined on \([0,1]^q\) by

\[
f \perp (h \circ f)(x_1, \ldots, x_q) := \begin{cases} f(x_1, \ldots, x_{q-1}, 1 - 2x_q), & 0 \leq x_q \leq \frac{1}{2}, \\ h \circ f(x_1, \ldots, x_{q-1}, 2x_q - 1), & \frac{1}{2} \leq x_q \leq 1,
\end{cases}
\]

is in \( F^q(L \cap X,*) \). Observe that the reversion of \( f \) and its concatenation with \( h \circ f \) are performed on the variable transverse to the free face (this makes sense precisely because \( h \) leaves \( M \cap X \) pointwise fixed).

Lemma 2.18 (cf. [7]). The correspondence

\[
\text{VAR}_{i,q} : \pi_q(L \cap X, M \cap X, *) \rightarrow \pi_q(L \cap X, *)
\]

\( \langle f \rangle_{L \cap X, M \cap X, *} \mapsto \langle f \perp (h \circ f) \rangle_{L \cap X, *} \)

gives a well-defined map which depends only on the homotopy class \( \langle \omega_i \rangle \in \pi_1(\mathbb{P}^1 \setminus \cup_i \lambda_i, \lambda) \). If \( q \geq 2 \), it is a homomorphism.

Map \( \text{VAR}_{i,q} \) is called a generalized homotopical variation operator associated to \( \langle \omega_i \rangle \). This terminology is justified by Theorem 2.19 below.

Operator \( \text{VAR}_{i,q} \) is linked by Hurewicz homomorphisms with the homological operator \( \text{var}_{i,q} \), and it is equivariant under the action of the fundamental group (cf. [7]).

(b) A new description of Theorem 2.17.

In this section (b), we work under Hypotheses 2.16.

Theorem 2.19 (cf. [7]). Under Hypotheses 2.16, the homotopical variation operator \( \text{VAR}_{i,n-1} \) of Chéniot–Libgober which is then well-defined (cf. Section 2.3) coincides with the generalized homotopical variation operator \( \text{VAR}_{i,n-1} \) (defined above).

This theorem, together with Theorem 2.17, implies the following result.

Theorem 2.20 (cf. [7]). Under Hypotheses 2.16, there is a natural isomorphism

\[
\pi_{n-1}(L \cap X,*) / \sum_i \text{Im} \text{VAR}_{i,n-1} \xrightarrow{\sim} \pi_{n-1}(X,*)
\]

Of course, Theorem 2.20 is equivalent to Theorem 2.17 and to the projective version of [21, Theorem 2.4].
(c) A conjecture generalizing the van Kampen theorem to non-singular quasi-projective varieties.

In this Section (c), we work under Hypotheses 2.5. We assume further that $\mathcal{M} \cap X \neq \emptyset$ (as this condition is equivalent to $\dim X \geq 2$, it will be automatically fulfilled when $d \geq 2$). We fix a base point $*$ in $\mathcal{M} \cap X$.

Chéniot and myself have the feeling that Theorem 2.20 (and hence Theorem 2.17 and the projective version of [21, Theorem 2.4]) may be generalized as follows.

**Conjecture 2.21** (cf. [7]). Under the hypotheses specified just above, if $X$ is non-singular then there are natural isomorphisms

\[
\pi_{d-1}(\mathcal{L} \cap X, *) / \sum_i \text{Im } \text{VAR}_{i,d-1} \xrightarrow{\sim} \pi_{d-1}(X, *) \quad \text{if } d \geq 3,
\]

\[
\pi_1(\mathcal{L} \cap X, *) / \bigcup_i \text{Im } \text{VAR}_{i,1} \xrightarrow{\sim} \pi_1(X, *) \quad \text{if } d = 2,
\]

where $\bigcup_i \text{Im } \text{VAR}_{i,1}$ is the normal subgroup of $\pi_1(\mathcal{L} \cap X, *)$ generated by $\bigcup_i \text{Im } \text{VAR}_{i,1}$.

In the special case $Y = \mathbb{P}^n$, so that $X = \mathbb{P}^n \setminus Z$, and provided $Z \neq \emptyset$, there is a natural isomorphism

\[
\pi_{n+c-2}(\mathcal{L} \cap X, *) / \sum_i \text{Im } \text{VAR}_{i,n+c-2} \xrightarrow{\sim} \pi_{n+c-2}(X, *) \quad \text{if } n + c \geq 4,
\]

where $c$ is the least of the codimensions of the irreducible components of $Z$ (notice that $n + c - 2 = d - 1$ when $c = 1$).

It is not difficult to prove that the subgroups by which the quotients are taken are contained in the kernels of the corresponding natural maps (which are epimorphisms by HST): see [7]. The reverse inclusion seems much more difficult and not proved at the moment. A first little approach is given by Theorem 2.8 (compare with the proof of the homological situation of [6]).

The van Kampen theorem on curves (Theorem 2.15) is a special case of this conjecture (cf. [7]). Notice also that, if proved, Conjecture 2.21 would provide a complete homotopical analogue of Theorem 2.13, and thus would gather in a generalized form the van Kampen theorem with a homotopical version of the SLT.

The generalized homotopical variation operators are also certainly a first step for further generalizations of van Kampen’s theorem to singular quasi-projective varieties, the influence of the singularities being measured by the ordinary or global rectified homotopical depth (cf. Section 1.3 (b)).

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