A CRITERION OF BLOCH FUNCTIONS AND LITTLE BLOCH FUNCTIONS

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Abstract. In this note, we give a characterization of Bloch functions and little Bloch functions, which extends results of Aulaskari-Lappan, Minda, and Aulaskari-Wulan.

An analytic function \( f \) in the unit disk \( \Delta \) is said to be a Bloch function if
\[
\sup_{z \in \Delta} (1 - |z|^2)|f'(z)| < \infty.
\]

An analytic function \( f \) in the unit disk \( \Delta \) is called a little Bloch function if
\[
\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0.
\]

Aulaskari and Lappan [1] and Minda [3] gave an alternative characterization of Bloch functions. They proved the following:

**Theorem ALM.** A function \( f \) analytic in the unit disk \( \Delta \) is not a Bloch function if and only if there exist a sequence \( \{z_n\} \subset \Delta \) with \( |z_n| \to 1 \) and a sequence \( \{\rho_n\} \) of positive numbers satisfying \( \frac{\rho_n}{1 - |z_n|^2} \to 0 \), such that the sequence \( \{f(z_n + \rho_n \xi) - f(z_n)\} \) converges locally uniformly to a nonconstant analytic function in \( C \).

For little Bloch functions, Aulaskari and Wulan [2] proved the following:

**Theorem AW.** A function \( f \) analytic on the unit disk \( \Delta \) is not a little Bloch function if and only if there exist a constant \( R > 0 \), a sequence \( \{z_n\} \subset \Delta \) with \( |z_n| \to 1 \), and a sequence \( \{\rho_n\} \) of positive numbers satisfying \( \frac{\rho_n}{1 - |z_n|^2} < \frac{1}{2R} \), such that the sequence \( \{f(z_n + \rho_n \xi) - f(z_n)\} \) converges locally uniformly to a nonconstant analytic function in \( |\xi| < R \).

In this paper, we investigate the possibility of introducing a sliding scale involving a parameter \( \alpha \) into these theorems in analogy to Pang’s generalization [4] of Zalcman’s Lemma [5]. Our first result is the following.

**Theorem 1.** Let \( f \) be an analytic function in the unit disk \( \Delta \), and \( \alpha \) a given real number with \( 0 \leq \alpha < 1 \). Then \( f \) is not a Bloch function if and only if there exist a sequence \( \{z_n\} \subset \Delta \) with \( |z_n| \to 1 \), and a sequence \( \{\rho_n\} \) of positive numbers satisfying \( \frac{\rho_n}{1 - |z_n|^2} \to 0 \), such that the sequence \( \{f(z_n + \rho_n \xi) - f(z_n)\} \) converges locally uniformly to \( a \xi \) in \( C \), where \( a \) is a constant with \( |a| = 1 \).

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Proof. Assume that \( f \) is not a Bloch function. Then there exists a sequence \( \{z_n^*\} \subset \Delta \) with \( |z_n^*| \to 1 \) such that
\[
(1 - |z_n^*|^2)|f'(z_n^*)| \to \infty, \quad \text{as} \quad n \to \infty. \tag{1}
\]
Without loss of generality, we can assume \( |z_n^*| > \frac{1}{2} \) for all \( n \in N \). Let
\[
r_n = \frac{2|z_n^*|}{1 + |z_n^*|}, \quad n \in N.
\]
Then \( |z_n^*| < r_n < 1 \) and \( r_n \to 1 \) as \( n \to \infty \). Also, we have
\[
r_n - |z_n^*| = \frac{2|z_n^*|}{1 + |z_n^*|} - |z_n^*| = \frac{|z_n^*|(1 - |z_n^*|)}{1 + |z_n^*|}
\]
and
\[
r_n + |z_n^*| = \frac{|z_n^*(3 + |z_n^*|)}{1 + |z_n^*|}.
\]
Choose \( \{z_n\} \subset \Delta \) such that
\[
M_n = \sup_{|z| \leq r_n} \left(1 - \frac{|z|^2}{r_n^2}\right)^{\frac{1}{1-n}} (|f'(z)|)^{\frac{1}{1-n}}
\]
\[
= \left(1 - \frac{|z_n^*|^2}{r_n^2}\right)^{\frac{1}{1-n}} (|f'(z_n^*)|)^{\frac{1}{1-n}}. \tag{2}
\]
Since \( |z_n^*| < r_n \), it follows that
\[
1 - \frac{|z_n^*|^2}{r_n^2} = \frac{1}{r_n^2} (r_n + |z_n^*|)(r_n - |z_n^*|) = \frac{(3 + |z_n^*|)(1 - |z_n^*|)}{4(1 + |z_n^*|)} \geq \frac{1}{4}(1 - |z_n^*|^2).
\]
Thus
\[
M_n = \left(1 - \frac{|z_n^*|^2}{r_n^2}\right)^{\frac{1}{1-n}} (|f'(z_n^*)|)^{\frac{1}{1-n}}
\]
\[
\geq \left(\frac{1}{4}\right)^{\frac{1}{1-n}} (1 - |z_n^*|^2)^{\frac{1}{1-n}} (|f'(z_n^*)|)^{\frac{1}{1-n}} \tag{3}
\]
\[
\geq \left(\frac{1}{4}\right)^{\frac{1}{1-n}} \to \infty.
\]
Now set
\[
\rho_n = \left(1 - \frac{|z_n^*|^2}{r_n^2}\right)^{\frac{1}{1-n}} \left(\frac{1}{|f'(z_n^*)|}\right)^{\frac{1}{1-n}}. \tag{4}
\]
Then, by (2) and (3), we have
\[
\frac{\rho_n}{(r_n - |z_n|)^{\frac{1}{1-n}}} = \left(\frac{1}{|f'(z_n^*)|(r_n - |z_n|)}\right)^{\frac{1}{1-n}} = \frac{1}{M_n} \left(\frac{r_n + |z_n|}{r_n^2}\right)^{\frac{1}{1-n}} \to 0.
\]
So
\[
\frac{\rho_n^{1-n}}{r_n - |z_n|} \to 0. \tag{5}
\]
In particular,
\[
\frac{\rho_n^{1-n}}{1 - |z_n|} \to 0, \tag{6}
\]
\[
\frac{\rho_n}{r_n - |z_n|} \to 0, \tag{7}
\]
and
\[
\frac{\rho_n}{1 - |z_n|} \to 0. \tag{8}
\]
Therefore, the functions
\[ g_n(\xi) = \frac{f(z_n + \rho_n \xi) - f(z_n)}{\rho_n^\alpha} \]
are defined on the disk |\xi| < \frac{r_n - |z_n|}{\rho_n} := R_n \to \infty by (7), as n \to \infty. Hence, according to (2) and (5), for each fixed R > 0, when |\xi| \leq R < R_n, we have
\[ |g'_n(\xi)| = |\rho_n^{1-\alpha}| \left| f'(z_n + \rho_n \xi) \right| \]
\[ \leq \frac{|f'(z_n)| \left( 1 - \frac{|z_n|}{r_n} \right)}{r_n + |z_n| + \rho_n R} \cdot \frac{\left( 1 - \frac{|z_n + \rho_n \xi|}{r_n} \right)}{r_n - |z_n| - \rho_n R}. \]

Since the last term of the inequality (9) tends to 1, \( \{g'_n(0)\} \) is a normal family on \( C \). Taking a subsequence and renumbering, we may assume that \( \{g'_n\} \) converges locally uniformly on compacta to an entire function \( G \), and |\( G \)| \leq 1. By Liouville’s Theorem, \( G \) is a constant; and since \( |g'_n(0)| = 1 \), we have \( G \equiv a \), where |a| = 1. It follows from
\[ g_n(\xi) - g_n(0) = \int_0^\xi g'_n(\eta) d\eta \to a \xi, \quad \text{as} \quad n \to \infty \]
and \( g_n(0) = 0 \) that \( \{g_n(\xi)\} \) tends to \( a \xi \) locally uniformly on any compact set in \( C \).

Conversely, suppose that there exist \( z_n, \rho_n \) with |\( z_n | < 1, |z_n| \to 1, \frac{1}{|1 - |z_n||} \to 0 \) such that \( g_n(\xi) = \{f(z_n + \rho_n \xi) - f(z_n)\}/\rho_n^\alpha \to g(\xi) \) locally uniformly in \( C \). Evidently,
\[ |g'_n(\xi)| \leq \rho_n^{1-\alpha} \cdot \frac{1}{(1 - |z_n|)} \cdot \frac{1}{(1 - \frac{|z_n + \rho_n \xi|}{r_n})} \cdot \frac{1}{(1 - |z_n + \rho_n \xi|^2)} \cdot \left| f'(z_n + \rho_n \xi) \right|. \]
If \( f \) is a Bloch function, the product of the last two terms on the right is bounded; the second term on the right tends to 1 by (8) and hence is also bounded; and the first term on the right tends to 0 by assumption as \( n \) tends to infinity. So \( g'(\xi) = 0 \) for all \( \xi \in C \), and therefore \( g \) is constant. This completes the proof of Theorem 1.

**Example 2.** Let \( f(z) = \frac{1}{1 - z^2} \); then \( f \) is analytic in the unit disk, and it is evident that \( f \) is not a Bloch function. Fix \( 0 \leq \alpha < 1 \), and set \( z_n = 1 - \frac{1}{n}, \rho_n = n^{\frac{1-\alpha}{2}}, \) so that \( n \rho_n^{1-\alpha} \to 0, n^2 \rho_n^{1-\alpha} = 1, \) and \( n \rho_n \to 0 \). Then
\[ \frac{f(z_n + \rho_n z) - f(z_n)}{\rho_n^\alpha} = \frac{z}{1 - n \rho_n z} \to z, \]
locally uniformly in \( C \).

**Remark 3.** Theorem 1 does not hold for \( \alpha > 1 \). Indeed, let \( \{z_n\} \) be an arbitrary sequence in the unit disk with |\( z_n | \to 1, \) and \( \{\rho_n\} \) an arbitrary sequence of positive numbers with \( \rho_n \to 0 \). Then with \( f \) as in the previous example, we have
\[ \frac{f(z_n + \rho_n z) - f(z_n)}{\rho_n^\alpha} = \frac{\rho_n^{1-\alpha} z}{(1 - z_n - \rho_n z)(1 - z_n)} \to \infty, \quad \text{for} \quad z \neq 0. \]
Thus Theorem 1 does not hold if \( \alpha > 1 \).

For the little Bloch functions, we have
Theorem 4. Let $f$ be an analytic function in the unit disk $\Delta$, and $\alpha$ a given real number with $0 < \alpha < 1$. Then $f$ is not a little Bloch function if and only if there exist a sequence $\{z_n\} \subset \Delta$ with $|z_n| \to 1$, a constant $K > 0$, and a sequence $\{\rho_n\}$ of positive numbers satisfying
\[
\frac{1}{|z_n|^2} < K, \quad \text{such that the sequence } \{f(z_n + \rho_n \xi) - f(z_n)\}
\]
converges locally uniformly to $a \xi$ in $\Delta$, where $a$ is a constant with $|a| = 1$.

Proof. The proof follows the pattern in [2]. Assume that $f$ is not a little Bloch function. Then there exist a constant $M > 0$, and a sequence $\{z_n^*\} \subset \Delta$ with $|z_n^*| \to 1$ such that
\[
(1 - |z_n^*|^2)|f'(z_n^*)| \geq 2M, \quad n = 1, 2, 3, \ldots
\]
If $f$ is not a Bloch function, the result follows at once from Theorem 1. So we suppose that $f$ is a Bloch function, and without loss of generality, we can assume $|z_n^*| > \frac{1}{2}$ for all $n \in \mathbb{N}$. Let
\[
r_n = \frac{2|z_n^*|}{1 + |z_n^*|}, \quad \text{and} \quad r_n' = \frac{|z_n^*|}{2 - |z_n^*|}, \quad n \in \mathbb{N}.
\]

Then $1/3 < r_n' < |z_n^*| < r_n < 1$ and $r_n \to 1, r_n' \to 1$ as $n \to \infty$. Also, we have
\[
r_n - r_n' = \frac{3|z_n^*||1 - |z_n^*|^2|}{(1 + |z_n^*|)(2 - |z_n^*|)}
\]
and
\[
1 - r_n = \frac{1 - |z_n^*|}{1 + |z_n^*|}.
\]

So
\[
3(1 - r_n) \geq r_n - r_n'. \quad (11)
\]
Choose $\{z_n\} \subset \Delta$ such that
\[
M_n = \sup_{r_n \leq |z_n| \leq r_n'} \left(1 - \frac{|z_n|}{r_n}\right)^{\frac{1}{1 - \alpha}} \left(1 - \frac{r_n'}{|z_n|}\right)^{\frac{1}{1 - \alpha}} (|f'(z_n)|)^{\frac{1}{1 - \alpha}}
\]
\[
= \left(1 - \frac{|z_n|}{r_n}\right)^{\frac{1}{1 - \alpha}} \left(1 - \frac{r_n'}{|z_n|}\right)^{\frac{1}{1 - \alpha}} (|f'(z_n)|)^{\frac{1}{1 - \alpha}}. \quad (12)
\]
Since $r_n' < |z_n^*| < r_n$, by (10)
\[
M_n \geq \left(1 - \frac{|z_n|}{r_n}\right)^{\frac{1}{1 - \alpha}} \left(1 - \frac{r_n'}{|z_n|}\right)^{\frac{1}{1 - \alpha}} (|f'(z_n^*)|)^{\frac{1}{1 - \alpha}}
\]
\[
= \left(\frac{1 - |z_n|}{2}\right)^{\frac{1}{1 - \alpha}} \left(\frac{1 - |z_n|}{2}\right)^{\frac{1}{1 - \alpha}} (|f'(z_n^*)|)^{\frac{1}{1 - \alpha}} \quad (13)
\]
\[
\geq \left(\frac{1}{2}\right)^{\frac{1}{1 - \alpha}} \left(1 - |z_n^*|\right)^{\frac{1}{1 - \alpha}} (|f'(z_n^*)|)^{\frac{1}{1 - \alpha}} > M \, \frac{1}{1 - \alpha}.
\]
According to (12) and (13), we have $r_n' < |z_n| < r_n$, $|f'(z_n)| \to \infty$, as $n \to \infty$. Set
\[
\rho_n = \frac{1}{M_n} \left(1 - \frac{|z_n|}{r_n}\right)^{\frac{1}{1 - \alpha}} \left(1 - \frac{r_n'}{|z_n|}\right)^{\frac{1}{1 - \alpha}} = \left(\frac{1}{|f'(z_n)|}\right)^{\frac{1}{1 - \alpha}}, \quad (14)
\]
then $\rho_n \to 0$. By (11), we have
\[
1 - |z_n| > 1 - r_n \geq \frac{1}{3}(r_n - r_n') \geq \frac{1}{3}(|z_n| - r_n'). \quad (15)
\]
It follows from (13), (15) and $1/3 < r_n' < |z_n| < r_n$ that
\[
\frac{\rho_n^{1-\alpha}}{1-|z_n|} = \frac{1}{(1-|z_n|)M_{\text{an}}}|(1-\frac{|z_n|}{r_n})^{\frac{1}{2}}(1-\frac{r_n'}{|z_n|})^{\frac{1}{2}} \leq \frac{M_{\text{an}}^{1-\alpha}(r_n-|z_n|)^{\frac{1}{2}}(\frac{1}{4}(|z_n|-r_n)^{\frac{1}{2}}(r_n-|z_n|)^{\frac{1}{2}}}{\sqrt{4M}} = K.
\]
So that from (14) and (16), we have
\[
\frac{\rho_n}{1-|z_n|} \leq K|\rho_n|^a \to 0, \quad \text{as} \ n \to \infty.
\]
Therefore, the functions
\[
g_n(\xi) = \frac{f(z_n + \rho_n\xi) - f(z_n)}{\rho_n}
\]
are defined on the disk $|\xi| < \frac{1-|z_n|}{\rho_n} := R'_n \to \infty$ by (17), as $n \to \infty$. Hence, according to (12) and (15), for each fixed $R' > 0$, when $|\xi| \leq R' < R_n$, we have
\[
|g_n'(\xi)| = \rho_n^{1-\alpha}||f'(z_n + \rho_n\xi)|| \leq \frac{1}{\rho_n^{1-\alpha}}|f'(z)| \to \frac{\rho_n^{1-\alpha}||f'||}{1-|z_n|} \quad \text{as} \ n \to \infty,
\]
where $||f|| = \sup_{|z|<1}(1-|z|^2)|f'(z)| < \infty$. The first term of the inequality (18) is bounded by $K$ from (16); and the last term of the inequality (18) tends to $||f||$ as $n \to \infty$, and hence is also bounded. Thus $\{g_n'\}$ is a normal family on $C$. Taking a subsequence and renumbering, we may assume that the $\{g_n'\}$ converge locally uniformly on compacta to an entire function $G$, which satisfies $|G(\xi)| \leq K||f||$. By Liouville’s Theorem, $G$ is a constant. Since $|g_n'(0)| = 1$, we have $G \equiv a$, where $|a| = 1$. It follows from
\[
g_n(\xi) - g_n(0) = \int_0^\xi g_n'(\xi)d\xi \to a\xi, \quad \text{as} \ n \to \infty
\]
and $g_n(0) = 0$ that $\{g_n(\xi)\}$ tends to $a\xi$ locally uniformly on any compact set in $C$.

Conversely, suppose that there exist $z_n$, $\rho_n$ with $|z_n| < 1$, $|z_n| \to 1$, $\frac{\rho_n}{1-|z_n|} \to 0$ such that $g_n(\xi) = \{f(z_n + \rho_n\xi) - f(z_n)\} / \rho_n \to g(\xi)$ locally uniformly in $C$. Evidently,
\[
|g_n'(\xi)| \leq \frac{\rho_n^{1-\alpha}}{(1-|z_n|)} \cdot \frac{1}{(1-\frac{\rho_n|\xi|}{|z_n|})} \cdot (1-|z_n| + \rho_n\xi)^2 |f'(z_n + \rho_n\xi)|.
\]
If $f$ is a little Bloch function, the product of the last two terms on the right tends to 0; the second term on the right tends to 1 by (17), and hence is bounded; and the first term on the right is bounded by (16). Thus $g'(\xi) = 0$ for all $\xi \in C$, so $g(\xi) \equiv \text{const.}$ This completes the proof of Theorem 2.

**Example 5.** Let $f(z) = \log \frac{1}{1-z}$; since $(1-|z|^2)|f'(z)| = \frac{1-|z|}{1-|z|^2}(1+|z|)$, it is evident that $f$ is a Bloch function, but not a little Bloch function. Fix $0 < \alpha < 1$, and set
\[ z_n = 1 - \frac{1}{n}, \quad \rho_n = n^{-\alpha_n}, \] so that \( n\rho_n \to 0 \) as \( n \to \infty \). Then
\[ \frac{f(z_n + \rho_n z) - f(z_n)}{\rho_n^\alpha} = n^{1-\alpha} \log \left( \frac{1}{1 - \frac{1}{n^{\alpha}(1-z)} \rho_n} \right) \]
tends to \( z \) as \( n \to \infty \).

**Remark 6.** Theorem 2 does not hold if \( \alpha = 0 \). Indeed, let \( \{z_n\} \) be an arbitrary sequence in the unit disk with \( |z_n| \to 1 \), and \( \{\rho_n\} \) an arbitrary sequence of positive numbers with \( \rho_n \to 0 \), and satisfying \( \rho_n / (1 - |z_n|) \to 0 \). Then with \( f \) as above, we have
\[ f(z_n + \rho_n z) - f(z_n) = \log \frac{1}{1 - \frac{\rho_n z}{1 - z_n}} \to 0 \]
as \( n \to \infty \). Thus Theorem 2 does not hold when \( \alpha = 0 \).

Indeed, when \( \alpha = 0 \), (17) above shows that \( \rho_n / (1 - |z_n|) \) need not tend to 0, so the functions \( g_n \) may not be defined on arbitrarily large disks about the origin. In this case, Theorem AW insures that the sequence \( \{g_n\} \) converges locally uniformly on some disk of fixed radius about the origin.

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**References**